Mathematical Tripos Part III Michaelmas 2018
Distribution Theory & Applications, Example sheet 2 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`.

1. Let $u, v \in \mathcal{D}'(\mathbf{R}^n)$, one of which has compact support. Show that the convolution $u * v$, as defined in your notes, is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbf{R}^n)$.

2. Show that if $u, v, w \in \mathcal{D}'(\mathbf{R}^n)$ and at least two of them have compact support, than the convolution is associative, i.e. $(u * v) * w = u * (v * w)$.

3. Let $\varphi \in \mathcal{D}(\mathbf{R})$ and choose $\epsilon > 0$ sufficiently small so that $\operatorname{supp}(\varphi) \subset I_\epsilon$, where $I_\epsilon = (-1/\epsilon, 1/\epsilon)$. Given that φ has a uniformly convergent Fourier series on I_ϵ in the form

$$\varphi(x) = \sum_{n \in \mathbf{Z}} c_n e^{\mathrm{i}\epsilon\pi n x}, \qquad c_n = \frac{\epsilon}{2} \int \varphi(x) e^{-\mathrm{i}\epsilon\pi n x}\, \mathrm{d}x,$$

prove the Fourier inversion theorem on $\mathcal{D}(\mathbf{R})$ by taking a suitable limit.

4. For $\varphi \in \mathcal{S}(\mathbf{R}^n)$ prove that $\sum_m \varphi(m) = \sum_n \hat{\varphi}(2\pi n)$. This is the famous Poisson summation formula.

5. If $u \in H^s(\mathbf{R}^n)$ show that $D^\alpha u \in H^{s-|\alpha|}(\mathbf{R}^n)$. If $s > t$ show that $H^s(\mathbf{R}^n) \subset H^t(\mathbf{R}^n)$.

6. Show that $\mathcal{S}(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n) = H^0(\mathbf{R}^n)$ and deduce that $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^s(\mathbf{R}^n)$, i.e. prove that for each $u \in H^s(\mathbf{R}^n)$ there is a sequence $\{\varphi_m\}_{m \geq 1}$ in $\mathcal{S}(\mathbf{R}^n)$ such that

$$\lim_{m \to \infty} \|u - \varphi_m\|_{H^s} = 0.$$

[Hint: Use Parseval's theorem.]

7. Prove that multiplication by a Schwartz function gives rise to a continuous map from $H^s(\mathbf{R}^n)$ to itself, i.e. $\|\varphi u\|_{H^s} \lesssim \|u\|_{H^s}$ for $\varphi \in \mathcal{S}(\mathbf{R}^n)$. You may assume Peetre's inequality: for $\lambda, \mu \in \mathbf{R}^n$ and $s \in \mathbf{R}$

$$\left(\frac{1 + |\lambda|^2}{1 + |\mu|^2} \right)^s \leq 2^{|s|} (1 + |\lambda - \mu|^2)^{|s|}.$$

8. For $u \in \mathcal{S}'(\mathbf{R}^n)$ show that $(D^\alpha u)\hat{} = \lambda^\alpha \hat{u}$ and $(x^\beta u)\hat{} = (-D)^\beta \hat{u}$ for all multi-indices α, β.

9. For $u \in \mathcal{E}'(\mathbf{R}^n)$ show that $\hat{u}(\lambda) = \langle u(x), e^{-\mathrm{i}\lambda \cdot x} \rangle$. Deduce that if $u \in \mathcal{E}'(\mathbf{R}^n)$ there exists some $t \in \mathbf{R}$ such that $u \in H^t(\mathbf{R}^n)$.

10. Suppose that $u_1, u_2 \in \mathcal{S}'(\mathbf{R}^n)$ and at least one of u_1 and u_2 has compact support. Show that their convolution $u_1 * u_2 \in \mathcal{S}'(\mathbf{R}^n)$ and $(u_1 * u_2)\hat{} = \hat{u}_1 \hat{u}_2$.

11. Show that $e^{-\epsilon x} H \to H$ in $\mathcal{S}'(\mathbf{R})$ as $\epsilon \to 0$. Hence show

$$\hat{H} = \pi \delta_0 - \mathrm{i}\,\mathrm{p.v.} \left(\frac{1}{x} \right)$$

in $\mathcal{S}'(\mathbf{R})$.

12. The Riemann-Lebesgue lemma states that if $u \in L^1(\mathbf{R})$ then $|\hat{u}(\lambda)| \to 0$ as $|\lambda| \to \infty$. Prove this result by considering the substitution $x = x' + \pi/\lambda$ in the integral defining $\hat{u}(\lambda)$.

13. If $u \in H^m(\mathbf{R}^n)$ with $m \in \mathbf{N}$, use Parseval's theorem to show that

$$\sum_{|\alpha| \leq m} \int |D^\alpha u|^2 \, \mathrm{d}x < \infty.$$

Prove the converse.

14. Let $\mathcal{O}(\mathbf{R}^n)$ denote the space of smooth functions that grow no faster than a polynomial. Show that $\mathcal{O}(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$. Fix $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\varphi(0) = 1$. For $u \in \mathcal{O}(\mathbf{R}^n)$ we define

$$\hat{u}_\epsilon(\lambda) = \int e^{-\mathrm{i}\lambda\cdot x}\varphi(\epsilon x)u(x)\,\mathrm{d}x.$$

Show that $\hat{u} = \lim_{\epsilon\to 0} \hat{u}_\epsilon$ in $\mathcal{S}'(\mathbf{R}^n)$.

15. Compute the Fourier transform in $\mathcal{S}'(\mathbf{R})$ of the function

$$u(x) = \frac{x}{1+x^2}.$$

For which $s \in \mathbf{R}$ is $u \in H^s(\mathbf{R})$?

16. Prove that $D^\alpha\delta_0 \in H^s(\mathbf{R}^n)$ if and only if $s < -|\alpha| - \frac{1}{2}n$.

17. Let $\Gamma = \{x \in \mathbf{R}^n : x \cdot n = 0\}$ be a hyperplane with surface element $\mathrm{d}\sigma$ and normal n. Let $\chi \in \mathcal{S}(\mathbf{R}^n)$ be a fixed Schwartz function and set $\mathrm{d}\mu = \chi\mathrm{d}\sigma$. This defines a distribution $\mu_\Gamma \in \mathcal{S}'(\mathbf{R}^n)$ by

$$\langle\mu_\Gamma, \varphi\rangle = \int_\Gamma \varphi\,\mathrm{d}\mu$$

for each $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Prove that

$$\hat{\mu}_\Gamma(\lambda) = \int_\Gamma e^{-\mathrm{i}\lambda\cdot x}\mathrm{d}\mu(x).$$

Classify the large $|\lambda|$ behaviour of $\hat{\mu}_\Gamma$. For which $s \in \mathbf{R}$ is $\mu_\Gamma \in H^s(\mathbf{R}^n)$?

18. Suppose that $u \in \mathcal{S}'(\mathbf{R}^n)$ and $\Delta^2 u + u \in H^s(\mathbf{R}^n)$. Prove that $u \in H^{s+4}(\mathbf{R}^n)$.

19. Compute the Fourier transforms of the functions

$$\text{(a) } \operatorname{sgn}(x), \quad \text{(b) } \arctan(x), \quad \text{(c) } x\log|x| - x, \quad \text{(d) } \exp\left(\mathrm{i}\omega x^2\right)$$

in $\mathcal{S}'(\mathbf{R})$, where $\omega \in \mathbf{R}$.

20. Show that the Fourier transform of $(x_1 + \mathrm{i}x_2)^{-1}$ is proportional to itself and find the constant of proportionality.

21. Suppose $f \in C(\mathbf{R})$ and $f(x) - 1/x = O(1/x^2)$ as $|x| \to \infty$. Prove that

$$\lim_{\epsilon\to 0}\left[\hat{f}(-\epsilon) - \hat{f}(+\epsilon)\right] = 2\pi\mathrm{i}.$$

22. If $\varphi \in \mathcal{D}(\mathbf{R}^n)$ and $\operatorname{supp}(\varphi) \subset B_\delta$ show that $\hat{\varphi}(z)$ is an entire function and there exist constants C_m such that

$$|\hat{\varphi}(z)| \le C_m\,(1+|z|)^{-m}\,e^{\delta|\mathsf{Im}\,z|}$$

for $m = 0, 1, 2, \ldots$ and $z \in \mathbf{C}^n$. Prove the converse.