1. Write the following distributions as sums of derivatives of continuous functions
   \[ (a) \delta_0, \quad (b) \text{sgn}(x), \quad (c) \text{p.f.} \left(\frac{1}{x^2}\right). \]

2. For an \( N \)th order polynomial \( P \) in \( n \)-variables set \( P^{(\alpha)}(\lambda) = \partial^\alpha P(\lambda) \). Show that
   \[ P(D)[\varphi \psi] = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} D^\alpha(\varphi) P^{(\alpha)}(D)(\psi) \]
   for all test functions \( \psi, \varphi \in \mathcal{D}(\mathbb{R}^n) \).

3. Let \( \{\Delta_i\}_{i \geq 1} \) and \( \{c_i\}_{i \geq 1} \) be as in the proof of the Malgrange-Ehrenpreis theorem. Prove that
   \[ \varphi \mapsto \langle E, \varphi \rangle = \sum_{i=1}^{\infty} \int_{\Delta_i} \left( \int_{|\lambda_n| = c_i} \hat{\varphi}(-\lambda', -\lambda_n) P(\lambda', \lambda_n) d\lambda_n \right) d\lambda' \]
   defines an element of \( \mathcal{D}'(\mathbb{R}^n) \).

4. Let \( A \) be a symmetric \( n \times n \) matrix with \( \det A \neq 0 \) and \( \text{Re} A \geq 0 \) (i.e. \( \text{Re} A \) is positive semi-definite). Construct, explicitly, a fundamental solution to the operator
   \[ L = \frac{\partial}{\partial t} - \sum_{i,j=1}^n A_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \]
   [Hint: if \( A \) is the identity then \( L \) is just the heat operator, so we know the fundamental solution in this case. The fundamental solution for the general case looks similar.]

5. Suppose \( L = P(\partial/\partial x_1, \ldots, \partial/\partial x_n) \) is an \( N \)th order elliptic partial differential operator with constant coefficients. Given \( (\lambda, c) \in S^{n-1} \times \mathbb{R} \), show that for \( R \) sufficiently large the function
   \[ u(x; \lambda, c) = \frac{1}{2\pi i} \oint_{|z| = R} \frac{e^{z(x \cdot \lambda - c)}}{z P(z \lambda)} dz \]
   satisfies \( L[u] = 1 \) with \( u = 0 \) on the surface \( x \cdot \lambda = c \). Here the integral is taken in the complex \( z \)-plane around a circle centred at the origin with radius \( R \). How big did \( R \) need to be?

6. Let \( P \) be an \( N \)th order polynomial in \( n \)-variables and suppose that there exists \( C, \delta > 0 \) such that \( |P^{(\alpha)}(\lambda)| \leq C|\lambda|^{-\delta|\alpha|}|P(\lambda)| \) for all multi-indices \( \alpha \) when \( |\lambda| \) is sufficiently large. Show
   \[ P(D)u \in H^s_{\text{loc}}(X) \Rightarrow u \in H^{s+4N}_{\text{loc}}(X). \]
   Deduce that \( P(D) \) is hypoelliptic, i.e. \( P(D)u \in C^\infty(X) \Rightarrow u \in C^\infty(X) \).

7. Give an example of an operator \( P(D) \) that is hypoelliptic but not elliptic.

8. Does there exist a polynomial \( P \) of positive degree that satisfies a \((C, \delta)\)-estimate given in question 6 with \( \delta > 1 \)? What can you say about \( P(D) \) if the polynomial \( P \) satisfies the \((C, \delta)\)-estimate with \( \delta = 1 \)?

9. If \( a \in \text{Sym}(X, \mathbb{R}^k, N) \) show that \( D_x^\alpha D_y^\beta a \in \text{Sym}(X, \mathbb{R}^k, N - |\beta|) \). If \( a_1, a_2 \) belong to \( \text{Sym}(X, \mathbb{R}^k; N_1) \) and \( \text{Sym}(X, \mathbb{R}^k; N_2) \) respectively, show that \( a_1a_2 \in \text{Sym}(X, \mathbb{R}^k; N_1 + N_2) \).
10. Define the differential operator

\[
L = \sum_{j=1}^{k} a_j(x, \theta) \frac{\partial}{\partial \theta_j} + \sum_{j=1}^{n} b_j(x, \theta) \frac{\partial}{\partial x_j} + c(x, \theta)
\]

where \(a_j \in \text{Sym}(X; \mathbb{R}^k; 0)\), \(b_j \in \text{Sym}(X, \mathbb{R}^k; -1)\) and \(c \in \text{Sym}(X, \mathbb{R}^k; -1)\). Show that the formal adjoint \(L^*\) has the same form with coefficients in the same spaces of symbols.

11. Consider the oscillatory integral defined by the phase function \(\Phi\) and the symbol \(a \in \text{Sym}(X, \mathbb{R}^k; N)\) where \(X \subset \mathbb{R}^n\). Suppose that \(\Phi\) is non-degenerate, i.e. the differentials \(d(\partial \Phi/\partial \theta_j)\), \(1 \leq j \leq k\), are linearly independent. Prove that the set

\[
M(\Phi) = \{(x, \theta) : x \in X, \theta \in \mathbb{R}^k \setminus \{0\}, \nabla \theta \Phi(x, \theta) = 0\}
\]

is an smooth \(n\)-dimensional submanifold of \(X \times (\mathbb{R}^k \setminus \{0\})\). If you have experience with geometry, prove that the set

\[
SP(\Phi) = \{ (x, \nabla_x \varphi(x, \theta)) : (x, \theta) \in M(\Phi)\}
\]

is a Lagrangian submanifold of \(T^*X \setminus \{0\}\) (the cotangent bundle over \(X\)). If you don’t have experience with geometry, but have access to the Internet, Google the phrase “Lagrangian submanifold”.

12. Let \(p = (\omega, p)\) and \(x = (t, \mathbf{x})\) be coordinates in \(\mathbb{R}^4\). Show that the distribution \(\Delta_F\) defined by

\[
\hat{\Delta}_F(p) = \lim_{\epsilon \to 0} \frac{1}{-\omega^2 + |p|^2 + m^2 - i\epsilon}
\]

is a fundamental solution to the Klein-Gordon operator \(\partial_t^2 - \Delta_x + m^2\). Set \(\omega_p = \sqrt{m^2 + |p|^2}\) and show

\[
\Delta_F(x) = \frac{i}{2(2\pi)^3} \int \frac{e^{-i|\mathbf{x}|} e^{i|\mathbf{x}|} \mathbf{x}}{\omega_p} \, dp
\]

when interpreted as the sum of an ordinary function and an oscillatory integral. Show that the oscillatory integral has symbol belonging to \(\text{Sym}(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R}^3; -1)\). Show that the singular support of the Feynman propagator is contained in the light cone \(|t| = |\mathbf{x}|\).

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1 This is the Feynman propagator. This distribution, and ones closely related to it, are of central importance in theoretical physics – see Quantum Field Theory courses.