The Unified Transform Method for Linear PDEs

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1 Introduction

1.1 Linearizable PDEs

There exist certain nonlinear evolution PDEs which can be mapped to linear evolution PDEs. The prototypical such linearizable PDE is the so-called Burger’s equation

$$u_t - uu_x = 2uu_x,$$  \hspace{1cm} (1.1)

which can be written in the conservation form

$$\partial_t u = \partial_x (u_x + u^2).$$

Employing the so-called Cole-Hopf transformation

$$u = \frac{q_x}{q},$$  \hspace{1cm} (1.2)

and using the identity

$$\partial_t \frac{q_x}{q} = \partial_t \partial_x \ln q = \partial_x \partial_t \ln q = \partial_x \frac{q_t}{q},$$

we find, upon an \(x\)-integration, the heat equation

$$q_t - q_{xx} = 0.$$  \hspace{1cm} (1.3)

1.2 Integrable PDEs

Burger’s equation arises in several applications involving nonlinear diffusive processes. A prototypical equation arising in several applications involving nonlinear dispersive processes is the celebrated Korteweg-deVries (KdV) equation

$$u_t + uu_x + u_{xxx} + uu_x = 0.$$  \hspace{1cm} (1.4)
Is there a transformation analogous to (1.2) which maps (1.4) to the corresponding linear equation
\[ q_t + q_x + q_{xxx} = 0, \quad (1.5) \]
called the Stokes equation? The answer to this question is negative. However, there does exist a more subtle linearization procedure, which is based on the existence of the so-called Lax pair formulation. Equations which admit such a formulation are called integrable PDEs. Such equations are rare, however many of them are physically significant (F. Calogero has presented a heuristic argument for the explanation of this fact). The simplest integrable nonlinear evolution PDE in one spatial dimension is the so-called nonlinear Schrödinger (NLS) equation
\[ iu_t + u_{xx} - 2|u|^2u = 0, \quad \sigma = \pm 1, \quad u \in \mathbb{C}. \quad (1.6) \]

An associated Lax pair is given by the following equations:
\[ \Psi_x + i\sigma_3\Psi = U\Psi, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u \\ \sigma\bar{u} & 0 \end{pmatrix}, \quad (1.7a) \]
\[ \Psi_t + 2ik^2\sigma_3\Psi = (2kU - iU_x\sigma_3 - iU^2\sigma_3)\Psi, \quad k \in \mathbb{C}. \quad (1.7b) \]

Suppose that \( u \) and hence \( U \) is given; equations (1.7) are two equations for the single matrix-valued function \( \Psi(x, t, k) \), hence in general these equations will not have a common solution. It can be shown that equations (1.7) are compatible iff \( \Psi_{xt} = \Psi_{tx} \), and this is indeed the case iff \( u \) satisfies the NLS equation (1.6).

The above discussion shows that an integrable PDE is equivalent to the associated Lax pair. Hence, instead of analyzing directly a given integrable nonlinear PDE, one can analyze its linear Lax pair. For example, for the solution of the initial value problem on the line, one first analyzes the \( x \)-part of the Lax pair. This yields a nonlinear Fourier transform pair. Then, the \( t \)-part yields the evolution of the nonlinear Fourier data.

Regarding the analogy between linear and nonlinear Fourier transforms, recall that for the linear version of the NLS, namely for the equation
\[ iq_t + q_{xx} = 0, \quad (1.8) \]
we have the following scheme: Given initial data \( q_0(x) \), we compute their Fourier transform (FT) defined by
\[ \hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx}q_0(x)dx, \quad k \in \mathbb{R}. \quad (1.9) \]

Equation (1.8) implies that the evolution of the FT of \( q(x, t) \) is given by \( \exp(-ik^2t)\hat{q}_0(k) \). Hence, the inverse Fourier transform formula implies
\[ q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - ik^2t}\hat{q}_0(k)dk, \quad x \in \mathbb{R}. \quad (1.10) \]
For NLS, the analogue of $\hat{q}_0(k)$ is the pair of functions \( \{a(k), b(k)\} \), where if
\[
u = \varepsilon q + O(\varepsilon^2), \quad a(k) \to 1 \quad \text{and} \quad b(k) \to \varepsilon \hat{q}_0(k), \quad \varepsilon \to 0.
\]
However, \( \{a, b\} \) cannot be computed explicitly in terms of \( q_0 \), but they satisfy a system of linear Volterra integral equations. The \( t \)-part of the Lax pair implies that the above functions, called \textit{spectral data}, evolve in \( t \) as \( \{a(k), e^{-ik^2t}b(k)\} \). Given these functions, it is possible to compute \( u(x,t) \), but in contrast to the explicit formula (1.10), one now finds a \( 2 \times 2 \) Riemann-Hilbert problem, which is equivalent to a Fredholm type integral equation.

In spite of the above technical difficulties the above novel method, called \textit{inverse scattering transform}, does yield useful information for \( u(x,t) \). For example, for the KdV equation, it can be shown that \( q_0(x) \) will decompose into a number of \textit{solitons} as \( t \to \infty \). A soliton is a localized structure with remarkable interaction properties resembling the interaction properties of particles.

### 1.3 Lax Pairs for Linear PDEs

It was shown in the early nineties by the late Israel Gel’fand and the author that linear evolution PDEs with constant coefficients also admit a Lax pair formulation. For example, the linear version of the NLS, i.e. equation (1.8), admits the Lax pair
\[
\begin{align*}
\mu_x - ik\mu &= q, \quad (1.11a) \\
\mu_t + ik^2\mu &= -kq + iq_x. \quad (1.11b)
\end{align*}
\]
Indeed, equations (1.11) can be rewritten in the form
\[
\begin{align*}
(e^{-ikx+ik^2t}\mu)_x &= e^{-ikx+ik^2t}q, \quad (1.12a) \\
(e^{-ikx+ik^2t}\mu)_t &= e^{-ikx+ik^2t}(-kq + iq_x). \quad (1.12b)
\end{align*}
\]
Hence, equations (1.11) are compatible iff
\[
(e^{-ikx+ik^2t}q)_t = \left[ e^{-ikx+ik^2t}(-kq + iq_x) \right]_x,
\]
which is valid iff \( q \) solves (1.8).

The implementation of the inverse scattering transform to the NLS is based on the derivation of a nonlinear FT pair via the analysis of equation (1.7a). This suggests that the usual FT pair can be derived via the analogous analysis of equation (1.11a). This is indeed the case: It was shown by Gel’fand and the author that the analysis of (1.11a), as well as the analysis of the two dimensional extension of (1.11a) associated with the two dimensional version of (1.8), yield \textit{novel} derivations of the usual one and two dimensional Fourier transforms. This approach gave rise to a new technique for deriving transforms, or equivalency for inverting certain integrals. The first significant application of this approach led to the inversion of the so-called attenuated Radon transform, which is the basic tool needed for the analytical investigation of an important medical imaging technique called single photon emission computerized tomography (SPECT).
1.4 Initial-Boundary Value Problems

In 1982, the late Julian Cole suggested to Mark Ablowitz and the author the investigation of the KdV formulated on the half-line, i.e. equation (1.4) with \(0 < x < \infty, \ t > 0\), supplemented with the initial and boundary conditions

\[
\begin{align*}
    u(x,0) &= u_0(x), \quad 0 < x < \infty; \\
    u(0,t) &= g_0(t), \quad t > 0,
\end{align*}
\]  

(1.13)

where \(u(x,t)\) has sufficient decay for all \(t\) as \(x \to \infty\) and \(u_0, g_0\) have sufficient smoothness and decay.

The difficulty associated with the above initial-boundary value problem (IBVP) becomes clear if one considers the corresponding linear problem, i.e. Stokes equation (1.5) formulated on the half line and supplemented with

\[
\begin{align*}
    q(x,0) &= q_0(x), \quad 0 < x < \infty; \\
    q(0,t) &= g_0(t), \quad t > 0.
\end{align*}
\]  

(1.14)

It is often stated that a separable linear PDE in \((x, t)\) formulated in a separable domain can be solved by either a transform in \(x\) or a transform in \(t\). For example, the linearized version of NLS, equation (1.8), supplemented with (1.14) can be solved by either the sine transform in \(x\) or by the Laplace transform in \(t\). However, there does not exist an \(x\)-transform for the Stokes equation (1.5) supplemented with equations (1.14). Thus, in this case the only classical transform is the Laplace transform in \(t\). However, this approach has several limitations: (a) it involves \(\exp[st + \lambda(s)x]\), where \(\lambda(s)\) solves the cubic algebraic equation \(s + \lambda + \lambda^3 = 0\), whereas an \(x\)-transform would involve \(\exp[ikx - w(k)t]\), where \(w(k)\) is given by the explicit expression \(w(k) = ik - ik^3\); (b) it requires \(t\) going to \(\infty\), which is not natural for an evolution PDE (this difficulty can be overcome by appealing to causality arguments); and (c) it does not generalize to KdV since the inverse scattering transform involves an \(x\) as opposed to a \(t\)-transform.

1.5 Synthesis as Opposed to Separation of Variables

Progress in the analysis of IBVPs for integrable nonlinear evolution PDEs was finally made by the author in the late nineties. This was based on the realization that IBVPs require the implementation of an appropriate synthesis instead of separation of variables. In this respect it is interesting to note that a Lax pair formulation provides a deeper type of separation of variables valid for nonlinear PDEs. However, the inverse scattering transform is based on the separate analysis of the two equations forming the Lax pair, thus the inverse scattering transform also follows the classical idea of separation of variables. This is to be contrasted with the approach introduced by the author which is based on the simultaneous analysis of both equations forming a Lax Pair. This corresponds to a synthesis instead of separation of variables.

An unexpected outcome of the new method was that it gave rise to a new method for analyzing linear PDEs: Let us return to equation (1.8) and the
associated Lax pair (1.11). The classical separation of variables of equation (1.8) yields the following pair of eigenvalue equations:

\[
\frac{d^2 X(x, k)}{dx^2} + k^2 X(x, k) = 0, \quad (1.15a)
\]

\[
\frac{dT(t, k)}{dt} - k^2 T(t, k) = 0, \quad k \in \mathbb{C}. \quad (1.15b)
\]

If a given IBVP for equation (1.8) admits appropriate \(x\) and \(t\) transform pairs, those transform pairs can be obtained by performing the spectral analysis of equations (1.15a) and (1.15b) respectively. Alternatively, these transform pairs can also be derived by performing the spectral analysis of equations (1.11a) and (1.11b) respectively. However, equations (1.11) offer a new possibility: the simultaneous spectral analysis of equations (1.11) yields a new transform which is valid only for equation (1.8) formulated in a given domain. Hence, this transform is "custom made" for this particular PDE and this particular domain.

After the emergence of the above so-called unified transform method, it was realized that for linear PDEs, it is possible to obtain the relevant transform directly without the need to introduce a Lax pair (although for certain elliptic PDEs the simultaneous spectral analysis of the Lax pair remains the simplest way of constructing the new transform). In what follows we will present the direct implementation of the new method to evolution PDEs on the half line.

## 2 Evolution PDEs on the Half-Line

### 2.1 The Fourier Transform

If \(f(x) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})\), then the FT pair is given by

\[
\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \quad k \in \mathbb{R}, \quad (2.1a)
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk, \quad x \in \mathbb{R}. \quad (2.1b)
\]

In the particular case that \(f(x)\) has support only for \(0 < x < L\), equations (2.1) yield the finite FT

\[
\hat{f}(k) = \int_{0}^{L} e^{-ikx} f(x) dx, \quad k \in \mathbb{C}, \quad (2.2a)
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk, \quad 0 < x < L. \quad (2.2b)
\]

If \(L \to \infty\), equations (2.2) become

\[
\hat{f}(k) = \int_{0}^{\infty} e^{-ikx} f(x) dx, \quad k \in \mathbb{C}^- = \{k \in \mathbb{C}, \quad \text{Im} \ k \leq 0\}. \quad (2.3a)
\]
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk, \quad 0 < x < \infty. \quad (2.3b) \]

In the particular case that \( f(x) \) is an odd function, equations (2.1) reduce to the sine transform
\[ \hat{f}_s(k) = \int_0^{\infty} \sin(kx) f(x) dx, \quad k \in \mathbb{R}, \quad (2.4a) \]
\[ f(x) = \frac{2}{\pi} \int_0^{\infty} \sin(kx) \hat{f}_s(k) dk, \quad 0 < x < \infty. \quad (2.4b) \]

It should be noted that in equation (2.2a) \( \exp(-ikx) \) is bounded for all finite \( k \), thus the function \( \hat{f}(k) \) is an entire function. Similarly, in equation (2.3a), \( |\exp(-ikx)| = \exp(kx) \) thus the function \( \hat{f}(k) \) is an analytic function for \( k \in \mathbb{C}^- \) and \( \hat{f}(k) \) is of \( O(1/k) \) as \( k \to \infty \).

### 2.2 Jordan’s Lemma

Let \( C_R \) denote the semi-circle of radius \( R \) in the upper half complex \( z \)-plane centered at the origin, i.e.
\[ C_R : \{ z = Re^{i\theta}, \quad 0 < \theta < \pi, \quad R > 0 \}. \]

Assume that the analytic function \( f(z) \) vanishes on \( C_R \) as \( R \to \infty \), namely
\[ |f(z)| < K(R), \quad z \in C_R \quad \text{and} \quad K(R) \to 0 \quad \text{as} \quad R \to \infty. \]

Then,
\[ \int_{C_R} e^{i\alpha z} f(z) dz \to 0 \quad \text{as} \quad R \to \infty, \quad \alpha > 0. \]

**Proof**
\[ \left| \int_{C_R} e^{i\alpha z} f(z) dz \right| = \left| \int_0^\pi e^{i\alpha R(\cos \theta + i\sin \theta)} f(z) iRe^{i\theta} d\theta \right| \leq \]
\[ K(R) R \int_0^\pi e^{-|R| \sin \theta} d\theta = 2RK(R) \int_0^{\pi/2} e^{-\alpha R \sin \theta} d\theta \]
\[ \leq 2RK(R) \int_0^{\pi/2} e^{-\alpha R R \frac{\theta}{\pi}} d\theta = \frac{\pi}{\alpha} K(R) [1 - e^{-\alpha R}]. \]

![Figure 2.1](image-url)
2.3 The Heat Equation on the Line

Consider the heat equation (1.3) formulated on the line, with initial data \( q_0(x) \), i.e.

\[
q(x, 0) = q_0(x), \quad -\infty < x < \infty.
\]  

(2.5)

Then,

\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^2t} \hat{q}_0(k) \, dk, \quad -\infty < x < \infty, \quad t > 0,
\]

where

\[
\hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} q_0(x) \, dx, \quad k \in \mathbb{R}.
\]  

(2.7)

Indeed, the only \((x, t)\) dependence of the RHS of (2.6) is of the form \( \exp(ikx - k^2t) \) which clearly satisfies the heat equation. Furthermore, evaluating (2.6) at \( t = 0 \) we find

\[
q(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{q}_0(k) \, dk = q_0(x),
\]

where for the second equality above we have used the definition (2.7) as well as the inverse FT formula (2.1b).

2.4 The heat equation on the half-line via the sine transform

Let \( q(x, t) \) satisfy the heat equation (1.3), as well as equations (1.14). Let \( \hat{q}_s(k, t) \) denote the sine transform of \( q(x, t) \), i.e.

\[
\hat{q}_s(k, t) = \int_{0}^{\infty} \sin(kx) q(x, t) \, dx, \quad k \in \mathbb{R}.
\]

Then,

\[
\frac{\partial}{\partial t} \hat{q}_s(k, t) = \int_{0}^{\infty} \sin(kx) \frac{\partial q}{\partial t}(x, t) \, dx = \int_{0}^{\infty} \sin(kx) \frac{\partial^2 q}{\partial x^2}(x, t) \, dx.
\]

Employing integration by parts twice and using \( q(0, t) = g_0(t) \), we find

\[
\frac{\partial}{\partial t} \hat{q}_s + k^2 \hat{q}_s = kg_0(t).
\]

Hence,

\[
\hat{q}_s(k, t)e^{k^2t} = \hat{q}_0(k) + k \int_{0}^{t} e^{k^2\tau} g_0(\tau) \, d\tau, \quad k \in \mathbb{R},
\]

(2.8)

where \( \hat{q}_0(k) \) denotes the sine transform of \( q_0(x) \).

Using the inverse sine transform formula (2.4b) we find

\[
q(x, t) = \frac{2}{\pi} \int_{0}^{\infty} \sin(kx) e^{-k^2t} \left[ \hat{q}_0(k) + k \int_{0}^{t} e^{k^2\tau} g_0(\tau) \, d\tau \right] \, dk, 0 < x < \infty, t > 0.
\]  

(2.9)
The starting point of the above derivation is the definition $\hat{q}_s(k, t)$. However, $q(x, t)$ is unknown, thus unless we appeal to a priori PDE estimates, it is not clear that $\hat{q}_s(k, t)$ even exists. Thus, the above procedure is valid only under the a priori assumption of existence. In order to eliminate this assumption it is necessary to show that the function $q(x, t)$ defined by the RHS of equation (2.9) satisfies the heat equation, the initial condition $q(x, t) = q_0(x)$, as well as the boundary condition $q(0, t) = g_0(t)$. It is straightforward to verify the former two conditions, but the lateral condition is more complicated. This is due to the fact that the RHS of (2.10) is not uniformly convergent at $x = 0$, unless $g_0(t) = 0$. This means that we cannot interchange the limit $x \to 0$ with the $k$-integral. This lack of uniform convergence also makes the representation (2.9) unsuitable for the numerical evaluation of the solution. It should be emphasized that the solution of any IBVP with non-homogeneous boundary conditions obtained by the classical transform method will have the same disadvantages.

2.5 The heat equation on the half-line via the unified transform method

The new method involves three steps.

1a. *Rewrite the given PDE as a one-parameter family of divergence forms.*

For the heat equation we find

$$\left( e^{-ikx+k^2t}q \right)_t - \left[ e^{-ikx+k^2t}(q_x + ikq) \right]_x = 0, \quad k \in \mathbb{C}. \quad (2.10)$$

1b. *Given a domain, using the above divergence form as well as Green’s theorem, derive the global relation, i.e. an equation coupling $q$ and its derivatives on the boundary of the domain.*

For the heat equation formulated on the half-line we find

\[ e^{k^2t} \int_0^\infty e^{-ikx} q(x, t) dx = \int_0^\infty e^{-ikx} q_0(x) dx - \int_0^t e^{k^2\tau} [q_x(0, \tau) + ikq(0, \tau)] d\tau, \quad k \in \mathbb{C}^-. \quad (2.11) \]

Let us introduce some notation:

\[ \hat{q}(k, t) = \int_0^\infty e^{-ikx} q(x, t) dx, \quad \hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x) dx, \quad k \in \mathbb{C}^- , \quad (2.12) \]

Figure 2.2
\[ \tilde{g}_0(k, t) = \int_0^t e^{k\tau} q(0, \tau) d\tau, \quad \tilde{g}_1(k, t) = \int_0^t e^{k\tau} q_x(0, \tau) d\tau, \quad k \in \mathbb{C}. \quad (2.13) \]

Then, the global relation (2.11) can be written in the form
\[ e^{k^2t} \hat{q}(k, t) = \hat{q}_0(k) - \tilde{g}_1(k^2, t) - ik \tilde{g}_0(k^2, t), \quad k \in \mathbb{C}^-. \quad (2.14) \]

An alternative way to obtain this equation is to apply the half-line Fourier transform (2.2a) to the heat equation.

2. Express the solution as an integral in the complex \( k \)-plane involving \( \hat{q}_0(k) \), as well as transforms of all the relevant boundary values.

For the heat equation formulated on the half-line we find
\[ q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t}[\tilde{g}_1(k^2, t) + ik \tilde{g}_0(k^2, t)] dk, \]
where the contour \( \partial D^+ \) is the boundary of the domain \( D^+ \) defined by
\[ D^+ : \{ k \in \mathbb{C}^+, \quad \text{Re} k^2 < 0 \}. \quad (2.16) \]

Indeed, solving the global relation (2.14) for \( \hat{q}(k, t) \) and then using the inverse Fourier transform formula (2.3b), we find an expression similar to (2.15) but with the contour of integration along the real line instead of \( \partial D^+ \). In order to deform from the real line to \( \partial D^+ \) we use Cauchy’s theorem: the function
\[ e^{ikx-k^2t} \tilde{g}_1(k^2, t) = e^{ikx} \int_0^t e^{-k^2(t-\tau)} q_x(0, \tau) d\tau \]
is an analytic function of \( k \) which is bounded as \( k \to \infty \) if \( k \) satisfies \( \text{Im} k \geq 0 \) and \( \text{Re} k^2 \geq 0 \). Furthermore, integration by parts implies that this function is of \( O(1/k^2) \) as \( k \to \infty \). Thus, Cauchy’s theorem in the domain bounded by the real
line and $\partial D^+$ implies that the integral of the above function can be deformed from $\mathbb{R}$ to $\partial D^+$. The situation is similar with the term $i k \exp[i k x - k^2 t] \tilde{g}_0(k^2, t)$, but now this function is of $O(1/k)$ as $k \to \infty$, thus we need to supplement Cauchy's theorem with Jordan's lemma.

3. For given boundary conditions, by employing the global relation as well as certain invariant transformations, eliminate from the integral representation obtained in step 2. the transforms of the unknown boundary values.

Consider for example the Dirichlet problem of the heat equation formulated on the half line, i.e. equation (1.3) supplemented with the initial and boundary conditions (1.14). In this case, the functions $\hat{q}_0$ and $\tilde{g}_0$ appearing in the global relation (2.14) are known but the functions $\hat{q}$ and $\tilde{g}_1$ are unknown. Our strategy will be to use equation (2.14) to eliminate $\tilde{g}_1$; in this procedure we ignore $\hat{q}$ since it will turn out that its contribution to $q(x, t)$ vanishes. The unknown $\tilde{g}_1$ appears in (2.15) on the contour $\partial D^+$ which is in $\mathbb{C}^+$, whereas the global relation (2.14) is valid in $\mathbb{C}^-$. The transformation $k \to -k$ has two important features: (a) maps $\mathbb{C}^-$ into $\mathbb{C}^+$, and (b) leaves the unknown $\tilde{g}_1(k^2, t)$ invariant. Thus, replacing in the global relation (2.14) $k$ with $-k$ and then solving the resulting equation for $\tilde{g}_1(k^2, t)$ we find

$$\tilde{g}_1 = i k \tilde{g}_0 + \tilde{g}_0(-k) + e^{k^2 t} \hat{q}(-k, t), \quad k \in \mathbb{C}^+. \quad (2.17)$$

Replacing in equation (2.15) $i k \tilde{g}_1$ with the RHS of (2.17) we find

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - k^2 t} \tilde{g}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{i k x - k^2 t} [2i k \tilde{g}_0(k^2, t) + \hat{g}_0(-k)] dk. \quad (2.18)$$

The term $\exp(k^2 t) \hat{q}(-k, t)$ gives rise to the term

$$-\frac{1}{2\pi} \int_{\partial D^+} e^{i k x} \hat{q}(-k, t) dk, \quad 0 < x < \infty, \quad t > 0,$$

which vanishes, since both $\exp(i k x)$ and $\hat{q}(-k, t)$ are bounded and analytic in $\mathbb{C}^+$, and $\hat{q}(-k, t)$ is of $O(1/k)$ as $k \to \infty$.

Remarks 2.1

(a) Let $W(x, t, k)$ be defined by

$$W = e^{-i k x + k^2 t} [qdx + (q_x + i k q)dt]. \quad (2.19)$$

Then

$$dW = \left\{ (e^{-i k x + k^2 t} q)_x - e^{-i k x + k^2 t} (q_x + i k q) \right\} dt \wedge dx. \quad (2.20)$$

Hence, equation (2.10) implies that in the domain of validity of the heat equation, $dW = 0$, thus the integral of $W$ along the boundary of this domain vanishes. For the particular case that this domain is given by $\{0 < x < \infty, \quad t > 0\}$, we find (2.11).
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(b) Suppose that the heat equation is valid for $0 < t < T$. Let

$$\tilde{g}_0(k) \doteq \bar{g}_0(k, T), \quad \tilde{g}_1(k) \doteq \bar{g}_1(k, T). \quad (2.21)$$

Then, equation (2.15) is equivalent to the equation

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} [\hat{g}_1(k^2) + i k \bar{g}_0(k^2)] dk. \quad (2.15)'$$

Indeed, the RHS of equation (2.15) and the RHS of equation (2.15)' differ by the term

$$\frac{1}{2\pi} \int_{\partial D^+} e^{ikx} \left[ \int_t^T e^{k^2(\tau-t)} q_x(0, \tau) d\tau + ik \int_t^T e^{k^2(\tau-t)} q(0, \tau) d\tau \right] dk,$$

and Cauchy’s theorem supplemented with Jordan’s lemma imply that the above term vanishes.

Similarly, equation (2.18) is equivalent to the equation

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} [2ik \hat{g}_0(k^2) + \hat{q}_0(-k)] dk. \quad (2.18)'$$

The advantage of (2.18)' is that the only $(x, t)$ dependence of the RHS of (2.18)' is of the form $\exp(ikx - k^2t)$, thus it immediately follows that the function $q$ defined in (2.18)' satisfies the heat equation. On the other hand, (2.18) is consistent with causality, since the function $q(x, t)$ cannot depend on the values of $g_0(\tau)$ for $\tau > t$.

(c) In deriving (2.15), the real line was deformed to $\partial D^+$. This deformation is always possible before using the global relation. However, after using the global relation we introduce $\hat{q}_0$ and then it is not always possible to return to the real axis. Actually, the cases where there does exist a classical transform, are precisely the cases where this “return” is possible.

In the particular case of (2.18), we note that $\hat{q}_0(-k)$ is bounded and analytic in $\mathbb{C}^+$, thus it is possible to return to the real axis,

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} [\hat{g}_0(k) - \hat{q}_0(-k)] dk - \frac{i}{\pi} \int_{-\infty}^{\infty} ke^{ikx-k^2t} \bar{g}_0(k^2, t) dk.$$

Splitting the integral along $\mathbb{R}$ to an integral from $-\infty$ to 0 plus an integral from 0 to $\infty$, and letting $k \to -k$ in the former integral we obtain (2.9). An easier way to obtain (2.9) is to note that the global relation together with the equation obtained from the global relation after replacing $k$ with $-k$ are the following equations:

$$e^{k^2t} q(k, t) = \hat{q}_0(k) - \hat{g}_1 - ik \bar{g}_0, \quad k \in \mathbb{C}^-, \quad (k^2t) q(-k, t) = \hat{q}_0(-k) - \hat{g}_1 + ik \bar{g}_0, \quad k \in \mathbb{C}^+. \quad (2.22)$$
If \( k \in \mathbb{R} \), both equations (2.22) are valid. Hence if \( g_0 \) is given, we subtract equations (2.22) and we obtain equation (2.8). Similarly, if \( q_x(0, t) \) is given, we add equations (2.22) and we obtain

\[ e^{k^2 t} \hat{q}_c(k, t) = \hat{q}_0(k) - \hat{g}_1(k^2, t), \quad k \in \mathbb{R}, \]

where \( \hat{q}_c \) and \( \hat{q}_0 \) denote the cosine transform of \( q(x, t) \) and \( q_0(x) \).

(d) Equation (2.18)' immediately implies that \( q(x, t) \) satisfies the heat equation. Furthermore, evaluating (2.18) at \( t = 0 \) we find

\[ q(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx} \hat{q}_0(-k) dk, \quad x > 0. \]

Jordan’s lemma implies that the second integral in the above expression vanishes and hence by recalling the definition of \( \hat{q}_0(k) \) and employing the inverse FT formula we find

\[ q(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{q}_0(k) dk. \]

Evaluating (2.18)' at \( x = 0 \) we find

\[ q(0, t) = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} e^{-k^2 t} \hat{q}_0(k) dk - \int_{\partial D^+} e^{-k^2 t} \hat{q}_0(-k) dk \right] - \frac{1}{2\pi} \int_{\partial D^+} 2ike^{-k^2 t} \hat{q}_0(k^2) dk. \]

By deforming the second integral to the real axis and then replacing \( k \) with \(-k\) we find that the first two terms in the RHS of (2.23) cancel. Furthermore, letting \( ik^2 = l \) in the last integral in the RHS of (2.23) we find

\[ q(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ilt} \left( \int_{0}^{T} e^{-it\tau} g_0(\tau) d\tau \right) dl = g_0(t). \]

2.6 The General Case

Let \( q(x, t) \) satisfy the linear evolution PDE

\[ q_t + w(-i\partial_x)q = 0, \quad (2.24a) \]

\[ 0 < x < \infty, \quad 0 < t < T, \quad (2.24b) \]

where \( T \) is a positive constant and \( w(k) \) is given by

\[ w(k) = \alpha_0 + \alpha_1 k + \cdots + \alpha_n k^n, \quad \{\alpha_j\}_{0}^{n-1} \text{ constants}. \quad (2.25a) \]

We assume that

\[ \text{Re } w(k) \geq 0 \text{ for } k \text{ real.} \quad (2.25b) \]

Equation (2.24) admits the solution

\[ q(x, t) = e^{i(kx - w(k)t)}. \]

The constraint (2.25b) ensures that the initial value problem of (2.24a) is well posed.

We recall that step 1a depends only on the given PDE, i.e. it is independent of the domain and the given initial and boundary conditions.
Step 1a

\[ (e^{-ikx+w(k)t}q)_t - (e^{-ikx+w(k)t}Q)_x = 0, \quad (2.26a) \]

where

\[ Q(x, t, k) = i \left. \frac{w(k) - w(l)}{k - l} \right|_{l=-i\partial_x} q. \quad (2.27a) \]

Indeed, simplifying (2.26a) and using \( q_t = -w(-i\partial_x)q \) we find

\[ (\partial_x - ik)Q = [w(k) - w(-i\partial_x)]q, \]

which is equation (2.27a).

Using the fact that \( (w(k) - w(l))/(k - l) \) is a polynomial of degree \( n - 1 \) in both \( k \) and \( l \), it follows that \( Q \) can be expressed in the form

\[ Q(x, t, k) = \sum_{j=0}^{n-1} c_j(k) \partial_x^j q(x, t). \quad (2.27b) \]

For a given PDE, the easiest way to determine \( w(k) \) is to require that the expression \( \exp(ikx - w(k)t) \) is a solution of this PDE.

**EXAMPLES**

1. Heat equation:

\[ w(k) = k^2, \quad Q = i k^2 - l^2 \quad Q = (ik + \partial_x)q, \quad Q = q_x + ikq. \quad (2.28) \]

2. Linear NLS:

\[ w(k) = ik^2, \quad Q = iq_x - kq. \quad (2.29) \]

3. Stokes equation:

\[ w(k) = ik - ik^3, \quad Q = -qxx - ikqx + (k^2 - 1)q. \quad (2.30) \]

4. \( q_t + q_{xxx} = 0 \):

\[ w(k) = -ik^3, \quad Q = -qxx - ikqx + k^2q. \quad (2.31) \]

5. \( q_t - q_{xxx} = 0 \):

\[ w(k) = ik^3, \quad Q = qxx + ikqx - k^2q. \quad (2.32) \]

6. \( q_t = q_{xx} + \beta q_x \):

\[ w(k) = k^2 - i\beta k, \quad Q = q_x + (ik + \beta)q. \quad (2.33) \]
Step 1b. This step depends on both the PDE and the domain. For the domain specified by (2.24b), equation (2.26a) implies
\[ e^{w(k)t} \hat{q}(k, t) = \hat{q}_0(k) - \tilde{g}(k, t), \quad k \in \mathbb{C}^-, \] (2.34a)
where \( \hat{q} \) and \( \hat{q}_0 \) are the FTs of \( q \) and \( q_0 \), see equations (2.12), and
\[ \tilde{g}(k, t) = \int_0^t e^{w(k)\tau} Q(0, \tau, k) d\tau = \sum_{j=0}^{n-1} c_j(k) \tilde{g}_j(w(k), t), \] (2.34b)
with
\[ \tilde{g}_j(k, t) = \int_0^t e^{k\tau} \partial_\tau^j q(0, \tau) d\tau, \quad j = 0, 1, \ldots, n-1, \quad k \in \mathbb{C}. \] (2.34c)

Step 2 Solving (2.34a) for \( \hat{q}(k, t) \), using the inverse FT formula, and deforming from \( \mathbb{R} \) to \( \partial D^+ \) we find
\[ q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-w(k)t} \tilde{g}(k, t) dk, \] (2.35a)
where
\[ D^+: \{ k \in \mathbb{C}^+, \quad \text{Re } w(k) < 0 \}, \] (2.35b)
and \( D^+ \) is to the left of the increasing direction of \( \partial D^+ \).

EXAMPLES For the examples 1-7, the domain \( D \) is depicted in the figures below.

1. \( \cos 2\theta < 0 \)

2. \( \sin 2\theta > 0 \)
3. $k_I(1 + k_I^2 - 3k_R^2) > 0$

4. $\sin 3\theta < 0$

5. $\sin 3\theta > 0$
6. $\beta > 0, \quad k_R^2 - k_I^2 + \beta k_I < 0$

Step 3. The transformations which leave $w(k)$ invariant, i.e. the transformations

$$k \to \nu(k), \quad w(k) = w(\nu(k)), \quad (2.36)$$

map the subdomains of $D$ among themselves. Hence, in examples 3 and 4, starting with the global relation (2.34a) in $D^-$, we obtain two equations valid in $D^+$, whereas in example 5 we find one equation valid in $D^+_1$ and one equation valid in $D^+_2$, where $D^+_1$ and $D^+_2$ denote the two subdomains of $D^+$. Thus, in examples 3 and 4 we need to prescribe one boundary condition for a well posed problem, whereas in example 5 we require two boundary conditions. In general, it is straightforward to show that the IBV problem defined in equations (2.24) requires $N(n)$ boundary conditions, where

$$N(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n+1}{2}, & n \text{ odd and } \alpha_n \neq -iC, \ C \text{ positive constant} \\ \frac{n-1}{2}, & n \text{ odd and } \alpha_n = -iC, \ C \text{ positive constant.} \end{cases}$$
For examples 3 and 4, \( n = 3 \) and \( \alpha_3 = -i \), thus \( N = 1 \).

The implementation of Step 3 uses only algebraic manipulations. The general procedure is illustrated by some simple examples.

**EXAMPLES**

The linearized NLS with \( q_x(0, t) = g_1(t) \)

In this case

\[
\tilde{g} = ig_1 - k\tilde{g}_0
\]

and the global relation (2.34a) is given by

\[
e^{ik^2t}\tilde{q}(k, t) = \tilde{q}_0(k) - i\tilde{g}_1 + k\tilde{g}_0, \quad k \in D^-.
\]

Replacing \( k \) with \(-k\), solving in the resulting equation for \( k\tilde{g}_0 \), and then substituting in (2.35a) we find

\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - ik^2t}\tilde{q}_0(k)dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - ik^2t}[\tilde{q}_0(-k) - 2i\tilde{g}_1(ik^2, t)]dk,
\]

where \( \partial D^+ \) is depicted in Figure 2.5 and \( \tilde{g}_1 \) is defined by

\[
\tilde{g}_1(ik^2, t) = \int_{0}^{t} e^{ik^2\tau}g_1(\tau)d\tau.
\]

**Equation (2.31) with \( q(0, t) = g_0(t) \).**

The global relation (2.34a) is given by

\[
e^{-ik^3t}\tilde{q}(k, t) = \tilde{q}_0(k) + i\tilde{g}_2 + ik\tilde{g}_1 - k^2\tilde{g}_0, \quad k \in D^-.
\]

Replacing \( k \) with \( \alpha k \) and with \( \alpha^2k \), \( \alpha = \exp(\frac{2i\pi}{3}) \), we find

\[
e^{-ik^3t}\tilde{q}(\alpha k, t) = \tilde{q}_0(\alpha k) + i\tilde{g}_2 + i\alpha k\tilde{g}_1 - \alpha^2k^2\tilde{g}_0, \quad k \in D^+, \quad (2.37a)
\]

\[
e^{-ik^3t}\tilde{q}(\alpha^2k, t) = \tilde{q}_0(\alpha^2k) + i\tilde{g}_2 + i\alpha^2k\tilde{g}_1 - \alpha k^2\tilde{g}_0, \quad k \in D^+, \quad (2.37b)
\]

Note that if \( k \in D^+ \), then

\[
\pi < \arg(\alpha k) < \frac{4\pi}{3}, \quad \frac{5\pi}{3} < \arg\alpha^2k < 2\pi,
\]

thus \( \alpha k \in D^+_1 \) and \( \alpha^2k \in D^+_2 \), where \( D^+_1 \) and \( D^+_2 \) denote the two subdomains of \( D^- \).

Equations (2.37a) and (2.37b) can be considered as two algebraic equations for the two unknown functions \( \tilde{g}_1 \) and \( \tilde{g}_2 \). Thus, we can solve these equations for \( \tilde{g}_1 \) and \( \tilde{g}_2 \) and then substitute the relevant expressions in

\[
\tilde{g}(k, t) = -i\tilde{g}_2 - ik\tilde{g}_1 + k^2\tilde{g}_0, \quad k \in D^+. \quad (2.37c)
\]
It is slightly more convenient to supplement (2.37a) and (2.37b) with (2.37c) and to eliminate \( \tilde{g}_1 \) and \( \tilde{g}_2 \) from these three equations: multiplying (2.37a) and (2.37b) by \(-\alpha\) and \(-\alpha^2\) respectively, and using the identity

\[
1 + \alpha + \alpha^2 = 0,
\]

we find

\[
\tilde{g}(k, t) = 3k^2 \tilde{g}_0 - \alpha \tilde{q}_0(\alpha k) - \alpha^2 \tilde{q}_0(\alpha^2 k) + e^{-ik^3 t}[\alpha \tilde{q}(\alpha k, t) + \alpha^2 \tilde{q}(\alpha^2 k, t)].
\]

As noted earlier, if \( k \in D^+ \), then \( \alpha k \in D^- \) and \( \alpha^2 k \in D^- \), thus \( \tilde{q}(\alpha k, t) \) and \( \tilde{q}(\alpha^2 k, t) \) are bounded and analytic and hence their contribution vanishes. Thus, equation (2.35a) implies

\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}(\alpha k^2 + i\beta k - \alpha k^3 - i\beta k)t \tilde{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx}(\alpha k^2 + i\beta k - \alpha k^3 - i\beta k)t \tilde{q}_0(k) dk,
\]

where \( \partial D^+ \) is depicted in Figure 2.7.

**The Stokes equation with \( q(0, t) = g_0(t) \).**

\[
\nu - \nu_3 = k - k^3:
\nu = \begin{cases} 
-\frac{k}{2} + \frac{1}{2}\sqrt{3k^2 - 4} & \Rightarrow \nu_1 \\
-\frac{k}{2} - \frac{1}{2}\sqrt{3k^2 - 4} & \Rightarrow \nu_2
\end{cases}.
\]

Note that

\[
\nu(k) \to \begin{cases} 
\frac{\alpha k}{\alpha^2 k}, & \alpha = e^{2\pi i}, \ k \to \infty.
\end{cases}
\]

Similar calculations with those in the case of equation (2.31) imply

\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}(\nu_3 - i\beta k)t \tilde{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx}(\nu_3 - i\beta k)t [3k^2 \tilde{g}_0(\nu_3 - i\beta k) - \alpha \tilde{q}_0(\alpha k) - \alpha^2 \tilde{q}_0(\alpha^2 k)] dk,
\]

where \( \partial D^+ \) is depicted in Figure 2.6.

**Equation (2.33) with \( \beta > 0 \) and \( q(0, t) = g_0(t) \).**

In this case

\[
\tilde{g} = \tilde{g}_1 + (ik + \beta)\tilde{g}_0.
\]

Equation (2.36) implies

\[
\nu^2 - i\beta \nu = k^2 - i\beta k: \nu = -k + i\beta.
\]

The global relation (2.34a) is given by

\[
e^{(k^2 - i\beta k)t} \tilde{q}(k, t) = \tilde{q}_0(k) - \tilde{g}_1 - (ik + \beta)\tilde{g}_0, \quad k \in D^-.
\]
Replacing \( k \) with \( \nu \) we find
\[
e^{(k^2 - i\beta k)t} \tilde{q}(-k + i\beta, t) = \tilde{q}_0(-k + i\beta) - \tilde{g}_1 + i k \tilde{g}_0, \quad k \in D^+.
\]
Solving this equation for \( \tilde{g}_1 \), using the resulting equation in (2.35a), and noting that the contribution of \( \hat{q}(-k + i\beta, t) \) vanishes, we find
\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - \nu(k^2 - i\beta k)t} \hat{q}_0(k)dk
- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - \nu(k^2 - i\beta k)t}[\hat{q}_0(-k + i\beta) + (2i k + \beta)\tilde{g}_0(k^2 - i\beta k, t)]dk,
\]
where \( \partial D^+ \) is depicted in Figure 2.9.

It is worth noting that
\[
\tilde{q}_0(-k + i\beta) = \int_0^t e^{ik^2x} e^{(k_1 - \beta)\tau} q_0(x)dx,
\]
thus \( \tilde{q}_0(-k + i\beta) \) is well defined on \( \partial D^+ \) where \( k_1 \geq \beta \), but, in general, it is not defined below \( \partial D^+ \), hence we cannot return to the real line.

The linearized NLS with \( q_x(0, t) + \gamma q(0, t) = g_R(t), \gamma \) constant.

If a boundary condition involves a combination of boundary values, then the expression for \( \tilde{g}(k, t) \) contains a removable singularity. The general case is illustrated with the aid of the above Robin boundary condition. Multiplying this boundary condition by \( \exp(-ik^2t) \) and integrating with respect to \( d\tau \) from 0 to \( t \), we find
\[
\tilde{g}_1(-ik^2, t) + \gamma \tilde{g}_0(-ik^2, t) = \tilde{g}_R(-ik^2, t), \quad k \in \mathbb{R}, \quad t > 0,
\]
where
\[
\tilde{g}_R(k, t) = \int_0^t e^{k^2\tau} g_R(\tau) d\tau, \quad k \in \mathbb{C}.
\]
Eliminating \( \tilde{g}_1 \) in the expression for \( \tilde{g} \) we find
\[
\tilde{g} = i \tilde{g}_1 - k \tilde{g}_0 = i \tilde{g}_R - (k + i\gamma)\tilde{g}_0.
\]
The global relation is
\[
e^{ik^2t} \tilde{q}(k, t) = \tilde{q}_0(k) - i\tilde{g}_R + (k + i\gamma)\tilde{g}_0, \quad k \in D^-.
\]
Replacing \( k \) with \(-k\), solving for \( \tilde{g}_0 \) and then substituting in the above expression for \( \tilde{g} \) we find
\[
\tilde{g}(k, t) = i \tilde{g}_R + \frac{k + k_0}{k - k_0} [e^{ik^2t} \tilde{q}(-k, t) - \tilde{q}_0(-k) + i \tilde{g}_R], \quad k_0 = i\gamma, \quad k \in D^+.
\]
If \( k_0 \notin D^+ \), then \( 1/(k - k_0) \) is non-singular and the contribution of the term involving \( \tilde{q}_0(-k) \) vanishes. Let us consider the case that \( k_0 \) is in the
domain $D^+$. Since $\tilde{g}(k,t)$ is an entire function, the point $k_0 \in D^+$ must be a removable singularity (this can be verified directly by letting $k = k_0$ in the equation obtained from (2.39) after replacing $k$ with $-k$). Let $C_{k_0}$ denote a circle of radius $\varepsilon$ centered at $k_0$, see Figure 2.10a, and let $\partial D^+_{k_0}$ denote the union of $\partial D^+$ with $C_{k_0}$, see Figure 2.10b. The integral of $\exp(ikx + ik^2t)\tilde{g}(k,t)$ around $C_{k_0}$ vanishes. Thus,

\[
\int_{\partial D^+} e^{ikx-ik^2t}\tilde{g}(k,t)dk = \int_{\partial D^+_{k_0}} e^{ikx-ik^2t} \left[i\tilde{g}_R + \frac{k+k_0}{k-k_0}(i\tilde{g}_R - \tilde{q}_0(-k))\right]dk \\
+ \int_{\partial D^+_{k_0}} e^{ikx} \frac{k+k_0}{k-k_0} \tilde{q}(-k,t)dk.
\]

The integrand of the second integral above is analytic in the shaded region of Figure 2.10b, thus this integral vanishes and $q(x,t)$ is given by

\[
q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-ik^2t} \tilde{q}_0(k)dk - \frac{1}{2\pi} \int_{\partial D^+_{k_0}} e^{ikx-ik^2t} \frac{[2ik\tilde{g}_R -(k+k_0)\tilde{q}_0(-k)]}{k-k_0}dk.
\]

By computing the relevant residue we find the following equivalent representation for $q$:

\[
q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-ik^2t} \tilde{q}_0(k)dk - \frac{1}{2\pi} \int_{\partial D^+_{k_0}} e^{ikx-ik^2t} \frac{[2ik\tilde{g}_R -(k+k_0)\tilde{q}_0(-k)]}{k-k_0}dk \\
+ ie^{ik_0x-ik_0^2t}[2i\tilde{q}_0(-k_0^2,t) - 2k_0\tilde{q}_0(-k_0)], \quad k_0 = i\gamma,
\]

where $\partial D^+$ is depicted in Figure 2.5

**Remark 2.3.** In the above examples, $\tilde{g}_0(w(k),t)$, $\tilde{g}_1(w(k),t)$ and $\tilde{g}_R(w(k),t)$ can of course be replaced by $\tilde{g}_0(w(k))$, $\tilde{g}_1(w(k))$ and $\tilde{g}_R(w(k))$ respectively.
2.7 Numerical Evaluations

For the simple cases when the transforms of the given data can be computed explicitly, the numerical evaluation of the solution obtained by the new method reduces to the computation of a single integral in the complex \( k \)-plane. Using simple contour deformations, it is possible to obtain an integrand which decays exponentially as \( k \to \infty \).

**EXAMPLE** Consider the heat equation on the half line with

\[
q(x, 0) = e^{-a^2 x}, \quad q(0, t) = \cos(bt), \quad a, b \text{ real constants}.
\]

Then,

\[
\hat{q}_0(k) = \int_0^\infty e^{-ikx-a^2 x} dk = \frac{1}{ik + a^2},
\]

\[
\tilde{g}_0(k, t) = \int_0^t e^{k\tau} \cos(b\tau) d\tau = \frac{1}{2} \left[ \frac{e^{(k+i b) t}}{k + ib} - 1 + \frac{e^{(k-i b) t}}{k - ib} - 1 \right].
\]

Hence (2.18) becomes

\[
2\pi q(x, t) = \int_{-\infty}^{\infty} \frac{e^{ikx-k^2 t}}{ik + a^2} dk
\]

- \[
\int_{\partial D^+} e^{ikx-k^2 t} \left[ \frac{1}{-ik + a^2} + \frac{ik}{k + ib} \left( e^{(k^2+ib)t} - 1 \right) + \frac{ik}{k - ib} \left( e^{(k^2-ib)t} - 1 \right) \right] dk.
\]

The term \( \exp(ikx) \) in the integrand of the second integral decays as \( k \to \infty \), but the term \( \exp(-k^2 t) \) oscillates. However, if we deform \( \partial D^+ \) to a contour \( L \) between the real line and \( \partial D^+ \), then we achieve exponential decay in both \( \exp(ikx) \) and \( \exp(-k^2 t) \):

\[
2\pi q(x, t) = \int_L \left\{ e^{ikx-k^2 t} \left[ \frac{1}{ik + a^2} + \frac{1}{ik - a^2} \right] + ike^{ikx} \left[ \frac{e^{ibt} - e^{-k^2 t}}{k + ib} + \frac{e^{-ibt} - e^{-k^2 t}}{k - ib} \right] \right\} dk,
\]

where \( L \) is depicted in Figure 2.11

![Figure 2.11](image)

3 Linear Evolution PDEs on the Interval

Let

\[
0 < x < L, \quad 0 < t < T,
\]

where \( L \) and \( T \) are positive constants. In what follows we implement steps 1b, 2, and 3.
Step 1b.
\[ e^{w(k)t} \tilde{q}(k,t) = \tilde{q}_0(k) - \tilde{g}(k,t) + e^{-ikL} \tilde{h}(k,t), \quad k \in \mathbb{C}, \]  
where \( \tilde{q}(k,t) \) and \( \tilde{q}_0 \) are the finite Fourier transforms of \( q(x,t) \) and \( q_0(x) \), i.e.
\[ \tilde{q}(k,t) = \int_{0}^{L} e^{-ikx} q(x,t) \, dx, \quad \tilde{q}_0(k) = \int_{0}^{L} e^{-ikx} q_0(x) \, dx, \quad k \in \mathbb{C}, \]  
\( \tilde{g}(k,t) \) is defined by (2.34b) and \( \tilde{h}(k,t) \) is defined by
\[ \tilde{h}(k,t) = \int_{0}^{t} e^{w(k)\tau} Q(L, \tau, k) d\tau = \sum_{j=0}^{n-1} \bar{c}_j(k) \tilde{h}_j(w(k), t), \]  
with
\[ \tilde{h}_j(k,t) = \int_{0}^{t} e^{k\tau} \partial_2^j q(L, \tau, k) d\tau, \quad j = 0, \ldots, n-1, \quad k \in \mathbb{C}. \]  

Step 2. Solving (3.2a) for \( \tilde{q}(k,t) \), using the inverse FT formula and deforming from \( \mathbb{R} \) to \( \partial D^+ \) in the integral involving \( \tilde{g} \), and from \( \mathbb{R} \) to \( \partial D^- \) in the integral involving \( h \), we find
\[ q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \tilde{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-w(k)t} \tilde{g}(k,t) dk \]
\[ -\frac{1}{2\pi} \int_{\partial D^-} e^{-ik(L-x)-w(k)t} \tilde{h}(k,t) dk, \]  
where
\[ D^- : \{ k \in \mathbb{C}^- , \quad \text{Re } w(k) < 0 \}, \]
and \( D^- \) is to the left of the increasing direction of \( \partial D^- \).

Step 3. The transformations (2.36) together with the global relation (3.2a) yield \( n \) equations. Since there exist \( 2n \) unknown boundary values (\( n \) at each end), we require \( n \) boundary conditions. However, we cannot distribute these conditions in an arbitrary manner: a well posed problem requires \( N(n) \) boundary conditions at \( x = 0 \) and \( n - N(n) \) at \( x = L \). Indeed, it can be shown that the terms arising from \( \tilde{q}(w(k), t) \) are bounded as \( k \to \infty \) in the relevant domains \( D^+ \) and \( D^- \) if the boundary conditions are distributed in the above way.
EXAMPLES

The heat equation with \( q(0,t) = g_0(t), q(L,t) = h_0(t) \)

The functions \( \tilde{g} \) and \( \tilde{h} \) are given by

\[
\tilde{g} = \tilde{g}_1 + i k \tilde{g}_0, \quad \tilde{h} = \tilde{h}_1 + i k \tilde{h}_0.
\]  

(3.4)

Then the global relation (3.2a) becomes

\[
e^{k^2t} \hat{q}(k,t) = G(k,t) - \tilde{g}_1 + e^{-ikL} \tilde{h}_1,
\]

(3.5a)

where the known function \( G \) is defined by

\[
G(k,t) = \hat{q}_0(k) - i k \tilde{g}_0(k^2,t) + i k e^{-ikL} \tilde{h}_0(k^2,t).
\]

(3.6)

Letting \( k \to -k \) in (3.5a) we obtain

\[
e^{k^2t} \hat{q}(-k,t) = G(-k,t) - \tilde{g}_1 + e^{ikL} \tilde{h}_1.
\]

(3.5b)

Solving equations (3.5) for \( \tilde{g}_1 \) and \( \tilde{h}_1 \) we find

\[
\tilde{g}_1 = \frac{1}{e^{ikL} - e^{-ikL}} \left\{ e^{ikL} G(k,t) - e^{-ikL} G(-k,t) + e^{k^2t} [e^{-ikL} \hat{q}(-k,t) - e^{ikL} \hat{q}(k,t)] \right\},
\]

(3.7a)

\[
\tilde{h}_1 = \frac{1}{e^{ikL} - e^{-ikL}} \left\{ G(k,t) - G(-k,t) + e^{k^2t} [\hat{q}(-k,t) - \hat{q}(k,t)] \right\}.
\]

(3.7b)

We substitute \( \tilde{g}_1 \) and \( \tilde{h}_1 \) in (3.4) and then insert the expression for \( \tilde{g} \) and \( \tilde{h} \) in (3.3a). We claim that the terms involving \( \hat{q}(\pm k,t) \) yield a zero contribution. Indeed, since this is a well posed BVP, the relevant terms are bounded as \( k \to \infty \).

Let us verify this explicitly: for the term in \( \tilde{g}_1 \), i.e. for the term

\[
e^{-ikL} \hat{q}(-k,t) - e^{ikL} \hat{q}(k,t),
\]

\( k \in \mathbb{C}^+ \), thus \( e^{-ikL} \) grows, and the above expression, as \( k \to \infty \), becomes

\[
-\hat{q}(-k,t) + e^{ikL} \int_0^L e^{i(kL-x)} q(x,t) dx,
\]

which is clearly bounded as \( k \to \infty \) in \( \mathbb{C}^+ \). Similarly for the term in \( \tilde{h}_1 \), i.e. for the term

\[
\hat{q}(-k,t) - \hat{q}(k,t)
\]

\( k \in \mathbb{C}^- \), thus \( e^{ikL} \) grows, and the above expression, as \( k \to \infty \), becomes

\[
\int_0^L e^{-i(kL-x)} q(x,t) dx - e^{-ikL} \hat{q}(k,t),
\]
which is clearly bounded as \( k \to \infty \) in \( \mathbb{C}^- \).

We also note that the zeros of \( \exp(ikL) - \exp(-ikL) \) occur on the real axis, and hence are outside \( D \) except for \( k = 0 \) which is clearly a removable singularity, since

\[
[e^{-ikL}\tilde{q}(-k, t) - e^{ikL}\tilde{q}(k, t)]_{k=0} = [\tilde{q}(-k, t) - \tilde{q}(k, t)]_{k=0} = 0.
\]

Thus, (3.3a) becomes

\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} \tilde{q}_{0}(k)dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} \left[ik\tilde{\gamma}_{0}(k^2, t) + \frac{e^{ikL}G(k, t) - e^{-ikL}G(-k, t)}{e^{ikL} - e^{-ikL}}\right]dk
\]

\[
- \frac{1}{2\pi} \int_{\partial D^-} e^{ikx-k^2t} \left[ik\tilde{\gamma}_{0}(k^2, t) + \frac{G(k, t) - G(-k, t)}{e^{ikL} - e^{-ikL}}\right]dk,
\]

where \( \partial D^+ \) and \( \partial D^- \) are detected in Figure 2.4

Remarks 3.1.

(a) The numerators of the two fractions in (3.8) vanish at \( k = 0 \), thus \( k = 0 \) is a removable singularity.

(b) It is possible to deform \( \partial D^+ \) and \( \partial D^- \) back to the real axis and then using the residue theorem the classical sine-sine solution can be rederived. A simpler way to obtain the classical solution representation is to subtract (3.5):

\[
2ie^{k^2t}\int_{0}^{L} \sin(kx)q(x, t)dx = (e^{ikL} - e^{-ikL})\tilde{h}_{1}(k^2, t) + G(-k, t) - G(k, t). \tag{3.9}
\]

The unknown function \( \tilde{h}_{1} \) can be eliminated by evaluating the above \( q \) at those values of \( k \) for which the coefficient of \( \tilde{h}_{1} \) vanishes:

\[
e^{ikL} - e^{-ikL} = 0, \quad k = \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots
\]

Hence (3.9) becomes

\[
2ie^{k^2t}\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)q(x, t)dx = G\left(-\frac{n\pi}{L}, t\right) - G\left(\frac{n\pi}{L}, t\right)
\]

and then the classical representation follows.

The heat equation with \( q_{x}(0, t) - \gamma q(0, t) = g_{R}(t), \quad q(L, t) = h_{0}(t), \quad \gamma > 0. \)

Multiplying the left boundary condition by \( \exp(k^2\tau) \) and integrating with respect to \( d\tau \) from 0 to \( t \) we find

\[
\tilde{g}_{1}(k^2, t) - \gamma \tilde{g}_{0}(k^2, t) = \tilde{g}_{R}(k^2, t),
\]
where \( \tilde{g}_R(k, t) \) is defined in (2.38b). Hence,
\[
\tilde{g}(k, t) = \tilde{g}_R(k^2, t) + (ik + \gamma)\tilde{g}_0(k^2, t).
\]
(3.10)

The global relation is
\[
e^{k^2t}\tilde{q}(k, t) = G(k, t) - (ik + \gamma)\tilde{g}_0 + e^{-ikL}\tilde{h}_1.
\]
(3.11a)

where the given function \( G \) is defined by
\[
G(k, t) = \tilde{q}_0(k) - \tilde{g}_R(k^2, t) + ike^{-ikL}\tilde{h}_0(k^2, t).
\]
(3.12)

Replacing \( k \) with \( -k \) in (3.11a) we find
\[
e^{k^2t}\tilde{q}(-k, t) = G(-k, t) - (-ik + \gamma)\tilde{g}_0 + e^{ikL}\tilde{h}_1.
\]
(3.11b)

Equations (3.11) imply
\[
\tilde{g}_0 = \frac{1}{\Delta(k)} \left\{ e^{ikL}G(k, t) - e^{-ikL}G(-k, t) + e^{k^2t}[e^{-ikL}\tilde{q}(-k, t) - e^{ikL}\tilde{q}(k, t)] \right\},
\]
\[
\tilde{h}_1 = \frac{1}{\Delta(k)} \left\{ (\gamma - ik)G(k, t) - (\gamma + ik)G(-k, t) + e^{k^2t}[(\gamma + ik)\tilde{q}(-k, t) - (\gamma - ik)\tilde{q}(k, t)] \right\},
\]
(3.12)

where
\[
\Delta(k) = (\gamma + ik)e^{ikL} - (\gamma - ik)e^{-ikL}.
\]
(3.13)

The zeros of \( \Delta(k) \) occur on the real axis, thus proceeding as with the earlier example and noting that \( k = 0 \) is again a removable singularity, we find
\[
q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t}\tilde{q}_0(k)dk
\]

\[
-\frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} \left\{ \tilde{g}_R(k^2, t) + \frac{ik + \gamma}{\Delta(k)} [e^{ikL}G(k, t) - e^{-ikL}G(-k, t)] \right\} dk
\]

\[
-\frac{1}{2\pi} \int_{\partial D^-} e^{ikx-k^2t} \left\{ ik\tilde{h}_0 + \frac{(\gamma - ik)G(k, t) - (\gamma + ik)G(-k, t)}{\Delta(k)} \right\} dk,
\]
(3.14)

where \( \partial D^+, \partial D^- \) are depicted by Figure 2.4 and \( \Delta(k) \) is defined by (3.13).

The representation (3.14) is similar with the representation (3.8) for the Dirichlet problem. This is to be contrasted with the classical representation for the Robin problem which involves a spectrum satisfying the transcendental equation \( \Delta(k) = 0 \).
Equation (2.31) with $g(0, t) = g_0(t), \ g(L, t) = h_0(t), \ q_k(L, t) = h_1(t)$.

The functions $\tilde{g}$ and $\tilde{h}$ are given by

$$
\tilde{g} = -\tilde{g}_2 - ik\tilde{g}_1 + k^2\tilde{g}_0, \quad \tilde{h} = -\tilde{h}_2 - ik\tilde{h}_1 + k^2\tilde{h}_0,
$$
(3.15)

thus the global relation becomes

$$
e^{-ikL}q(k, t) = G(k, t) + \tilde{g}_2 + ik\tilde{g}_1 - e^{-ikL}\tilde{h}_2,
$$
(3.16a)

where the known function $G(k, t)$ is given by

$$
G(k, t) = q_0(k) - k^2\tilde{g}_0(-ik^3, t) + e^{-ikL}\tilde{h}_1(-ik^3, t) + k^2\tilde{h}_0(-ik^3, t).
$$
(3.17)

Replacing in (3.16a) $k$ with $\alpha^2 k$, $\alpha = \exp(2i\pi/3)$ we find

$$
e^{-ikL}q(\alpha^2 k, t) = G(\alpha^2 k, t) + \tilde{g}_2 + i\alpha k\tilde{g}_1 - e^{-i\alpha^2 kL}\tilde{h}_2,
$$
(3.16b)

Equations (3.16) can be written in the form

$$
\begin{pmatrix}
1 & 1 & -e^{-ikL} \\
1 & \alpha & -e^{-i\alpha kL} \\
1 & \alpha^2 & -e^{-i\alpha^2 kL}
\end{pmatrix}
\begin{pmatrix}
\tilde{g}_2 \\
\text{i} k \tilde{g}_1 \\
\tilde{h}_2
\end{pmatrix}
= \begin{pmatrix}
e^{-ikL}q(k, t) - G(k, t) \\
e^{-i\alpha kL}q(\alpha k, t) - G(\alpha k, t) \\
e^{-i\alpha^2 kL}q(\alpha^2 k, t) - G(\alpha^2 k, t)
\end{pmatrix}, \quad k \in \mathbb{C}.
$$
(3.17)

Hence the unknown functions $\tilde{g}_2, i\alpha k\tilde{g}_1, \tilde{h}_2$ can be expressed in terms of the known function $G$ as well as the unknown functions $\tilde{q}(k, t), \tilde{q}(\alpha k, t), \tilde{q}(\alpha^2 k, t)$.

The terms involving the latter functions are bounded in the relevant subdomains of (3.17) are in $D^+$, $D^-$, and $D_2^+$. Thus, these terms will yield a zero contribution provided that the zeros of the determinant of the matrix appearing in the LHS of (3.17) are in $D^-$. This determinant is given by

$$
\Delta(k) = (1 - \alpha)(-\alpha e^{-ikL} + \alpha^2 e^{-i\alpha kL} + e^{-i\alpha^2 kL}).
$$
(3.18)

The zeros of $\Delta(k)$ can be found by making use of the following result: Let $F(z)$ be defined by

$$
F(z) = e^z + a_1 e^{\lambda_1 z} + \cdots + a_6 e^{\lambda_6 z},
$$
where $\{a_j, \lambda_j\}_{j=1}^6$ are complex constants and assume that the polygon with vertices at the points $\{1, \lambda_1, \ldots, \lambda_6\}$ is not degenerate. Then, the zeros of $F(z)$ are clustered along the rays emanating from the origin with direction orthogonal to the sides of the polygon. Furthermore, these zeros can only accumulate at infinite on these rays.

Letting $-ikL = z$, it follows that the vertices $1, \alpha, \alpha^2$ yield the triangle and the lines depicted in Figure 3.2a. The relevant lines are at angles $\pi/3, \pi$ and $5\pi/3$. Hence the corresponding lines in the complex $k$-plane are at angles $5\pi/6, 3\pi/2$ and $\pi/6$. Thus, these lines are in $D^-$ and hence the contribution of the term involving $\tilde{q}(k, t), \tilde{q}(\alpha k, t), \tilde{q}(\alpha^2 k, t)$ vanishes.
4 Elliptic PDEs in the Interior of a Convex Polygon

It is straightforward to implement the first two steps introduced in section 2.5 to the Laplace, modified Helmholtz, and Helmholtz equations in the interior of a convex polygon. Furthermore, in certain simple cases it is also possible to implement the third step. For brevity of presentation we will concentrate on the modified Helmholtz equation

\[ \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} - 4\beta^2 q = 0, \quad \beta > 0. \] (4.1)

Letting

\[ z = x + iy, \quad \bar{z} = x - iy, \] (4.2)

we find

\[ \partial z = \frac{1}{2}(\partial x - i\partial y), \quad \partial \bar{z} = \frac{1}{2}(\partial x + i\partial y), \] (4.3)

and hence the modified Helmholtz equation can be rewritten in the form

\[ \frac{\partial^2 q}{\partial z \partial \bar{z}} - \beta^2 q = 0. \] (4.1')

Recall that step 1a is independent of the domain. If \( q \) satisfies (4.1) then the following identity is valid:

\[ \left\{ e^{i\beta(\frac{1}{4}-\lambda)x + \beta(\frac{1}{4}+\lambda)y} \left[ q_x + i\beta(\lambda - \frac{1}{\lambda})q \right] \right\}_x = \]

\[ \left\{ e^{i\beta(\frac{1}{4}-\lambda)x + \beta(\frac{1}{4}+\lambda)y} \left[ -q_y + \beta(\lambda + \frac{1}{\lambda})q \right] \right\}_y, \lambda \in \mathbb{C}. \] (4.4)

The easiest way to derive (4.4) is to consider the formal adjoint of (4.1), which actually coincides with (4.1). If \( q \) and \( \bar{q} \) solve (4.1), then

\[ (\bar{q}q_x - q\bar{q}_x)_x = (q\bar{q}_y - \bar{q}q_y)_y. \] (4.5)
Letting

\[ \bar{q} = e^{k_1 x + k_2 y}, \quad k_1^2 + k_2^2 = 4\beta^2 \]

and using the parametrization

\[ k_1 = 2\beta \sin k, \quad k_2 = 2\beta \cos k, \quad k = e^{i\lambda}, \]

equation (4.5) becomes equation (4.4).

Using the complex variables \((z, \bar{z})\), equation (4.4) becomes

\[
\left[ e^{-i\beta(\lambda z - \bar{z})} (q_z + i\lambda \beta \bar{q}) \right]_z + \left[ e^{-i\beta(\lambda z - \bar{z})} (q_{\bar{z}} + i\lambda \beta q) \right]_{\bar{z}} = 0, \lambda \in \mathbb{C}. \quad (4.4)' \]

Step 1b depends on the domain of the validity of the given equation. Let \(\Omega\) denote the interior of the convex polygon formed by the complex numbers \(\{z_1, z_2, \cdots, z_n\}\), see Figure 4.1.

![Figure 4.1](image)

Employing equation (4.4) and Green’s theorem, as well as equation (4.4)', and the complex form of Green’s theorem, we find the following global relation:

\[
\sum_{j=1}^{n} \bar{q}_j(\lambda) = 0, \quad \lambda \in \mathbb{C}\setminus\{0\}, \quad (4.5a)
\]

where \(\bar{q}_j(\lambda)\) is defined by

\[
\bar{q}_j(\lambda) = i \int_{z_j}^{z_{j+1}} e^{i\beta(\frac{1}{\lambda} - \lambda) x + \beta(\frac{1}{\lambda} + \lambda) y} \left\{ [q_z + i\beta(\lambda - \frac{1}{\lambda}) q]dy + [-q_y + \beta(\lambda + \frac{1}{\lambda}) q]dx \right\}
\]

\[
= \int_{z_j}^{z_{j+1}} e^{-i\beta(\lambda z - \bar{z})} \left[ (q_z + i\lambda q)dz - (q_z + \frac{\beta}{i\lambda} q)dz \right], j = 1, \cdots, n. \quad (4.5b)
\]

By replacing \(\lambda\) with \(1/\lambda\) in equations (4.4) and (4.4)', and by employing Green’s theorem, we also find the following additional global relation:

\[
\sum_{j=1}^{n} \bar{q}_j(\lambda) = 0, \quad \lambda \in \mathbb{C}\setminus\{0\}, \quad (4.6a)
\]
where
\[
\hat{q}_j(\lambda) = -i \int_{z_j}^{z_{j+1}} e^{-i\beta(\frac{1}{2} - \lambda)x + \beta(\frac{1}{2} + \lambda)y} \left\{ \left[ q_x - i\beta \left( \lambda - \frac{1}{\lambda} \right) q \right] dy + \left[ -q_y + \beta \left( \lambda + \frac{1}{\lambda} \right) q \right] dx \right\} = \int_{z_j}^{z_{j+1}} e^{i\beta(\lambda z - \bar{z})} \left[ (q_x - i\beta \lambda q)dz - \left( q_z - \frac{\beta}{i\lambda} q \right) d\bar{z} \right].
\] (4.6b)

Remark 4.1. Equations (4.4) and (4.4)' imply that the differential form \( W \) is closed, where \( W \) is defined by
\[
W = ie^{i\beta(\frac{1}{2} - \lambda)x + \beta(\frac{1}{2} + \lambda)y} \left\{ \left[ q_x + i\beta \left( \lambda - \frac{1}{\lambda} \right) q \right] dy + \left[ -q_y + \beta \left( \lambda + \frac{1}{\lambda} \right) q \right] dx \right\} = e^{-i\beta(\lambda z - \bar{z})} \left[ (q_x + i\beta \lambda q)dz - \left( q_z + \frac{\beta}{i\lambda} q \right) d\bar{z} \right].
\] (4.7)

Step 2 can be implemented by either employing the classical Green’s representation formula, or by performing the spectral analysis of the differential form \( W \). Either of these approaches yields the formula
\[
q(z, \bar{z}) = \frac{1}{4i\pi} \sum_{j=1}^{n} \int_{l_j} e^{i\beta(\lambda z - \bar{z})} \hat{q}_j(\lambda) \frac{d\lambda}{\lambda}, \quad z \in \Omega,
\] (4.8)

where \( \{l_j\}_{1}^{n} \) are the rays in the complex \( \lambda \)-plane oriented towards infinity and defined by
\[
l_j = \{ \lambda \in \mathbb{C} : \arg \lambda = - \arg(z_{j+1} - z_j), \quad j = 1, \ldots, n, \quad z_{n+1} = z_1 \}. \] (4.9)

Remark 4.2. Using the identities
\[
q_z dz = \frac{1}{2} (q_T + iq_N) ds,
\]
\[
q_{\bar{z}} d\bar{z} = \frac{1}{2} (q_T - iq_N) ds,
\]
where \( (q_T, q_N) \) denote the (tangential, normal) components of the derivative along a curve parameterized by the arc length \( s \), the second of the equations (4.5b) becomes
\[
\hat{q}_j(\lambda) = \int_{z_j}^{z_{j+1}} e^{-i\beta(\lambda z - \bar{z})} \left[ q_N + \beta \frac{dz}{ds} + \frac{1}{\lambda} \frac{d\bar{z}}{ds} \right] ds, j = 1, \ldots, n.
\]
EXAMPLE Let the complex valued function $q(x, y)$ satisfy equation (4.1) in the quarter plane $0 < \arg z < \pi/2$ with oblique Robin boundary conditions, namely the combination of $q$ and of the derivative of $q$ along the direction making an angle $\gamma_1$ with the $y$-axis, as well as the combination of $q$ and of the derivative of $q$ in the direction making an angle $\gamma_2$ with the $x$-axis are prescribed, see Figure 4.2:

\[
\begin{align*}
\alpha_1 q(0, y) - q_y(0, y) \sin \gamma_1 + q_x(0, y) \cos \gamma_1 &= g_1(y), \quad 0 < y < \infty, \quad (4.10a) \\
\alpha_2 q(x, 0) - q_y(x, 0) \sin \gamma_2 + q_x(x, 0) \cos \gamma_2 &= g_2(x), \quad 0 < x < \infty. \quad (4.10b)
\end{align*}
\]

This general case is analyzed elsewhere. Here in order to minimize computations we only give the details for the Newman problem, i.e. we consider the case

\[
\begin{align*}
q_x(0, y) &= g_1(y), \quad 0 < y < \infty, \quad (4.11a) \\
q_y(x, 0) &= g_2(x), \quad 0 < x < \infty. \quad (4.11b)
\end{align*}
\]

In this case, $z_1 = i\infty$, $z_2 = 0$, $z_3 = \infty$. Thus,

\[-\arg(z_2 - z_1) = \frac{\pi}{2}, \quad -\arg(z_3 - z_2) = 0; \]

hence $l_1$ and $l_2$ are the rays along the positive imaginary axis and the positive real axis respectively.

Equations (4.5b) yield

\[
\begin{align*}
\hat{q}_1(\lambda) &= -i \int_0^\infty e^{i\beta(x+y)} y \left[ q_x(0, y) + i\beta \left( \lambda - \frac{1}{\lambda} \right) q(0, y) \right] dy, \quad \text{Re} \lambda \leq 0, \\
\hat{q}_2(\lambda) &= i \int_0^\infty e^{i\beta(x-y)} x \left[ -q_y(x, 0) + i\beta \left( \lambda + \frac{1}{\lambda} \right) q(x, 0) \right] dx, \quad \text{Im} \lambda \leq 0. \quad (4.12a)
\end{align*}
\]

Using (4.11) in (4.12) we find

\[
\hat{q}_1(\lambda) = \beta \left( \lambda - \frac{1}{\lambda} \right) U_1(\lambda) - iG_1(\lambda), \quad \text{Re} \lambda \leq 0, \quad (4.13a)
\]
\[ \hat{q}_2(\lambda) = \beta \left( \lambda + \frac{1}{\lambda} \right) U_2(-i\lambda) - iG_2(-i\lambda), \quad \text{Im} \, \lambda \leq 0, \quad (4.13b) \]

where \( G_1 \) and \( G_2 \) are the known functions

\[ G_1(\lambda) = \int_0^\infty e^{\beta(\lambda + \frac{1}{\lambda})} \xi g_1(y)dy, \quad \text{Re} \, \lambda \leq 0, \quad (4.14a) \]

\[ G_2(-i\lambda) = \int_0^\infty e^{-i\beta(\lambda - \frac{1}{\lambda})} \xi g_2(x)dx, \quad \text{Im} \, \lambda \leq 0, \quad (4.14b) \]

whereas \( U_1 \) and \( U_2 \) are the unknown functions

\[ U_1(\lambda) = \int_0^\infty e^{\beta(\lambda + \frac{1}{\lambda})} \xi q(0, y)dy, \quad \text{Re} \, \lambda \leq 0, \quad (4.15a) \]

\[ U_2(-i\lambda) = \int_0^\infty e^{-i\beta(\lambda - \frac{1}{\lambda})} \xi q(x, 0)dx, \quad \text{Im} \, \lambda \leq 0. \quad (4.15b) \]

The global relations (4.5a) and (4.6a) yield

\[ \beta \left( \lambda - \frac{1}{\lambda} \right) U_1(\lambda) + i\beta \left( \lambda + \frac{1}{\lambda} \right) U_2(-i\lambda) - iG_1(\lambda) - iG_2(-i\lambda) = 0, \quad \lambda \in III, \quad (4.16a) \]

\[ \beta \left( \lambda - \frac{1}{\lambda} \right) U_1(\lambda) - i\beta \left( \lambda + \frac{1}{\lambda} \right) U_2(i\lambda) + iG_1(\lambda) + iG_2(i\lambda) = 0, \quad \lambda \in II, \quad (4.16b) \]

where \( I, \ldots, IV \) denote the first, \ldots, fourth quadrant of the complex \( \lambda \)-plane.

We will express \( \hat{q}_1 \) and \( \hat{q}_2 \) in terms of the unknown function \( U_2(i\lambda) \): equation (4.16b) yields

\[ U_1(\lambda) = i \frac{(\lambda^2 + 1)}{\lambda^2 - 1} U_2(i\lambda) - i \frac{G_1(\lambda) + G_2(i\lambda)}{\beta(\lambda - \frac{1}{\lambda})}, \quad \lambda \in II. \quad (4.17a) \]

Eliminating \( U_1 \) from equations (4.16), replacing \( \lambda \) with \( -\lambda \) in the resulting equation and then solving for \( U_2(-i\lambda) \) we find

\[ U_2(-i\lambda) = -U_2(i\lambda) - \frac{1}{\beta(\lambda + \frac{1}{\lambda})}[2G_1(-\lambda) + G_2(-i\lambda) + G_2(i\lambda)], \quad \lambda \in \mathbb{R}^+. \quad (4.17b) \]
The unknown function $U_2(i\lambda)$ yields the following contribution

$$-i\beta \int_{\partial I} \left( \lambda + \frac{1}{\lambda} \right) U_2(i\lambda) e^{i\beta(\lambda - \frac{1}{\lambda})x - \beta(\lambda + \frac{1}{\lambda})y} \frac{d\lambda}{\lambda},$$

where $\partial I$ denotes the boundary of $I$. The integrand of the above integral is bounded and analytic in $I$, thus its contribution vanishes. Hence,

$$q(z, \bar{z}) = \frac{1}{4\pi} \int_0^{\infty} e^{i\beta(\lambda z - \frac{1}{\lambda})} \hat{q}_1(\lambda) \frac{d\lambda}{\lambda} + \frac{1}{4\pi} \int_0^{\infty} e^{i\beta(\lambda z - \frac{1}{\lambda})} \hat{q}_2(\lambda) \frac{d\lambda}{\lambda}, \quad (4.18a)$$

where

$$\hat{q}_1(\lambda) = -2iG_1(\lambda) - iG_2(\lambda), \quad \hat{q}_2(\lambda) = -2iG_2(\lambda - i\lambda) - 2iG_1(-\lambda - i\lambda).$$

Using analyticity we can rewrite $\hat{q}_1, \hat{q}_2$ in the symmetric form

$$\hat{q}_1(\lambda) = -2i[G_1(\lambda) + G_1(-\lambda)], \quad \hat{q}_2(\lambda) = -2i[G_2(\lambda) + G_2(-\lambda)]. \quad (4.18b)$$

Hence,

$$q(z, \bar{z}) = -\frac{1}{2\pi} \int_0^{\infty} e^{i\beta(\lambda z - \frac{1}{\lambda})}[G_1(\lambda) + G_1(-\lambda)] \frac{d\lambda}{\lambda}$$

$$-\frac{1}{2\pi} \int_0^{\infty} e^{i\beta(\lambda z - \frac{1}{\lambda})}[G_2(\lambda) + G_2(-\lambda)] \frac{d\lambda}{\lambda}. \quad (4.19)$$

Equation (4.19) can also be derived by using the classical cosine transform and then deforming in the complex $k$-plane. However, the BVP with the boundary condition (4.10) cannot be solved by a classical transform.

**Numerical Evaluation**

As in the case of evolution PDEs, if the functions $G_1$ and $G_2$ can be evaluated explicitly, then the numerical computation of $q$ involves a single integral. Consider, for example, the case of

$$g_1(y) = e^{-a^2 y}, \quad a > 0.$$ 

Then,

$$G_1(\lambda) = \frac{1}{a^2 - \beta (\lambda + \frac{1}{\lambda})}$$

and the first integral in the RHS of (4.19) yields

$$I_1 = \int_0^{\infty} e^{i\beta(\lambda z - \frac{1}{\lambda})} \left[ \frac{1}{a^2 - \beta (\lambda + \frac{1}{\lambda})} + \frac{1}{a^2 + \beta (\lambda + \frac{1}{\lambda})} \right] \frac{d\lambda}{\lambda}$$

$$= \int_0^{\infty} e^{-\beta(\rho^2 + \frac{1}{\rho^2})} \left[ \frac{1}{a^2\rho - i\beta (\rho^2 - 1)} + \frac{1}{a^2\rho + i\beta (\rho^2 - 1)} \right] d\rho. \quad (4.20)$$
Noting that 
\[ e^{-\beta(\rho z + \bar{z})} = e^{-\beta(\rho + \bar{1})x} e^{-i\beta(\rho - \bar{1})y}, \]
\[ e^{-\beta(\rho z + \bar{z})} = e^{-\beta(\rho + \bar{1})x} e^{-i\beta(\rho - \bar{1})y}, \]
\[ e^{-\beta(\rho z + \bar{z})} = e^{-\beta(\rho + \bar{1})x} e^{-i\beta(\rho - \bar{1})y}, \]
it follows that the x-exponential decays as \( \rho \to \infty \), but the y-exponential oscillates. However, if we deform the contour to the lower half complex \( \rho \)-plane, then both the x and the y-exponentials decay.

Similar considerations are valid for the second integral in the rhs of (4.19).

**A Semi-Analytical Spectral Method for Elliptic Equations in the Interior of Polygons.**

The analysis of the global relations yields a new numerical technique for characterizing the “generalized Dirichlet to Neumann map”. This technique provides the analogue of the boundary integral method, but now the analysis takes place in the Fourier space instead of the physical space. For the implementation of this technique one must chose: (a) a suitable basis for expanding the unknown boundary values, and (b) an appropriate set of complex values at which to evaluate the global relations. Several such choices have already appeared in the literature; it appears that the best choice is (a) the unknown boundary values are expanded in terms of Legendre polynomials, and (b) the collocation points are on certain rays in the complex \( \lambda \)-plane. In this connection we note that Smitheman and the author have recently presented an analytical formula for the Fourier transform of the Legendre polynomials [?].

**EXAMPLE**  Consider the simplest polygon, namely the square with corners at
\[ (-1, 1), \quad (-1, -1), \quad (1, -1), \quad (1, 1) \]

![Figure 4.4](image-url)
\[ q^{(1)} = q(-1, y) = \cosh(1) \cosh(\sqrt{3}y) + \cosh(\sqrt{3}) \cosh(y), \quad -1 < y < 1; \]  
\[ q^{(3)} = q(1, y) = \cosh(1) \cosh(\sqrt{3}y) + \cosh(\sqrt{3}) \cosh(y), \quad -1 < y < 1; \]  
\[ q^{(2)} = q(x, -1) = \cosh(1) \cosh(\sqrt{3}x) + \cosh(\sqrt{3}) \cosh(x), \quad -1 < x < 1; \]  
\[ q^{(4)} = q(x, 1) = \cosh(1) \cosh(\sqrt{3}x) + \cosh(\sqrt{3}) \cosh(x), \quad -1 < x < 1. \]  

Then,
\[ q(x, y) = q(-x, y), \quad q(x, y) = q(x, -y), \quad q(x, y) = q(y, x), \]  

Thus,
\[ q_2^{(3)} = -q_x^{(1)}, \quad q_y^{(4)} = -q_y^{(2)}, \quad q_y^{(2)} = q_x^{(1)}(x, y) g_{\epsilon(y, x)} \]
Hence, the global relation (4.5a) becomes

\[
\cos \left( \lambda - \frac{1}{\lambda} \right) \hat{N}_1(\lambda) + \cos \left( i\lambda - \frac{1}{i\lambda} \right) \hat{N}_1(-i\lambda) = \left( \lambda - \frac{1}{\lambda} \right) \sin \left( \lambda - \frac{1}{\lambda} \right) \hat{D}_1(\lambda) + \left( i\lambda - \frac{1}{i\lambda} \right) \sin \left( i\lambda - \frac{1}{i\lambda} \right) \hat{D}_1(-i\lambda), \quad \lambda \in \mathbb{C}\{0\}. \quad (4.26a)
\]

The simplest way to obtain the global relation (4.6a) is to take the Schwarz conjugate of equation (4.26a) (i.e. to take the complex conjugate of (4.26a) and then to replace $\bar{\lambda}$ with $\lambda$). This yields the equation

\[
\cos \left( \lambda - \frac{1}{\lambda} \right) \hat{N}_1(\lambda) + \cos \left( i\lambda - \frac{1}{i\lambda} \right) \hat{N}_1(i\lambda) = \left( \lambda - \frac{1}{\lambda} \right) \sin \left( \lambda - \frac{1}{\lambda} \right) \hat{D}_1(\lambda) + \left( i\lambda - \frac{1}{i\lambda} \right) \sin \left( i\lambda - \frac{1}{i\lambda} \right) \hat{D}_1(i\lambda), \quad \lambda \in \mathbb{C}\{0\}. \quad (4.26b)
\]

Using $N$ basis functions to approximate $q^{(1)}_x$, equations (4.26) yield 2 equations for $N$ unknowns.

Plots of the relative error $E_\infty$, as well as of the matrix condition number as a function of $N$, for $N/2$, $N$, $3N/2$ and $2N$ collocation points, are presented in Figure 4.5. The rectangular collocation matrix was inverted by using the “backslash” command in Matlab. It is clear that over-determining the linear system by a factor of 2 is sufficient to achieve very good matrix conditioning.

![Figure 4.5](image_url) Numerical solution of the global relations (4.26) for the symmetric Dirichlet boundary value problem (4.23)