Mathematical Tripos Part III Michaelmas 2018
Distribution Theory & Applications, Example sheet 1 Dr A.C.L. Ashton

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1. Construct a non-zero element of $\mathcal{D}(\mathbf{R})$ that vanishes outside $(0,1)$. Construct a non-zero element of $\mathcal{D}(\mathbf{R}^n)$ that vanishes outside the ball $B_\epsilon = \{x \in \mathbf{R}^n : |x| < \epsilon\}$.

2. Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives

$$\varphi(x+h) = \sum_{|\alpha| \le N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subset X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o\left(|h|^N\right)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h|\to 0} \left[\frac{\sup_x |\partial^\alpha R_N(x,h)|}{|h|^N} \right] = 0$$

for each multi-index α.

[Hint: you may find it convenient to use the following form of remainder

$$R_N(x,h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!}(N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th)\, \mathrm{d}t,$$

and note that $(N+1)! \sum_{|\alpha|=N+1} h^\alpha/\alpha! = (h_1 + \cdots + h_n)^{N+1}$. *]*

3. Which elements of $\mathcal{D}(X)$ can be represented as a power series on X?

4. Prove the C^∞ Urysohn lemma: if K is a compact subset of $X \subset \mathbf{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ on a neighbourhood of K.

[Hint: Let χ be a characteristic (indicator) function on a neighbourhood of K and smooth it off by taking the convolution with a test function that approximates δ_0.]

5. Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(X).$$

Show that that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\operatorname{ord}(T) = m$ what can you say about $\operatorname{ord}(\partial^\alpha T)$?

6. Let $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$. Prove that for each multi-index α

$$\partial^\alpha (Tf) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in $\mathcal{D}'(X)$ (take the case $X = \mathbf{R}$ if you're still getting used to multi-indices).

7. Let $\{x_k\}_{k\ge 1}$ be a sequence in X with no limit point in X. Consider the family of linear maps $u_\alpha : \mathcal{D}(X) \to \mathbf{C}$ defined by

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^\infty \partial^\alpha \varphi(x_k)$$

for each multi-index α. For which α is $u_\alpha \in \mathcal{D}'(X)$? What is $\operatorname{ord}(u_\alpha)$?

8. Find the most general solution to the equations

$$\text{(a)}\ \ u' = 1, \qquad \text{(b)}\ \ xu' = \delta_0, \qquad \text{(c)}\ \ \left(e^{2\pi \mathrm{i} x} - 1\right) u' = 0$$

in $\mathcal{D}'(\mathbf{R})$.

9. Define the distribution $u \in \mathcal{D}'(\mathbf{R}^2)$ by the locally integrable function

$$u(x,y) = \begin{cases} 1, & x \geq y \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\partial_x^2 u - \partial_y^2 u = 0$ in $\mathcal{D}'(\mathbf{R}^2)$. Can you give a physical interpretation of this result?

10. Compute $\Delta(|x|^{2-n})$ in $\mathcal{D}'(\mathbf{R}^n)$ for $n \geq 3$, i.e. compute

$$\langle \Delta(|x|^{2-n}), \varphi \rangle = \langle |x|^{2-n}, \Delta\varphi \rangle = \int \frac{\Delta\varphi}{|x|^{n-2}}\,\mathrm{d}x$$

for arbitrary $\varphi \in \mathcal{D}(\mathbf{R}^n)$. Note that $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\Delta = \sum_i (\partial/\partial x_i)^2$.
[Hint: use $\int \mathrm{d}x = \int_{|x|\leq\epsilon} \mathrm{d}x + \int_{|x|>\epsilon} \mathrm{d}x$ and treat each integral separately.]

11. Let $\{f_k\}_{k\geq 1}$ be the sequence of smooth functions defined by

$$f_k(x) = \frac{1}{\pi} \frac{k}{(kx)^2+1}.$$

Prove that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbf{R})$. Compute the limits

$$\text{(a)} \lim_{k\to\infty} k^2 x e^{-k^2x^2}, \qquad \text{(b)} \lim_{k\to\infty} k^3 e^{\mathrm{i}kx}, \qquad \text{(c)} \lim_{k\to\infty} \frac{\sin(kx)}{\pi x}$$

in $\mathcal{D}'(\mathbf{R})$.

12. Compute the limit

$$\lim_{k\to\infty} \left[\frac{1}{2} + \sum_{m=1}^{k} \cos(\pi m x) \right]$$

in $\mathcal{D}'(-1,1)$.

13. We define the *principal value* of $1/x$, written p.v.$(1/x)$, by

$$\left\langle \text{p.v.}\left(\frac{1}{x}\right), \varphi \right\rangle = \lim_{\epsilon\to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\mathbf{R})$. Prove that p.v.$(1/x) \in \mathcal{D}'(\mathbf{R})$ and ord(p.v.$(1/x)) = 1$. Show that

$$\lim_{\epsilon\to 0} \left(\frac{1}{x - \mathrm{i}\epsilon} \right) = \text{p.v.}\left(\frac{1}{x}\right) + \mathrm{i}\pi\delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

14. Show that $(\log|x|)' = \text{p.v.}(1/x)$ in $\mathcal{D}'(\mathbf{R})$.

15. Let $f = f(z)$ be complex analytic on $X \subset \mathbf{C} \simeq \mathbf{R}^2$. Suppose f has zeros at $\{z_i\}$ in X with multiplicities $\{m_i\}$. Prove that

$$\Delta \log|f| = 2\pi \sum_i m_i \delta_{z_i}$$

in $\mathcal{D}'(X)$, where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian on $\mathbf{R}^2$.

16. Define the distribution $u \in \mathcal{E}'(\mathbf{R}^3)$ the locally integrable function

$$u(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Prove that $-\sum_i x_i(\partial u/\partial x_i) = \mathrm{d}\sigma_2$ in $\mathcal{E}'(\mathbf{R}^3)$, where $\mathrm{d}\sigma_2$ is the surface element on the sphere $S^2 \subset \mathbf{R}^3$.