II: Ginzburg Landau Vortices and the Abelian Higgs model

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These are informal notes for the second part of the course, which concerns vortices in Ginzburg Landau theory. Precise statements and proofs can be found in the book "Vortices and Monopoles" by Jaffe and Taubes (Birkhauser 1982).

II.1 Ungauged Ginzburg Landau vortices

Consider a Higgs field $\Phi : \mathbb{R}^2 \to \mathbb{C}$ in 2 spatial dimensions with potential energy

$$GL(\Phi) \;=\; \frac{1}{2}\int_{\mathbb{R}^2} \left\{ \left|\nabla\Phi\right|^2 \,+\, \frac{\lambda}{4}\left(1 - |\Phi|^2\right)^2 \right\} \mathrm{d}^2x \,. \tag{II.1}$$

Here, $\lambda > 0$ is a real parameter. Minimizers, or more generally stationary points, will satisfy the Euler Lagrange equation

$$-\triangle\Phi \,-\, \frac{\lambda}{2}\Phi\left(1 - |\Phi|^2\right) \;=\; 0\,. \tag{II.2}$$

It is natural to impose the boundary condition $\lim_{|x|\to\infty}|\Phi(x)| = 1$ in view of the form of the self-interaction potential. The basic vortex solutions have the following form in polar coordinates $re^{i\theta} = x^1 + ix^2$:

$$\Phi(x) \;=\; f_N(r)e^{iN\theta}\,, \tag{II.3}$$

where f_N is satisfies

$$\begin{aligned} -f_N'' - \frac{f_N'}{r} + \frac{N^2}{r^2} f_N \;&=\; \frac{\lambda}{2}\left(1 - f_N^2\right) f_N\,, \\ \lim_{r\to 0} f_N(r) \;=\; 0\,, \quad \lim_{r\to\infty} f_N(r) \;&=\; 1\,. \end{aligned} \tag{II.4}$$

The potential itself has a finite piece $\int\left(1 - f_N^2\right)^2 r\,\mathrm{d}r\,\mathrm{d}\theta < \infty$, but the angular gradient terms make the $\mathbb{R}^2$ integral diverge at spatial infinity:

$$\int\left|\nabla\Phi\right|^2 \mathrm{d}^2x \;=\; \int\left(f_N'^2(r) \,+\, \frac{N^2}{r^2} f_N^2(r)\right) r\,\mathrm{d}r\,\mathrm{d}\theta \;\overset{r\to\infty}{\sim}\; 2\pi N^2 \int_0^\infty \frac{\mathrm{d}r}{r} \;\to\; \infty\,. \tag{II.5}$$

The boundary condition at spatial infinity means that the Higgs field defines a map from large circles $\|x\| = R$ to the unit circle (as $R \to \infty$):

$$\frac{\Phi}{|\Phi|} : \;\; \left\{x : \|x\| = R\right\} \;\to\; S^1 \;=\; \left\{\Phi : |\Phi| = 1\right\} \;\subset\; \mathbb{C}\,. \tag{II.6}$$

One can associate to such maps an integer called the *winding number*, given as an integral

$$w \;:=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R} \mathrm{d}(\arg\Phi) \;=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle\,, \tag{II.7}$$

for sufficiently regular fields. For the basic vortex, defined in polar co-ordinates by (II.3) and (II.4), the winding number clearly equals the number N.

Exercise: use the Stokes-Green identity to obtain an alternative formula for the winding number as a two dimensional integral. (See (II.30) below.)

Exercise: check that if Φ is a smooth solution of (II.2) then $w = 1 - |\Phi|^2$ verifies $-\Delta w + \lambda|\Phi|^2 w = 2|\nabla\Phi|^2 \geq 0$. Deduce from the maximum principle that as long as $\lambda \geq 0$, this solution satisfies $0 \leq |\Phi| \leq 1$ in any disc (or arbitrary regular domain) such that $0 \leq |\Phi| \leq 1$ holds on the boundary of the domain.

II.2 Gauged Ginzburg Landau vortices

Electromagnetic fields on $\mathbb{R}^2$ can be described by a real 1-form:

$$A \;=\; A_1\,\mathrm{d}x^1 \;+\; A_2\,\mathrm{d}x^2 \in \Omega^1(\mathbb{R}^2) \tag{II.8}$$

The corresponding magnetic field B is given by the two-dimensional curl $B = \partial_1 A_2 - \partial_2 A_1$. To understand the coupling of electromagnetic fields to complex scalar fields such as the Higgs field we introduce the formalism of gauge theories.

II.2.1 Covariant derivatives and gauge invariance

The *covariant derivative* of a (complex scalar) $\Phi : \mathbb{R}^2 \to \mathbb{C}$ is defined as

$$D\Phi \;:=\; \sum_{j=1}^{2} \left(\frac{\partial \Phi}{\partial x^j} \;-\; iA_j\,\Phi \right) \mathrm{d}x^j \;=:\; \sum_{j=1}^{2} \big(\nabla_A\big)_j\,\Phi\,\mathrm{d}x^j \,. \tag{II.9}$$

This definition can be motivated by the *gauge principle* - the potential (II.1) for Φ has a *global gauge invariance* $GL(\Phi) = GL\big(e^{i\chi}\Phi\big)$ for $\chi =$ const which can be extended to a *local gauge invariance* in which the phase factor depends upon x, i.e. $\chi = \chi(x)$, by introducing the additional field (II.8) with transformaion rule

$$\Phi \;\mapsto\; e^{i\chi}\,\Phi \quad\Rightarrow\quad A \;\mapsto\; A \,+\, \mathrm{d}\chi \,. \tag{II.10}$$

The covariant derivative (II.9) then transforms to

$$\begin{aligned} \tilde{D}\tilde{\Phi} \;&=\; \sum_{j=1}^{2} \left(\frac{\partial}{\partial x^j} \;-\; i\tilde{A}_j \;-\; i\partial_j\chi \right) \tilde{\Phi}\,\mathrm{d}x^j \;=\; \sum_{j=1}^{2} \left(\frac{\partial}{\partial x^j} \;-\; iA_j \right) \big(e^{i\chi}\,\Phi\big)\,\mathrm{d}x^j \\ &=\; e^{i\chi} \sum_{j=1}^{2} \left(\frac{\partial}{\partial x^j} \;-\; iA_j \right) \Phi\,\mathrm{d}x^j \;=\; e^{i\chi}\,D\Phi \,. \end{aligned} \tag{II.11}$$

Since $\big|\tilde{D}\tilde{\Phi}\big|^2 = \big|D\Phi\big|^2$, one can easily construct gauge invariant theories from the former which also involve a gauge invariant field strength B:

$$V_\lambda(A,\Phi) \;=\; \frac{1}{2}\int_{\mathbb{R}^2} \left\{ B^2 \,+\, \big|D\Phi\big|^2 \,+\, \frac{\lambda}{4}\,\big(1 \,-\, |\Phi|^2\big) \right\} \mathrm{d}^2x \;=\; V_\lambda(\tilde{A},\tilde{\Phi}) \tag{II.12}$$

$$B\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \;=\; \mathrm{d}A \;=\; \left(\frac{\partial A_2}{\partial x^1} \,-\, \frac{\partial A_1}{\partial x^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{II.13}$$

The integral (II.12) defines the Abelian Higgs functional V_λ.

II.2.2 Geometric interpretation of gauge fields

We briefly relate gauge theories to the formalism of connections on vector bundles described in differential geometry. The set $\mathcal{F}$ of physical fields - in this case complex-valued functions $\Phi : \mathbb{R}^2 \to \mathbb{C}$ - forms a vector space[1]. One can multiply elements of $\mathcal{F}$ by real-valued functions $f \in C^\infty(M)$ defined on the spatial domain $M = \mathbb{R}^2$, i.e. $f \cdot \Phi \in \mathcal{F}$ for $\Phi \in \mathcal{F}$ - they have the structure of a module over $C^\infty(M)$. We define an inner product

$$\langle\cdot\,,\,\cdot\rangle \;:\; \mathcal{F}\times\mathcal{F} \;\to\; C^\infty(M)\,, \qquad \langle\Phi\,,\,\Psi\rangle \;:=\; \frac{1}{2}\,\big(\Phi^*\,\Psi \,+\, \Phi\,\Psi^*\big)\,. \tag{II.14}$$

The covariant derivative operator (II.9) is constructed (as a map from $\mathcal{F}$ to the space $\Omega^1(\mathcal{F})$ of 1-forms taking values in $\mathcal{F}$) which satisfies:

$$D\big(f\,\Phi\big) \;=\; \mathrm{d}f\cdot\Phi \,+\, f\cdot D\Phi \qquad \text{Leibnitz proprety} \tag{II.15}$$

$$\mathrm{d}\langle\Phi\,,\,\Psi\rangle \;=\; \langle D\Phi\,,\,\Psi\rangle \,+\, \langle\Phi\,,\,D\Psi\rangle \qquad \text{unitarity} \tag{II.16}$$

[1] Ignoring boundary conditions

In the geometrical view of this formalism, one regards the fields as sections of a *vector bundle* over the spatial manifold (base space) M, in this case $\mathbb{R}^2$. A vector bundle V with base space $\mathbb{R}^2$ and fibre $\mathbb{C}^j$ means that we have

- total space $V := \mathbb{R}^2 \times \mathbb{C}^j$
- projection map $\pi : V \to \mathbb{R}^2$

The set $\pi^{-1}(x) \cong \mathbb{C}^j$ is the fibre over $x \in M$, and has the structure of a vector space. (If curves in M are not contractible to points (e.g. on the torus), then V may well have more structure than a mere product space.) A crucial point is that we do **not** have a pre-determined way to relate points on different fibres, i.e. there is no given identification of $\pi^{-1}(x)$ and $\pi^{-i}(y)$. A *connection* gives a way to relate fibres over different points in the base space, i.e.

$$\mathcal{F} \;=\; \Gamma(V) \;:=\; \text{sections of } V \;=\; \Big\{\text{functions } s : M \to V \;:\; \pi\big(s(x)\big) = x\Big\}\,. \tag{II.17}$$

Given a pointwise inner product on $\mathcal{F}$, a unitary connection is characterized by properties (II.15) and (II.16).
A connection gives a way to "lift" a curve $t \mapsto x(t) \in M$ to a curve in total space $t \mapsto \Phi(t) \in V$, given $\Phi(0)$ such that

$$\pi\big(\Phi(t)\big) \;=\; x(t) \qquad \forall\, t \in \mathbb{R}\,. \tag{II.18}$$

From this point of view, a connection corresponds to the specification of a family of horizontal subspaces $H_p \subset T_pV$ with $\dim H_p = \dim M$, which can be used to define an identification of fibres over different points of the base space lying on a curve. In our case at $p = (x, \Phi) \in V$, introduce the 2 (real) dimensional target space

$$H_p \;:=\; \Big\{(\dot{x}, \dot{\Phi}) \in T_pV :\quad \dot{\Phi} \,-\, iA_j\,\dot{x}^j\Phi \;=\; 0\Big\}\,, \tag{II.19}$$

then $t \mapsto \Phi(t)$ is a lift of the curve $t \to x(t) \in M$ if

$$\begin{aligned} \frac{\mathrm{d}\Phi}{\mathrm{d}t} \,-\, iA_j\,\Phi\,\frac{\mathrm{d}x^j}{\mathrm{d}t} \;&=\; \frac{\partial\Phi}{\partial x^j}\,\frac{\mathrm{d}x^j}{\mathrm{d}t} \,-\, iA_j\,\Phi\,\frac{\mathrm{d}x^j}{\mathrm{d}t} \;=\; 0 \\ \Rightarrow\quad \dot{x}\cdot D\Phi \;&=\; 0\,, \end{aligned} \tag{II.20}$$

using the chain rule.
The section $\Phi(t)$ so constructed is a *horizontal lift* of the curve $x(t)$. The phrase *parallel transport* is also used. For a full development of this formalism, see "Bleecker, *Gauge theory and variational principle*" or "Jost, *Riemannian geometry and geometric analysis*".
The notion of curvature can be introduced in analogous fashion to Riemannian geometry, as the commutator of covariant derivatives:

$$D_jD_k\Phi \,-\, D_kD_j\Phi \;=\; -i\Big(\partial_jA_k \,-\, \partial_kA_j\Big)\,\Phi \tag{II.21}$$

In $\mathbb{R}^2$, there is only one algebraically independent component

$$D_1D_2\Phi \,-\, D_2D_1\Phi \;=\; -iB\,\Phi\,. \tag{II.22}$$

II.2.3 Finite energy critical points of the gauge invariant potential

We are interested in field configurations (A, Φ) which make the potential $V_\lambda(A, \Phi)$ given by (II.12) stationary and finite, i.e. impose boundary conditions

$$\lim_{\|x\|\to\infty} B(x) \;=\; 0\,, \qquad \lim_{\|x\|\to\infty} \big|D\Phi(x)\big| \;=\; 0\,, \qquad \lim_{\|x\|\to\infty} \big|\Phi(x)\big|^2 \;=\; 1\,. \tag{II.23}$$

Critical points have to satisfy the Euler Lagrange equations:

$$\begin{aligned} -\big((D_1)^2 + (D_2)^2\big)\,\Phi \;&=\; \frac{\lambda}{2}\,\big(1 \,-\, |\Phi|^2\big)\,\Phi \\ \partial_2\,B \;&=\; -\langle i\Phi\,,\, D_1\Phi\rangle \\ \partial_1\,B \;&=\; +\langle i\Phi\,,\, D_2\Phi\rangle \end{aligned} \tag{II.24}$$

Single vortex solutions are of the form

$$\Phi(x) \;=\; f_N(r)\, e^{iN\theta}\,, \qquad A(x) \;=\; N\,\alpha_N(r)\,\mathrm{d}\theta\,, \tag{II.25}$$

where the exclusively r dependent functions f_N, α_N are determined by ordinary differential equations resulting from (II.24), subject to the boundary conditions $\lim_{r\to\infty} f_N(r) = \lim_{r\to\infty}\alpha_N(r) = 1$ and $\lim_{r\to 0} f_N(r) = \lim_{r\to 0}\alpha_N(r) = 0$, which can be seen to be the natural conditions for finite energy (exercise).

The winding number can alternatively be computed in terms of the gauge field A: indeed, assume that $\exists\ \delta > 0, C > 0$ such that

$$\big|D\Phi\big| \;+\; \big|1 \,-\, |\Phi|^2\big| \;\leq\; \frac{C}{(1\,+\,r)^{1+\delta}}\,. \tag{II.26}$$

then

$$\begin{aligned} N \;&=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle \;=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R} A \\ &=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R}\big(A_1\,\mathrm{d}x^1 \;+\; A_2\,\mathrm{d}x^2\big) \\ &=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} B\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,. \end{aligned} \tag{II.27}$$

The proof is based on the identity

$$\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle \;=\; \oint_{\|x\|=R}\langle i\Phi\,,\,D\Phi \,+\, iA\,\Phi\rangle \;=\; \oint_{\|x\|=R}\underbrace{\langle i\Phi\,,\,D\Phi\rangle}_{=\mathcal{O}(R^{-\delta})} \,+\, \oint_{\|x\|=R}\underbrace{\big|\Phi\big|^2}_{\to 1}\,A\,. \tag{II.28}$$

and then using Stoke's theorem to arrive at the magnetic field B. In the geometric picture, N gives the degree of the line bundle of which Φ in section. For finite action solutions to the Euler Lagrange equations (II.24), the fields $|B|$, $|D\Phi|$ and $\big|1 - |\Phi|^2\big|$ decay exponentially fast as $\|x\| \to \infty$, and so the above calculation is certainly justified.

Finally, regarding Φ as a function $\mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$, we can use the Jacobian of the derivative to introduce the *topological charge density*:

$$\begin{aligned} j^0(\Phi) \;&=\; \det\begin{pmatrix} \frac{\partial\Phi^1}{\partial x^1} & \frac{\partial\Phi^1}{\partial x^2} \\ \frac{\partial\Phi^2}{\partial x^1} & \frac{\partial\Phi^2}{\partial x^2}\end{pmatrix} \;=\; \left\langle i\,\frac{\partial\Phi}{\partial x^1}\,,\,\frac{\partial\Phi}{\partial x^2}\right\rangle \\ &=\; \frac{1}{2}\left\{\frac{\partial}{\partial x^1}\,\left\langle i\Phi\,,\,\frac{\partial\Phi}{\partial x^2}\right\rangle \,-\, \frac{\partial}{\partial x^2}\,\left\langle i\Phi\,,\,\frac{\partial\Phi}{\partial x^1}\right\rangle\right\} \\ &= \frac{1}{2}\epsilon_{ab}\epsilon_{ij}\,\frac{\partial\Phi^a}{\partial x^i}\,\frac{\partial\Phi^b}{\partial x^j} \end{aligned} \tag{II.29}$$

to derive another expression for N by Stoke's theorem:

$$\begin{aligned} \frac{1}{2\pi}\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle \;&=\; \frac{1}{2\pi}\oint_{\|x\|=R}\left\{\left\langle i\Phi\,,\,\frac{\partial\Phi}{\partial x^1}\right\rangle\,\mathrm{d}x^1 \;+\; \left\langle i\Phi\,,\,\frac{\partial\Phi}{\partial x^2}\right\rangle\,\mathrm{d}x^2\right\} \\ &=\; \frac{1}{\pi}\int_{\|x\|\leq R} j^0(\Phi)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,, \end{aligned}$$

which implies that

$$N \;=\; \lim_{R\to\infty}\frac{1}{2\pi}\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle \;=\; \frac{1}{\pi}\int_{\mathbb{R}^2} j^0(\Phi)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,. \tag{II.30}$$

II.3 $\lambda = 1$: the Bogomolny argument for self-dual vortices

For $\lambda = 1$ the gauge invariant potential can be put into the form

$$V_1(A, \Phi) \;=\; \frac{1}{2} \int_{\mathbb{R}^2} \left\{ \left(B - \frac{1 - |\Phi|^2}{2} \right)^2 + 4 \left| \bar{\partial}_A \Phi \right|^2 \right\} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \;+\; N\pi \,, \tag{II.31}$$

where the differential operator $\bar{\partial}_A$ is defined by

$$\bar{\partial}_A \, \Phi \;:=\; \frac{1}{2} \left(D_1 \Phi \,+\, i D_2 \Phi \right) . \tag{II.32}$$

In order to prove the identity (II.31), we first of all expand the two expressions in the integrand on the right hand side:

$$\left(B - \frac{1 - |\Phi|^2}{2} \right)^2 \;=\; B^2 \,+\, \frac{1}{4} \left(1 - |\Phi|^2\right)^2 \,-\, B \left(1 - |\Phi|^2\right) \tag{II.33}$$

$$4 \left| \bar{\partial}_A \Phi \right|^2 \;=\; \left| D_1 \Phi \right|^2 \,+\, \left| D_2 \Phi \right|^2 \,+\, 2 \left\langle D_1 \Phi \,,\, i D_2 \Phi \right\rangle \tag{II.34}$$

The $\mathbb{R}^2$ integral of the mixed term $B\big(1 - |\Phi|^2\big)$ in (II.33) can be rewritten by integration by parts

$$\begin{aligned} \int_{\|x\| \leq R} B\big(1 - |\Phi|^2\big) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \;&=\; \int_{\|x\| \leq R} \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right) \left(1 - |\Phi|^2\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &=\; - \int_{\|x\| \leq R} A_2 \, \frac{\partial}{\partial x^1} \left(1 - |\Phi|^2\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &\qquad +\; \int_{\|x\| \leq R} A_1 \, \frac{\partial}{\partial x^2} \left(1 - |\Phi|^2\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &\qquad +\; \oint_{\|x\| = R} \left(1 - |\Phi|^2\right) \left(A_1 \, \mathrm{d}x^1 \,+\, A_2 \, \mathrm{d}x^2 \right) . \end{aligned} \tag{II.35}$$

When extending the integral to $\mathbb{R}^2$, i.e. sending the the circle radius $R \to \infty$, the surface term vanishes and we are left with

$$\int_{\mathbb{R}^2} B\big(1 - |\Phi|^2\big) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \;=\; 2 \int_{\mathbb{R}^2} \left\{ A_2 \left\langle \Phi \,,\, \frac{\partial \Phi}{\partial x^1} \right\rangle \,-\, A_1 \left\langle \Phi \,,\, \frac{\partial \Phi}{\partial x^2} \right\rangle \right\} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,. \tag{II.36}$$

Up to a Jacobian $j^0(\Phi)$, this is cancelled by the $\mathbb{R}^2$ integral of the $D_1 \leftrightarrow D_2$ mixing terms which arise due to (II.34):

$$\begin{aligned} 2 \int_{\mathbb{R}^2} \left\langle D_1 \Phi \,,\, i D_2 \Phi \right\rangle \mathrm{d}x^1 \wedge \mathrm{d}x^2 \;&=\; -2 \int_{\mathbb{R}^2} \left\{ A_1 \left\langle -i\Phi \,,\, i \frac{\partial \Phi}{\partial x^2} \right\rangle \,+\, \left\langle \frac{\partial \Phi}{\partial x^1} \,,\, A_2 \, \Phi \right\rangle \right\} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &\qquad -\, 2 \int_{\mathbb{R}^2} j^0(\Phi) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned} \tag{II.37}$$

Altogether, (II.31), (II.36) and (II.37) imply that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} \left\{ 4 \left| \bar{\partial}_A \Phi \right|^2 \,+\, \left(B - \frac{1 - |\Phi|^2}{2} \right)^2 \right\} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \;&=\; V_1(A, \Phi) \,-\, \int_{\mathbb{R}^2} j^0(\Phi) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &=\; V_1(A, \Phi) \,-\, N\pi \,. \end{aligned} \tag{II.38}$$

cf. (II.30) for the $j^0(\Phi)$ integral.

Remark II.1 *The identity (II.38)should be regarded as analogous to the identity*

$$\frac{1}{2} \int_{\mathbb{R}} \frac{d\theta}{dx}^2 + 2(1 - \cos\theta) \, dx \;=\; \frac{1}{2} \int_{\mathbb{R}} (\frac{d\theta}{dx} - 2 \sin\frac{\theta}{2})^2 \, dx \,+\, 8$$

which provides the identification of the kinks set of kinks $\{\theta_K(\cdot - X)\}_{X \in \mathbb{R}}$ *with arbitrary centre* X *as the set of all minimizers of the sine-Gordon potential energy with boundary conditions* $\theta \to 2\pi$ *as* $x \to +\infty$ *and* $\theta \to 0$ *as* $x \to -\infty$.

We rewrite the identity as

$$V_1(A,\Phi) \;=\; \frac{1}{2}\int_{\mathbb{R}^2} \big\|\mathcal{B}(A,\Phi)\big\|^2 \,\mathrm{d}x^1\wedge \mathrm{d}x^2 \;+\; N\pi$$
$$\mathcal{B}(A,\Phi) \;=\; \Big(B \,-\, \tfrac{1}{2}\big(1 \,-\, |\Phi|^2\big)\,,\, 2\,\bar\partial_A\Phi\Big)\,, \qquad \text{(II.39)}$$

with a two component Bogomolny operator $\mathcal{B}(A,\Phi)$. It shows that if there exists a solution of $\mathcal{B}(A,\Phi) = 0$ then it minimizes the energy amongst configurations of winding number $N\in\mathbb{N}$, and we say the Bogomolny bound $V_1 \geq N\pi$ is saturated.

$$\Big\{(A,\Phi):\; \mathcal{B}(A,\Phi)=0\Big\} \;\subset\; \Big\{(A,\Phi):\; V_1(A,\Phi)=N\pi\Big\}\,. \qquad \text{(II.40)}$$

If we change signs in the computations (II.34) and (II.37), we obtain

$$V_1(A,\Phi) \;=\; \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left(B \,-\, \frac{1\,-\,|\Phi|^2}{2}\right)^2 \,+\, 4\,\big|\partial_A\Phi\big|^2\right\}\mathrm{d}x^1\wedge\mathrm{d}x^2 \;-\; N\pi$$
$$\partial_A\Phi \;:=\; \frac{1}{2}\,\big(D_1\Phi \,-\, iD_2\Phi\big)\,. \qquad \text{(II.41)}$$

When $N>0$, the first version (II.39) is appropriate and we will only consider this case. If $N<0$, one could use the related formula (II.41) with sign changes to deduce completely analogous results.

II.4 The Self-dual case: Finite energy solutions to the Bogomolny equations

Theorem: *Fix $N\in\mathbb{N}_0$, then for any choice $\mathbf{z}=(z_1,z_2,\ldots,z_N)$ of N unordered, possibly repeated points $z_j\in\mathbb{C}\cong\mathbb{R}^2$, there is a smooth, uniquely determined pair of functions a, ϕ with*

(i) $V_1\big(a(\cdot;\mathbf{z}),\phi(\cdot,\mathbf{z})\big) \;=\; N\pi$

(ii) $\mathcal{B}\big(a(\cdot;\mathbf{z}),\phi(\cdot,\mathbf{z})\big) \;=\; 0$

(iii) $a(\cdot;\mathbf{z})$ and $\phi(\cdot,\mathbf{z})$ solve the Euler Lagrange equations (II.24) at $\lambda=1$. Conversely, all the solutions of (II.24) $\Big|_{\lambda=1}$ with finite action and winding number N solve $\mathcal{B}=0$ and are gauge equivalent to one of the $\big(a(\cdot;\mathbf{z}),\phi(\cdot,\mathbf{z})\big)$.

(iv) $\Phi\approx c_j(z-z_j)^{n_j}$ as $z\to z_j$ where $c_j\in\mathbb{C}$ and n_j is the number of times z_j appears in the list $\mathbf{z}$.

(v) $\exists$ positive numbers k_1,k_2,k_3 and $\delta<1$ such that

$$\big|D\Phi(x)\big| \;\leq\; k_1\,\big(1\,-\,|\Phi|^2\big) \;\leq\; k_2\,e^{-(1+\delta)\|x\|}$$
$$0 \;\leq\; B=\big(1\,-\,|\Phi|^2\big) \;\leq\; k_3\,e^{-(1+\delta)\|x\|}\,. \qquad \text{(II.42)}$$

(vi) The winding number N can be expressed as

$$N \;=\; \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\|x\|=R}\langle i\Phi\,,\,\mathrm{d}\Phi\rangle \;=\; \frac{1}{\pi}\int_{\mathbb{R}^2}\mathrm{d}\Phi\wedge\mathrm{d}\Phi \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}B\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,. \qquad \text{(II.43)}$$

(vii) At $N=0$, all the solutions of (II.24) $\Big|_{\lambda=1}$ are gauge equivalent to $a=0$ and $\phi=1$.

(viii) For $N<0$, there are completely analogous solutions as we discussed above.

(ix) Note that the zeros of ϕ are unchanged by gauge transformations since $\big|e^{i\chi}\big|=1$.

To interpret generic (a,ϕ) solutions physically, keep in mind that $\Phi=f_N(r)e^{iN\theta}$ behaves like $\Phi\approx c_N r^N e^{iN\theta} = c_N z^N$ as $r\to 0$. Thus regard the $a(\cdot;\mathbf{z})$ and $\phi(\cdot,\mathbf{z})$ as being *non-linear superpositions* of N vortices of arbitrary locations $z_1,\ldots,z_N\in\mathbb{R}^2$. At the very special self dual value $\lambda=1$, single vortices can be interpreted as non-interacting. At the level of the energy, the total energy of these solutions is $N\pi$ which is precisely N times the energy π of one vortex.

II.4.1 Sketch of the proof

For the complete proof of this theorem, see chapter III of the book of "Jaffe Taubes". We will just discuss the most important ingredients.

(i) The second component of the Bogomolny equation $\mathcal{B} = 0$, namely,

$$\begin{aligned} \bar\partial_A \Phi &= \frac{1}{2}\left(D_1 + iD_2\right)\Phi = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)\Phi - \frac{i}{2}\left(A_1 + iA_2\right)\Phi \\ &=: \frac{\partial \Phi}{\partial \bar z} - i\bar\alpha\,\Phi = 0\,, \qquad \text{where } \bar\alpha = \frac{A_1 + iA_2}{2}\,, \end{aligned} \tag{II.44}$$

almost takes the form of the holomorphicity condition $\frac{\partial \Phi}{\partial \bar z} = 0$. To get rid of the $\bar\alpha$ perturbation, use the method of integrating factors. If we can solve $\frac{\partial w}{\partial \bar z} = i\bar\alpha$, then

$$\frac{\partial}{\partial \bar z}\left(e^{-iw}\,\Phi\right) = e^{-w}\left(\frac{\partial \Phi}{\partial \bar z} - i\bar\alpha\,\Phi\right) = 0\,, \tag{II.45}$$

i.e. $f = e^{-w}\Phi$ is holomorphic. Now recall the proposition from complex analysis that, if a function f is holomorphic $\frac{\partial f}{\partial \bar z} = 0$, then the zeros $\{z_j \in \mathbb{C} : f(z_j) = 0\}$ of f are isolated. Near z_j, f has the local form

$$f(z) \approx c_j(z - z_j)^{n_j}\,, \qquad n_j \equiv \text{multiplicity},\ c_j \in \mathbb{C}\,. \tag{II.46}$$

from which we can read off the local behaviour of Φ near its zeros. The solution of $\frac{\partial}{\partial \bar z} = i\bar\alpha$ can be written as

$$w(z) = \frac{1}{2\pi i}\int_{|u-z|<\varepsilon} \frac{i\bar\alpha(u,\bar u)}{u - z}\,\mathrm{d}u \wedge \mathrm{d}\bar u \tag{II.47}$$

since $\frac{1}{u-z}$ is the Green function of the $\frac{\partial}{\partial \bar z}$ operator

$$\frac{\partial}{\partial \bar z}\frac{1}{u - z} = 2\pi i\,\delta(z - u)\,, \tag{II.48}$$

see the introduction to *Principles of algebraic geometry* by "Griffiths & Harris".

(ii) Once Φ is known, A_1, A_2 are determined by $\frac{\partial \Phi}{\partial \bar z} - i\bar\alpha\Phi = 0$:

$$\bar\alpha = -i\frac{\partial}{\partial \bar z}\ln\Phi\,, \qquad \alpha = +i\frac{\partial}{\partial z}\ln\overline{\Phi}\,. \tag{II.49}$$

Using a polar decomposition for Φ, the phase Θ is given by the arguments of $z - z_j$,

$$\Phi = \exp\left(\frac{1}{2}\left(u + i\Theta\right)\right) \Rightarrow \Theta(z) = 2\sum_{j=1}^{N}\arg(z - z_j)\,, \tag{II.50}$$

and we can solve for the radial part u using (II.49):

$$A_1 = \frac{1}{2}\left(\partial_2 u + \partial_1\Theta\right)\,, \qquad A_2 = -\frac{1}{2}\left(\partial_1 u - \partial_2\Theta\right)\,. \tag{II.51}$$

(iii) The magnetic field is in principle determined by (II.51), but it is a bit delicate to treat because Θ is not a smooth function due to (II.50), in particular its second derivatives do not commute on the full complex plane, as indicated by an application of the Stokes-Green identity:

$$\int_{|z-z_j|\le\varepsilon}\left(\frac{\partial^2\Theta}{\partial x^1\,\partial x^2} - \frac{\partial^2\Theta}{\partial x^2\,\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 = \oint_{|z-z_j|=\varepsilon}\mathrm{d}\Theta = 4\pi\,n_j\,. \tag{II.52}$$

More precisely, (II.52) demonstrates their difference to equal a sum of delta functions:

$$\frac{\partial^2\Theta}{\partial x^1\,\partial x^2} - \frac{\partial^2\Theta}{\partial x^2\,\partial x^1} = 4\pi\sum_{j=1}^{N}\delta(z - z_j) \tag{II.53}$$

From (II.51) and (II.53), we can compute the magnetic field as

$$\begin{aligned} B &= \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}\,\triangle u + \frac{1}{2}\left(\frac{\partial^2\Theta}{\partial x^1\,\partial x^2} - \frac{\partial^2\Theta}{\partial x^2\,\partial x^1}\right) \\ &= -\frac{1}{2}\,\triangle u + 2\pi\sum_{j=1}^{N}\delta(z - z_j)\,. \end{aligned} \tag{II.54}$$

On the other hand, the first component Bogomolny equation fixes B as

$$B = \frac{1}{2}\left(1 - |\Phi|^2\right) = \frac{1}{2}\left(1 - e^u\right), \tag{II.55}$$

so putting (II.54) and (II.55) together, we arrive at the non-linear partial differential equation:

$$-\triangle u + e^u - 1 = -4\pi\sum_{j=1}^{N}\delta(z - z_j) \tag{II.56}$$

Explicit solutions (II.56) are unknown so far, but one can bring (II.56) into a finite form by regularization: If we introduce

$$v := u - u_0\,, \qquad u_0 := -\sum_{j=1}^{N}\ln\left(1 + \frac{\mu}{|z - z_j|^2}\right), \qquad \mu \gg 1\,, \tag{II.57}$$

then (II.56) reads

$$-\triangle v + e^{u_0}\,e^{v} = 1 - 4\sum_{j=1}^{N}\ln\left(1 + \frac{\mu}{|z - z_j|^2}\right) =: g_0\,. \tag{II.58}$$

This modified equation (II.58) has a unique C^∞ solution for any $z_1, \dots, z_N$ obtained by minimizing the functional

$$I[v] = \int_{\mathbb{R}^2}\left\{\frac{1}{2}\left|\nabla v\right|^2 + (g_0 - 1)\,v + e^u\,(e^v - 1)\right\}\mathrm{d}^2x \tag{II.59}$$

see chapter III in the book of Jaffe and Taubes.

A generalization to curved spaces In the article of Witten, (PRL (1977) Vol.38, 121-124), the Bogomolny argument was carried out on the Poincare unit disc, i.e. the domain

$$\Sigma := \left\{z \in \mathbb{C} :\ |z| < 1\right\} \tag{II.60}$$

with metric

$$\mathrm{d}s^2 = \frac{8\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)}{(1 - |z|^2)^2} =: e^{2\rho}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \tag{II.61}$$

Let Φ denote a section of $V \to \Sigma$ with $\Phi(x) \in \mathbb{C}$ and $A = A_1\mathrm{d}x + A_2\mathrm{d}y$ a connection 1 form, moreover define

$$\begin{aligned} \Omega^0(\Sigma) &\equiv \text{set of functions } \Sigma \to \mathbb{R} \equiv \text{zero forms} \\ \Omega^1(\Sigma) &\equiv \left\{A_1\,\mathrm{d}x + A_2\,\mathrm{d}y\right\},\ A_i : \Sigma \to \mathbb{R} \equiv \text{1 forms} \\ \Omega^2(\Sigma) &\equiv \left\{f\,\mathrm{d}x \wedge \mathrm{d}y\right\},\ f : \Sigma \to \mathbb{R} \equiv \text{2 forms}\,. \end{aligned} \tag{II.62}$$