Hybrid Photonics Computing. Example Sheet 1.

Due February 11, 2020

Ising formulation of NP-hard combinatorial optimisation problems

Write down the Ising formulation of the following problems.

1. Vertex Cover. Given an undirected graph $G = (V, E)$, what is the smallest number of vertices that can be "coloured" such that every edge is incident to a coloured vertex? Let x_v be a binary variable on each vertex, which is 1 if it is coloured, and 0 if it is not coloured. Write down the Ising Hamiltonian in terms of x_v that needs to be minimised in order to solve this problem. What is the number of spins required?

2. Knapsack with Integer Weights. We have a list of N objects, labeled by indices α, with the weight of each object given by w_α, and its value given by c_α, and we have a knapsack which can only carry weight W. If x_α is a binary variable denoting whether (1) or not (0) object α is contained in the knapsack, the total weight in the knapsack is

$$\mathcal{W} = \sum_{\alpha=1}^{N} w_\alpha x_\alpha$$

and the total cost is

$$\mathcal{C} = \sum_{\alpha=1}^{N} c_\alpha x_\alpha.$$

The NP-hard knapsack problem asks us to maximize $\mathcal{C}$ subject to the constraint that $\mathcal{W} \leq W$. It has a large variety of applications, particularly in economics and finance.

Let y_n for $1 \leq n \leq W$ denote a binary variable which is 1 if the final weight of the knapsack is n, and 0 otherwise. Write down the Ising Hamiltonian for this problem.

(*Hint: Enforce the conditions that the weight can only take on one value and that the weight of the objects in the knapsack equals the value we claimed it did.*)

Dynamical systems on networks

3. Consider a dynamical system on a k-regular network (i.e., one in which every node has the same degree k) satisfying

$$\frac{dx_i}{dt} = f(x_i) + \sum_j A_{ij} g(x_i, x_j), \tag{1}$$

and in which the initial condition is uniform over nodes, so that $x_i(0) = x_0$ for all i.

a) Show that $x_i(t) = x(t)$ for all i where

$$\frac{dx_i}{dt} = f(x) + kg(x, x), \tag{2}$$

and hence that one need solve only one equation to solve the dynamics.

b) Show that for stability around a fixed point at $x_i = x^*$ for all i we require that

$$\frac{1}{k} > -\frac{1}{f'(x^*)} \left[\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) g(u, v) \right]_{u=v=x^*}. \tag{3}$$

4. Consider a dynamical system on an undirected network, with one variable per node obeying

$$\frac{dx_i}{dt} = f(x_i) + \sum_j A_{ij}[g(x_i) - g(x_j)]. \tag{4}$$

Suppose that the system has a symmetric fixed point at $x_i = x^*$ for all i.

a) Show, using results derived in class, that the fixed point is always stable if $f'(x^*) < 0$ and the largest degree k_{max} in the network satisfies

$$\frac{1}{k_{max}} > -2\left[\frac{\mathrm{d}g}{\mathrm{d}x} / \frac{\mathrm{d}f}{\mathrm{d}x}\right]_{x=x^*}. \tag{5}$$

b) Suppose that $f(x) = rx(1-x)$ and $g(x) = ax^2$, where r and a are positive constants. Show that there are two symmetric fixed points for this system, but that one of them is always unstable.

c) Give a condition on the maximum degree in the network that will ensure the stability of the other fixed point.

5. The dynamical systems we have considered in class have all been on undirected networks, but systems on directed networks are possible too. Consider a dynamical system on a directed network in which the sign of the interaction along an edge attached to a node depends on the direction of the edge, ingoing edges having positive sign and outgoing edges having negative sign. An example of such a system is a food web of predator-prey interactions, in which an ingoing edge indicates in-flow of energy to a predator from its prey and outgoing edge indicates out-flow from prey to predator. Such a system can be represented by a dynamics of the form

$$\frac{dx_i}{dt} = f(x_i) + \sum_j (A_{ij} - A_{ji}) g(x_i, x_j), \tag{6}$$

where g is a symmetric function of its arguments: $g(u,v) = g(v,u)$.

a) Consider a system of this form in which the in- and out-degrees of every node are equal to the same constant k. Show that such a system has a symmetric fixed point $x_i^* = x^*$ for all i satisfying $f(x^*) = 0$.

b) Writing $x_i = x^* + \epsilon_i$, linearize around this fixed point to show that in the vicinity of the fixed point the vector $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, ...)$ satisfies

$$\frac{\mathrm{d}\boldsymbol{\epsilon}}{\mathrm{d}t} = (\alpha \boldsymbol{I} + \beta \boldsymbol{M})\boldsymbol{\epsilon}, \tag{7}$$

where $\boldsymbol{M} = \boldsymbol{A} - \boldsymbol{A}^T$. Determine the values of the constants α and β.

c) Show that the matrix $\boldsymbol{M}$ has the property $\boldsymbol{M}^T = -\boldsymbol{M}$. Matrices with this property are called *skew-symmetric.*

d) If $\mathbf{v}$ is a right eigenvector of a skew-symmetric matrix $\boldsymbol{M}$ with eigenvalue μ, show that $\mathbf{v}^T$ is a left eigenvector with eigenvalue $-\mu$. Hence by considering the equality

$$\mu = \frac{\mathbf{v}^\dagger \mu \mathbf{v}}{\mathbf{v}^\dagger \mathbf{v}} = \frac{\mathbf{v}^\dagger \mathbf{M} \mathbf{v}}{\mathbf{v}^\dagger \mathbf{v}}, \tag{8}$$

where $\mathbf{v}^\dagger$ is the Hermitian conjugate of $\mathbf{v}$, show that the complex conjugate of the eigenvalue is $\mu^* = -\mu$ and hence that all eigenvalues of a skew-symmetric matrix are imaginary.

e) Show that the dynamical system is stable if $\mathrm{Re}(\alpha + \beta\mu_r) < 0$ for all eigenvalues μ_r of the matrix $\mathbf{M}$, and hence that the condition for stability is simply $\alpha < 0$.

The last result means that the coupled dynamical system is stable at the symmetric fixed point if and only if the individual nodes are stable in the absence of interaction with other nodes.

6. The largest (most positive) eigenvalue κ_1 of the adjacency matrix of a k-regular graph, a Poisson random graph with mean degree c, and a star graph with n nodes, is k, $c+1$, and $\sqrt{n-1}$, respectively. Verify that the inequalities $\kappa_1 \geq \langle k \rangle$ and $\kappa_1 \geq \sqrt{\langle k^2 \rangle}$ are satisfied in each of these cases.

7. Following the arguments presented in class, the stability of a fixed point in certain dynamical systems on networks depends on the spectrum of eigenvalues of the adjacency matrix. Suppose we have a network that takes the form of an $L \times L$ square lattice, with each node labeled by its position vector $\mathbf{r} = (i, j)$ where $i, j = 1...L$ are the row and column indicies of the node. And suppose also that the system has periodic (toroidal) boundary conditions along its edges, such that the node at position $(i, 1)$ is adjacent to the node at (i, L) and the node at $(1, j)$ is adjacent to (L, j).

a) Consider the vector $\mathbf{v}$ with one element for each node such that $v_r = \exp(i\mathbf{k}^T\mathbf{r})$ for some vector $\mathbf{k}$. Show that $\mathbf{v}$ is an eigenvector of the adjacency matrix provided

$$\mathbf{k} = \frac{2\pi}{L}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{9}$$

where n_1 and n_2 are integers.

b) What range of values is permitted for the integers n_1 and n_2? Hence find the largest and smallest eigenvalues of the adjacency matrix.

8. Consider a network with an oscillator on every node. The state of the oscillator on node i is represented by a phase angle θ_i and the system is governed by dynamical equations of the form

$$\frac{\mathrm{d}\theta_\mathrm{i}}{\mathrm{d}t} = \omega + \sum_j A_{ij} g(\theta_i - \theta_j), \tag{10}$$

where ω is a constant and the function $g(x)$ has $g(0) = 0$ and $g(x + 2\pi) = g(x)$ for all x.

a) Show that the synchronized state $\theta_i = \theta^* = \omega t$ for all i is a solution of the dynamics.

b) Consider a small perturbation away from the synchronized state $\theta_i = \theta^* + \epsilon_i$ and show that the vector $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, ...)$ satisfies

$$\frac{\mathrm{d}\boldsymbol{\epsilon}}{\mathrm{d}t} = g'(0)\mathbf{L}\boldsymbol{\epsilon}, \tag{11}$$

where $\mathbf{L}$ is the graph Laplacian.

c) Hence show that the synchronized state is stable against small perturbations if and only if $g'(0) < 0$.