Mathematical Tripos Part III Michaelmas 2017
Distribution Theory & Applications, Example sheet 3 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`.

1. Write the following distributions as sums of derivatives of continuous functions

$$\text{(a)}\ \delta_0, \qquad \text{(b)}\ \operatorname{sgn}(x), \qquad \text{(c)}\ \text{p.f.}\left(\frac{1}{x^2}\right).$$

2. For an Nth order polynomial P in n-variables set $P^{(\alpha)}(\lambda) = \partial^\alpha P(\lambda)$. Show that

$$P(D)[\varphi\psi] = \sum_{|\alpha|\le N} \frac{1}{\alpha!} D^\alpha(\varphi) P^{(\alpha)}(D)(\psi)$$

for all test functions $\psi, \varphi \in \mathcal{D}(\mathbf{R}^n)$.

3. Let $\{\Delta_i\}_{i\ge 1}$ and $\{c_i\}_{i\ge 1}$ be as in the proof of the Malgrange-Ehrenpreis theorem. Prove that

$$\varphi \mapsto \langle E, \varphi\rangle = \sum_{i=1}^\infty \int_{\Delta_i} \left(\int_{\mathsf{Im}\,\lambda_n = c_i} \frac{\hat{\varphi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)}\, \mathrm{d}\lambda_n \right) \mathrm{d}\lambda'$$

defines an element of $\mathcal{D}'(\mathbf{R}^n)$.

4. Let A be a symmetric $n\times n$ matrix with $\det A \neq 0$ and $\mathsf{Re}\, A \ge 0$ (i.e. $\mathsf{Re}\, A$ is positive semi-definite). Construct, explicitly, a fundamental solution to the operator

$$L = \frac{\partial}{\partial t} - \sum_{i,j=1}^n A_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

[Hint: if A is the identity then L is just the heat operator, so we know the fundamental solution in this case. The fundamental solution for the general case looks similar.]

5. Suppose $L = P(\partial/\partial x_1, \dots, \partial/\partial x_n)$ is an Nth order elliptic partial differential operator with constant coefficients. Given $(\lambda, c) \in S^{n-1} \times \mathbf{R}$, show that for R sufficiently large the function

$$u(x; \lambda, c) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=R} \frac{e^{z(x\cdot\lambda - c)}}{zP(z\lambda)}\, \mathrm{d}z$$

satisfies $L[u] = 1$ with $u = 0$ on the surface $x\cdot\lambda = c$. Here the integral is taken in the complex z-plane around a circle centred at the origin with radius R. How big did R need to be?

6. Let P be an Nth order polynomial in n-variables and suppose that there exists $C, \delta > 0$ such that $|P^{(\alpha)}(\lambda)| \le C|\lambda|^{-\delta|\alpha|}|P(\lambda)|$ for all multi-indices α when $|\lambda|$ is sufficiently large. Show

$$P(D)u \in H^s_{\mathrm{loc}}(X) \Rightarrow u \in H^{s+\delta N}_{\mathrm{loc}}(X).$$

Deduce that $P(D)$ is hypoelliptic, i.e. $P(D)u \in C^\infty(X) \Rightarrow u \in C^\infty(X)$.

7. Give an example of an operator $P(D)$ that is hypoelliptic but not elliptic.

8. Does there exist a polynomial P of positive degree that satisfies a (C,δ)-estimate given in question 6 with $\delta > 1$? What can you say about $P(D)$ if the polynomial P satisfies the (C,δ)-estimate with $\delta = 1$?

9. If $a \in \mathrm{Sym}(X, \mathbf{R}^k, N)$ show that $D^\alpha_x D^\beta_\theta a \in \mathrm{Sym}(X, \mathbf{R}^k, N - |\beta|)$. If a_1, a_2 belong to $\mathrm{Sym}(X, \mathbf{R}^k; N_1)$ and $\mathrm{Sym}(X, \mathbf{R}^k; N_2)$ respectively, show that $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

10. Define the differential operator

$$L = \sum_{j=1}^{k} a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

where $a_j \in \mathrm{Sym}(X;\mathbf{R}^k;0)$, $b_j \in \mathrm{Sym}(X,\mathbf{R}^k;-1)$ and $c \in \mathrm{Sym}(X,\mathbf{R}^k;-1)$. Show that the formal adjoint L^* has the same form with coefficients in the same spaces of symbols.

11. Consider the oscillatory integral defined by the phase function Φ and the symbol $a \in \mathrm{Sym}(X,\mathbf{R}^k;N)$ where $X \subset \mathbf{R}^n$. Suppose that Φ is *non-degenerate*, i.e. the differentials $\mathrm{d}(\partial\Phi/\partial\theta_j)$, $(1 \leq j \leq k)$, are linearly independent. Prove that the set

$$M(\Phi) = \{(x,\theta) : x \in X, \theta \in \mathbf{R}^k \setminus \{0\}, \nabla_\theta\Phi(x,\theta) = 0\}$$

is an smooth n-dimensional submanifold of $X \times (\mathbf{R}^k \setminus \{0\})$. If you have experience with geometry, prove that the set

$$SP(\Phi) = \{(x, \nabla_x\Phi(x,\theta)) : (x,\theta) \in M(\Phi)\}$$

is a Lagrangian submanifold of $T^*X \setminus \{0\}$ (the cotangent bundle over X). If you don't have experience with geometry, but have access to the Internet, Google the phrase "Lagrangian submanifold".

12. Let $p = (\omega, \mathbf{p})$ and $x = (t, \mathbf{x})$ be coordinates in $\mathbf{R}^4$. Show that the distribution[1] Δ_F defined by

$$\hat{\Delta}_F(p) = \lim_{\epsilon\to 0} \frac{1}{-\omega^2 + |\mathbf{p}|^2 + m^2 - \mathrm{i}\epsilon}$$

is a fundamental solution to the Klein-Gordon operator $\partial_t^2 - \Delta_{\mathbf{x}} + m^2$. Set $\omega_{\mathbf{p}} = \sqrt{m^2 + |\mathbf{p}|^2}$ and show

$$\Delta_F(x) = \frac{\mathrm{i}}{2(2\pi)^3} \int \frac{e^{-\mathrm{i}\omega_{\mathbf{p}}|t| + \mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\omega_{\mathbf{p}}}\, \mathrm{d}\mathbf{p}$$

when interpreted as the sum of an ordinary function and an oscillatory integral. Show that the oscillatory integral has symbol belonging to $\mathrm{Sym}(\mathbf{R}^+ \times \mathbf{R}^3, \mathbf{R}^3; -1)$. Show that the singular support of the Feynman propagator is contained in the light cone $|t| = |\mathbf{x}|$.

[1] This is the Feynman propagator. This distribution, and ones closely related to it, are of central importance in theoretical physics – see Quantum Field Theory courses.