Symmetric Dynamical Systems **J.H.P. Dawes**
Part III 2007–08

Example Sheet 4

Starred questions are not necessarily harder, just less central to subsequent course material. Send comments and queries to `J.H.P.Dawes@damtp.cam.ac.uk`.

1. Show that absolute irreducibility implies irreducibility.

2. Show that the representation of the group D_n, $n \geq 3$, on $\mathbb{R}^2 \simeq \mathbb{C}$, and generated by

$$\rho(z) = \mathrm{e}^{2\pi \mathrm{i}/n} z, \qquad m_x(z) = \bar{z},$$

is absolutely irreducible.

3. The group orbit $\mathcal{G}x$ of a point $x \in \mathbb{R}^n$ is defined to be $\mathcal{G}x = \{gx : g \in \mathcal{G}\}$. Show that $\Sigma_{gx} = g\Sigma_x g^{-1}$, i.e. that points on the same group orbit have conjugate isotropy subgroups.

4. (a) Let H be a subgroup of $\mathcal{G}$ acting on $\mathbb{R}^n$. Show that if $x \in \mathrm{Fix}(H)$ then $H \subseteq \Sigma_x$.
(b) Consider $\mathbb{Z}_4$ acting on $\mathbb{R}^2 \simeq \mathbb{C}$, generated by $\rho(z) = \mathrm{i}z$: show that H may be strictly smaller than Σ_x.

* 5. Let H be an isotropy subgroup of $\mathcal{G}$. show that $N_{\mathcal{G}}(H)$ contains all symmetries that map $\mathrm{Fix}(H)$ onto itself. Deduce that the dynamics within $\mathrm{Fix}(H)$ is $N_{\mathcal{G}}(H)/H$ – equivariant.

6. (a) For the worked example with the 2D representation of D_3, compute the stability of the new branch near $\mu = 0$ using the truncation containing only linear and quadratic terms. Show the new branch is not stable for small $|\mu|$.
(b) For the worked example with the 2D representation of D_4, show that if both branches bifurcate supercritically (i.e. into $\mu > 0$) then exactly one of them is stable.

7. Consider a bifurcation problem with D_5 symmetry, acting via its 2D absolutely irreducible representation, given in question 2. Let $\Sigma = \{I, m_x\}$ be the isotropy subgroup of the new solution branch created at the bifurcation point. By computing the low-order Taylor series terms in the amplitude equation, show that the new branch bifurcates as a pitchfork even though $N(\Sigma)/\Sigma \cong \{I\}$.

8. *Hopf bifurcation with $\mathbb{Z}_2$ symmetry.* Let $\mathbb{Z}_2 = \{I, \kappa\}$ act on $\mathbb{R}$ nontrivially, i.e. $\kappa(x) = -x$. Show that a generic Hopf bifurcation with this symmetry is symmetric under $\mathbb{Z}_2 \times S^1$ acting on $\mathbb{R}^2$ by

$$\kappa \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$
$$\tau_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Apply the Equivariant Hopf Theorem to conclude there exists a unique branch of periodic orbits. Give the spatiotemporal symmetry group of these orbits.

9. *Pattern formation on a rotating hexagonal lattice.* Consider a steady-state bifurcation problem on a hexagonal lattice in a rotating system; the symmetry group is now $\mathbb{Z}_6 \ltimes T^2$ generated by ρ and $\tau_{\mathbf{p}}$. Show that the amplitude equations are now

$$\dot{z}_1 \quad = \quad \mu z_1 + \varepsilon \bar{z}_2 \bar{z}_3 - a z_1 |z_1|^2 - b z_1 |z_2|^2 - c z_1 |z_3|^2$$

plus symmetric versions for z_2 and z_3. Find the axial isotropy subgroups and solution branches.

Now restrict to the subspace where all the amplitudes are real, so that the equivariant ODEs (truncated at cubic order) take the form

$$\begin{aligned}
\dot{x}_1 &= x_1[\mu - a x_1^2 - b x_2^2 - c x_3^2] + x_2 x_3 \\
\dot{x}_2 &= x_2[\mu - a x_2^2 - b x_3^2 - c x_1^2] + x_1 x_3 \\
\dot{x}_3 &= x_3[\mu - a x_3^2 - b x_1^2 - c x_2^2] + x_1 x_2
\end{aligned}$$

Show that the branch of solutions (x, x, x) undergoes a secondary Hopf bifurcation when $x = \mu(b + c - 2a)/(4a + b + c)$, (assuming the usual nondegeneracy conditions), which gives rise to periodic orbits with period $2\pi/\omega$ where $\omega = 2\sqrt{3}\mu(c - b)/(4a + b + c)$.

[Hint: use the structure of the Jacobian matrix, or isotypic decomposition, to find the eigenvalues.]

10. *[Part III 2004, Paper 60, Question 1].* Consider the group $\Gamma = \mathbb{O} \times \mathbb{Z}_2^c$ of rotations and reflections of a cube in $\mathbb{R}^3$, centred at the origin and aligned with the coordinate axes. A representation of Γ on $\mathbb{R}^3$ is defined by the following matrices representing elements that generate the group:

$$\kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad r_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

(a) Show that this representation of Γ is absolutely irreducible.

(b) By determining their isotropy subgroups show that three (group orbits of) distinct equilibria are guaranteed to bifurcate from the origin in a generic Γ-symmetric steady-state bifurcation problem.

(c) Determine the normal form amplitude equations for the bifurcation, up to cubic order.

[Hint: to simplify notation define κ_y, κ_z and r_z by analogy with the matrices above.]

* 11. The *trace formula*

$$\dim \text{Fix}(\Sigma) \quad = \quad \frac{1}{|\Sigma|} \sum_{g \in \Sigma} \text{tr}(g)$$

provides an explicit way of computing the dimension of the fixed point subspace of an isotropy subgroup $\Sigma \subseteq \mathcal{G}$, where $|\Sigma|$ is the number of elements in Σ, and $\text{tr}(g)$ means the trace of the matrix representing g. Check that the formula confirms the results in lectures for steady-state bifurcations with D_3 and D_4 symmetry. Check also your results of Question 10(b) above.

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