Part III — Classical and Quantum Solitons

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Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantization, they give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, on gauge theory vortices in two dimensions, and on Skyrmions in three dimensions.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Reference

N. Manton and P. Sutcliffe, *Topological Solitons*, CUP, 2004

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0 Introduction

Given a classical field theory, if we want to "quantize" it, we find the vacuum of the theory, and then do perturbation theory around this vacuum. If there are multiple vacua, then we arbitrarily pick a vacuum, and expand around that.

However, these field theories with multiple vacua often contain *soliton* solutions. These are localized, smooth solutions of the classical field equations, and they "connect multiple vacuums". To quantize these solitons solutions, we fix such a soliton, and use it as the "background". We then do perturbation theory around these solutions, but this is rather tricky to do. Thus, in a lot of the course, we will just look at the classical part of the theory.

Recall that when quantizing our field theories in perturbation theory, we obtain particles in the quantum theory, despite the classical theory being completely about fields. It turns out solitons also behave like particles, and they are a *new* type of particles. These are non-perturbative phenomenon. If we want to do the quantum field theory properly, we have to include these solitons in the quantum field theory. In general this is hard, and so we are not going to develop this a lot.

What does it mean to say that solitons are like particles? In relativistic field theories, we find these solitons have a classical energy. We define the "mass" M of the soliton to be the energy in the "rest frame". Since this is relativistic, we can do a Lorentz boost, and we obtain a moving soliton. Then we obtain an energy-momentum relation of the form

$$E^2 - \mathbf{P}\cdot\mathbf{P} = M^2.$$

This is a Lorentz-invariant property of the soliton. Together with the fact that solitons are localized, this is a justification for thinking of them as particles.

These particles differ from the particles of perturbative quantum fields, as they have rather different properties. Interesting solitons have a *topological* character different from the classical vacuum. Thus, at least naively, they cannot be thought of perturbatively.

There are also non-relativistic solitons, and they usually don't have interpretations as particles. These appear, for example, as defects in solids. We will not be interested in these much.

What kinds of theories have solitons? To obtain solitons, we definitely need a non-linear field structure and/or non-linear equations. Thus, free field theories with quadratic Lagrangians such as Maxwell theory do not have solitons. We need interaction terms.

Note that in QFT, we did interactions using the interaction picture. We split the Hamiltonian into a "free field" part, which we solve exactly, and the "interaction" part. However, to quantize solitons, we need to solve the full interacting Lagrangian *exactly*.

Having interactions is not enough for solitons to appear. To obtain solitons, we also need some non-trivial vacuum topology. In other words, we need more than one vacuum. This usually comes from symmetry breaking, and often gauge symmetries are involved.

In this course, we will focus on three types of solitons.

– In one (space) dimension, we have kinks. We will spend 4 lectures on this.

- In two dimensions, we have vortices. We will spend 6 lectures on this.
- In three dimensions, there are monopoles and Skyrmions. We will only study Skyrmions, and will spend 6 lectures on this.

These examples are all relativistic. Non-relativistic solitons include *domain walls*, which occur in ferromagnets, but we will not study these.

In general, solitons appear in all sorts of different actual, physical scenarios such as in condensed matter physics, optical fibers, superconductors and exotic magnets. "Cosmic strings" have also been studied. Since we are mathematicians, we probably will not put much focus on these applications. However, we can talk a bit more about Skyrmions.

Skyrmions are solitons in an *effective field theory* of interacting pions, which are thought to be the most important hadrons because they are the lightest. This happens in spite of the lack of a gauge symmetry. While pions have no baryon number, the associated solitons have a topological charge identified with baryon number. This baryon number is conserved for topological reasons.

Note that in QCD, baryon number is conserved because the quark number is conserved. Experimenters have tried extremely hard to find proton decay, which would be a process that involves baryon number change, but we cannot find such examples. We have very high experimental certainty that baryon number is conserved. And if baryon number is topological, then this is a very good reason for the conservation of baryon number.

Skyrmions give a model of low-energy interactions of baryons. This leads to an (approximate) theory of nucleons (proton and neutron) and larger nuclei, which are bound states of any number of protons and neutrons.

There is a whole other set of Skyrmions studied, called Magnetic Skyrmions, which are two-dimensional. These are structure in exotic magnets, and they have actually been seen.

For these ideas to work out well, we need to eventually do quantization. For example, Skyrmions by themselves do not have spin. We need to quantize the theory before spins come out. Also, Skyrmions cannot distinguish between protons and neutrons. These differences only come up after we quantize.

1 ϕ^4 kinks

1.1 Kink solutions

In this section, we are going to study ϕ^4 *kinks*. This is in $1+1$ dimensions, and involves a single scalar field $\phi(x,t)$. In higher dimensions, we often need many fields to obtain solitons, but in the case of 1 dimension, we can get away with a single field.

In general, the *Lagrangian density* of such a scalar field theory is of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

for some potential $U(\phi)$ polynomial in ϕ. Note that in $1+1$ dimensions, any such theory is renormalizable. Here we will choose the Minkowski metric to be

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $\mu, \nu = 0, 1$. Then the *Lagrangian* is given by

$$L = \int_{-\infty}^{\infty} \mathcal{L}\,\mathrm{d}x = \int_{-\infty}^{\infty} \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right)\,\mathrm{d}x,$$

and the *action* is

$$S[\phi] = \int L\,\mathrm{d}t = \int \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t.$$

There is a non-linearity in the field equations due to a potential $U(\phi)$ with *multiple vacua*. We need multiple vacua to obtain a soliton. The kink stability comes from the *topology*. It is very simple here, and just comes from counting the discrete, distinct vacua.

As usual, we will write

$$\dot{\phi} = \frac{\partial\phi}{\partial t}, \quad \phi' = \frac{\partial\phi}{\partial x}.$$

Often it is convenient to (non-relativistically) split the Lagrangian as

$$L = T - V,$$

where

$$T = \int \frac{1}{2}\dot{\phi}^2\,\mathrm{d}x, \quad V = \int \left(\frac{1}{2}\phi'^2 + U(\phi)\right)\,\mathrm{d}x.$$

In higher dimensions, we separate out $\partial_\mu\phi$ into $\dot{\phi}$ and $\nabla\phi$.

The classical field equation comes from the condition that $S[\phi]$ is stationary under variations of ϕ. By a standard manipulation, the field equation turns out to be

$$\partial_\mu\partial^\mu\phi + \frac{\mathrm{d}U}{\mathrm{d}\phi} = 0.$$

This is an example of a Klein–Gordon type of field equation, but is non-linear if U is not quadratic. This is known as the *non-linear Klein–Gordon equation*.

We are interested in a soliton that is a static solution. For a static field, the time derivatives can be dropped, and this equation becomes

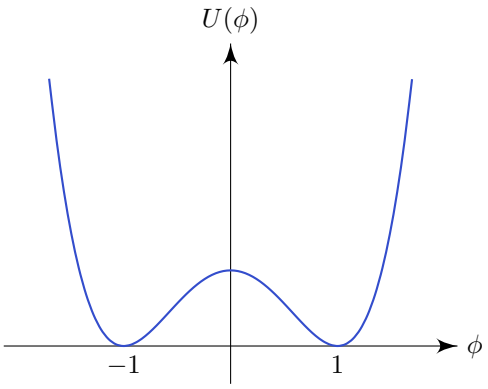
$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \frac{\mathrm{d}U}{\mathrm{d}\phi}.$$

Of course, the important part is the choice of U! In ϕ^4 theory, we choose

$$U(\phi) = \frac{1}{2}(1-\phi^2)^2.$$

This is mathematically the simplest version, because we set all coupling constants to 1. If we want to quantize this (or just want to be more sophisticated), then we should include some parameters.

The importance of this U is that it has two minima:



The two classical vacua are

$$\phi(x) \equiv 1, \quad \phi(x) \equiv -1.$$

This is, of course, not the only possible choice. We can, for example, include some parameters and set

$$U(\phi) = \lambda(m^2 - \phi^2)^2.$$

If we are more adventurous, we can talk about a ϕ^6 theory with

$$U(\phi) = \lambda\phi^2(m^2 - \phi^2)^2.$$

In this case, we have 3 minima, instead of 2. Even braver people can choose

$$U(\phi) = 1 - \cos\phi.$$

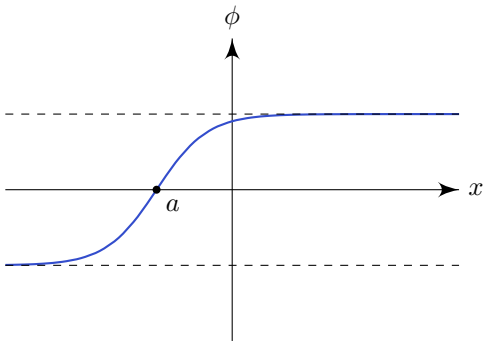
This has *infinitely* many minima. The field equation involves a $\sin\phi$ term, and hence this theory is called the *sine-Gordon theory* (a pun on the name Klein–Gordon, of course).

The sine-Gordon theory is a special case. While it seems like the most complicated potential so far, it is actually *integrable*. This implies we can find explicit exact solutions involving multiple, interacting solitons in a rather easy way. However, integrable systems is a topic for another course, namely the Part II Integrable Systems course.

For now, we will focus on our simplistic ϕ^4 theory. As mentioned, there are two vacuum field configurations, both of zero energy. We will in general use the term "*field configuration*" to refer to fields at a given time that are not necessarily solutions to the classical field equation, but in this case, the vacua are indeed solutions.

If we wanted to quantize this ϕ^4 theory, then we have to pick one of the vacua and do perturbation theory around it. This is known as *spontaneous symmetry breaking*. Of course, by symmetry, we obtain the same quantum theory regardless of which vacuum we expand around.

However, as we mentioned, when we want to study solitons, we have to involve *both* vacua. We want to consider solutions that "connect" these two vacua. In other words, we are looking for solutions that look like



This is known as a *kink solution*.

To actually find such a solution, we need the full field equation, given by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -2(1-\phi^2)\phi.$$

Instead of solving this directly, we will find the kink solutions by considering the energy, since this method generalizes better.

We will work with a general potential U with minimum value 0. From Noether's theorem, we obtain a conserved energy

$$E = \int \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\right)\,\mathrm{d}x.$$

For a static field, we drop the $\dot{\phi}^2$ term. Then this is just the V appearing in the Lagrangian. By definition, the field equation tells us the field is a stationary point of this energy. To find the kink solution, we will in fact find a *minimum* of the energy.

Of course, the global minimum is attained when we have a vacuum field, in which case $E = 0$. However, this is the global minimum only if we don't impose any boundary conditions. In our case, the kinks satisfy the boundary conditions "$\phi(\infty) = 1$", "$\phi(-\infty) = -1$" (interpreted in terms of limits, of course). The kinks will minimize energy subject to these boundary conditions.

These boundary conditions are important, because they are "topological". Eventually, we will want to understand the dynamics of solitons, so we will want to consider fields that evolve with time. From physical considerations, for any fixed t, the field $\phi(x,t)$ must satisfy $\phi(x,t) \to$ vacuum as $x \to \pm\infty$, or else the

field will have infinite energy. However, the vacuum of our potential U is discrete. Thus, if ϕ is to evolve continuously with time, the boundary conditions must not evolve with time! At least, this is what we expect classically. Who knows what weird tunnelling can happen in quantum field theory.

So from now on, we fix some boundary conditions $\phi(\infty)$ and $\phi(-\infty)$, and focus on fields that satisfy these boundary conditions. The trick is to write the potential in the form

$$U(\phi) = \frac{1}{2}\left(\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\right)^2.$$

If U is non-negative, then we can always find W in principle — we take the square root and then integrate it. However, in practice, this is useful only if we can find a simple form for W. Let's assume we've done that. Then we can write

$$\begin{aligned}
E &= \frac{1}{2}\int \left(\phi'^2 + \left(\frac{\mathrm{d}W}{\mathrm{d}\phi}\right)^2\right)\,\mathrm{d}x \\
&= \frac{1}{2}\int \left(\phi' \mp \frac{\mathrm{d}W}{\mathrm{d}\phi}\right)^2\,\mathrm{d}x \pm \int \frac{\mathrm{d}W}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x \\
&= \frac{1}{2}\int \left(\phi' \mp \frac{\mathrm{d}W}{\mathrm{d}\phi}\right)^2\,\mathrm{d}x \pm \int \mathrm{d}W \\
&= \frac{1}{2}\int \left(\phi' \mp \frac{\mathrm{d}W}{\mathrm{d}\phi}\right)^2\,\mathrm{d}x \pm (W(\phi(\infty)) - W(\phi(-\infty))).
\end{aligned}$$

The second term depends purely on the boundary conditions, which we have fixed. Thus, we can minimize energy if we can make the first term vanish! Note that when completing the square, the choice of the signs is arbitrary. However, if we want to set the first term to be 0, the second term had better be non-negative, since the energy itself is non-negative! Hence, we will pick the sign such that the second term is ≥ 0, and then the energy is minimized when

$$\phi' = \pm\frac{\mathrm{d}W}{\mathrm{d}\phi}.$$

In this case, we have

$$E = \pm(W(\infty) - W(-\infty)).$$

These are known as the *Bogomolny equation* and the *Bogomolny energy bound*. Note that if we picked the other sign, then we cannot solve the differential equation $\phi' = \pm\frac{\mathrm{d}W}{\mathrm{d}\phi}$, because we know the energy must be non-negative.

For the ϕ^4 kink, we have

$$\frac{\mathrm{d}W}{\mathrm{d}\phi} = 1 - \phi^2.$$

So we pick

$$W = \phi - \frac{1}{3}\phi^3.$$

When $\phi = \pm 1$, we have $W = \pm\frac{2}{3}$. We need to choose the $+$ sign, and then we know the energy (and hence mass) of the kink is

$$E \equiv M = \frac{4}{3}.$$

We now solve for ϕ. The Bogomolny equation is

$$\phi' = 1 - \phi^2.$$

Rearranging gives

$$\frac{1}{1-\phi^2}\mathrm{d}\phi = \mathrm{d}x,$$

which we can integrate to give

$$\phi(x) = \tanh(x - a).$$

This a is an arbitrary constant of integration, labelling the intersection of the graph of ϕ with the x-axis. We think of this as the "*location*" of the kink.

Note that there is not a unique solution, which is not unexpected by translation invariance. Instead, the solutions are labeled by a parameter a. This is known as a *modulus* of the solution. In general, there can be multiple moduli, and the space of all possible values of the moduli of solitons is known as the *moduli space*. In the case of a kink, the moduli space is just $\mathbb{R}$.

Is this solution stable? We obtained this kink solution by minimizing the energy within this topological class of solutions (i.e. among all solutions with the prescribed boundary conditions). Since a field cannot change the boundary conditions during evolution, it follows that the kink must be stable.

Are there other soliton solutions to the field equations? The solutions are determined by the boundary conditions. Thus, we can classify all soliton solutions by counting all possible combinations of the boundary conditions. We have, of course, two vacuum solutions $\phi \equiv 1$ and $\phi \equiv -1$. There is also an *anti-kink* solution obtained by inverting the kink:

$$\phi(x) = -\tanh(x - b).$$

This also has energy $\frac{4}{3}$.

1.2 Dynamic kink

We now want to look at kinks that move. Given what we have done so far, this is trivial. Our theory is Lorentz invariant, so we simply apply a Lorentz boost. Then we obtain a field

$$\phi(x,t) = \tanh\gamma(x - vt),$$

where, as usual

$$\gamma = (1 - v^2)^{-1/2}.$$

But this isn't all. Notice that for small v, we can approximate the solution simply by

$$\phi(x,t) = \tanh(x - vt).$$

This looks like a kink solution with a modulus that varies with time slowly. This is known as the *adiabatic* point of view.

More generally, let's consider a "moving kink" field

$$\phi(x,t) = \tanh(x - a(t))$$

for some function $a(t)$. In general, this is not a solution to the field equation, but if $\dot{a}$ is small, then it is "approximately a solution".

We can now explicitly compute that

$$\dot{\phi} = -\frac{\mathrm{d}a}{\mathrm{d}t}\phi'.$$

Let's consider fields of this type, and look at the Lagrangian of the field theory. The kinetic term is given by

$$T = \int \frac{1}{2}\dot{\phi}^2 \,\mathrm{d}x = \frac{1}{2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 \int \phi'^2 \,\mathrm{d}x = \frac{1}{2}M\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2.$$

To derive this result, we had to perform the integral $\int \phi'^2 \,\mathrm{d}x$, and if we do that horrible integral, we will find a value that happens to be equal to $M = \frac{4}{3}$. Of course, this is not a coincidence. We can derive this result from more general principles to see that the result of integration is manifestly M.

The remaining part of the Lagrangian is less interesting. Since it does not involve taking time derivatives, the time variation of a is not seen by it, and we simply have a constant

$$V = \frac{4}{3}.$$

Then the original field Lagrangian becomes a particle Lagrangian

$$L = \frac{1}{2}M\dot{a}^2 - \frac{4}{3}.$$

Note that when we first formulated the field theory, the action principle required us to find a field that extremizes the action *among all fields*. However, what we are doing now is to restrict to the set of kink solutions only, and then when we solve the variational problem arising from this Lagrangian, we are extremizing the action among fields of the form $\tanh(x - a(t))$. We can think of this as motion in a "valley" in the field configuration space. In general, such a solution will not also extremize the action among all fields. However, as we said, it will do so "approximately" if $\dot{a}$ is small.

We can obtain an effective equation of motion

$$M\ddot{a} = 0,$$

which is an equation of motion for the variable $a(t)$ *in the moduli space*.

Of course, the solution is just given by

$$a(t) = vt + \text{const},$$

where v is an arbitrary constant, which we interpret as the velocity. In this formulation, we do not have any restrictions on v, because we took the "non-relativistic approximation". This approximation breaks down when v is large.

There is a geometric interpretation to this. We can view the equation of motion $M\ddot{a} = 0$ as the *geodesic equation* in the moduli space $\mathbb{R}$, and we can think of the coefficient M as specifying a Riemannian metric on the moduli space. In this case, the metric is (a scalar multiple of) the usual Euclidean metric $(\mathrm{d}a)^2$.

This seems like a complicated way of describing such a simple system, but this picture generalizes to higher-dimensional systems and allows us to analyze multi-soliton dynamics.

We might ask ourselves if there are multi-kinks in our theory. There aren't in the ϕ^4 theory, because we saw that the solutions are classified by the boundary conditions, and we have already enumerated all the possible boundary conditions. In more complicated theories like sine-Gordon theory, multiple kinks are possible.

However, while we cannot have two kinks, we can have a kink followed by an anti-kink, or more of these pairs. This actually lies in the "vacuum sector" of the theory, but it still looks like it's made up of kinks and anti-kinks, and it is interesting to study these.

1.3 Soliton interactions

We now want to study interactions between kinks and anti-kinks, and see how they cause each other to move. So far, we were able to label the position of the particle by its "center" a, and thus we can sensibly talk about how this center moves. However, this center is well-defined only in the special case of a pure kink or anti-kink, where we can use symmetry to identify the center. If there is some perturbation, or if we have a kink and an anti-kink, it is less clear what should be considered the center.

Fortunately, we can still talk about the momentum of the field, even if we don't have a well-defined center. Indeed, since our theory has translation invariance, Noether's theorem gives us a conserved charge which is interpreted as the momentum.

In general, for a single scalar field in $1+1$ dimensions, the Lagrangian density can be written in the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi).$$

Applying Noether's theorem to the translation symmetry, we obtain the *energy-momentum tensor*

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}.$$

Fixing a time and integrating over all space, we obtain the conserved energy and conserved momentum. These are

$$\begin{aligned} E &= \int_{-\infty}^{\infty} T^0{}_0 \,\mathrm{d}x = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi)\right)\,\mathrm{d}x,\\ P &= -\int_{-\infty}^{\infty} T^0_1 \,\mathrm{d}x = -\int_{-\infty}^{\infty} \dot{\phi}\phi' \,\mathrm{d}x. \end{aligned}$$

We now focus on our moving kink in the adiabatic approximation of the ϕ^4 theory, where $U(\phi) = \frac{1}{2}(1-\phi^2)^2$. Then the field is given by

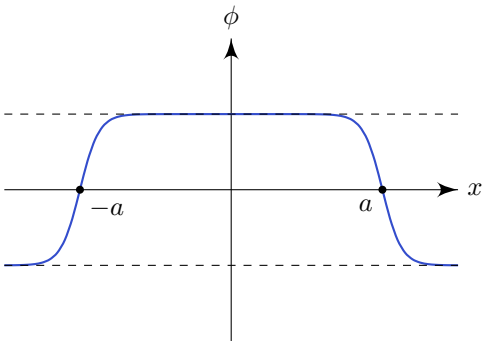
$$\phi = \tanh(x - a(t)).$$

Doing a similar integral as before, we find that the momentum is just

$$P = M\dot{a}.$$

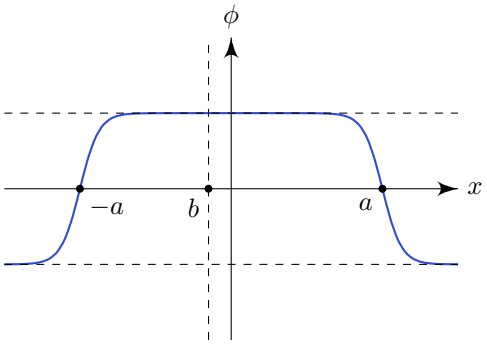
This is just as we would expect for a particle with mass M!

Now suppose what we have is instead a kink-antikink configuration



Here we have to make the crucial assumption that our kinks are well-separated. Matters are a lot worse when they get close to each other, and it is difficult to learn anything about them analytically. However, by making appropriate approximations, we can understand well-separated kink-antikink configurations.

We first pick a point b lying between the kink and the anti-kink:



The choice of b is arbitrary, but we should choose it so that $-a \ll b \ll a$. We will later see that, at least to leading order, the result of our computations does not depend on b if it is in this range. We will declare that the parts to the left of b belongs to the kink, and the parts to the right of b belong to the anti-kink. Then by integrating the energy-momentum tensor in these two regions, we can obtain the momentum of the kink and the anti-kink separately.

We will focus on the kink only. Its momentum is given by

$$P = -\int_{-\infty}^{b} T^0_1 \, \mathrm{d}x = -\int_{-\infty}^{b} \dot{\phi}\phi' \, \mathrm{d}x.$$

Since T^μ_ν is conserved, we know $\partial_\mu T^\mu{}_\nu = 0$. So

$$\frac{\partial}{\partial t} T^0{}_1 + \frac{\partial}{\partial x} T^1{}_1 = 0.$$

By Newton's second law, the force F on the kink is given by the rate of change

of the momentum:

$$\begin{aligned} F = \frac{\mathrm{d}}{\mathrm{d}t}P &= -\int_{-\infty}^{b} \frac{\partial}{\partial t} T^0{}_1 \, \mathrm{d}x \\ &= \int_{-\infty}^{b} \frac{\partial}{\partial x} T^1{}_1 \, \mathrm{d}x \\ &= T^1{}_1\big|_b \\ &= \left(-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + \frac{1}{2}(1-\phi^2)^2 \right)_b . \end{aligned}$$

Note that there is no contribution at the $-\infty$ end because it is vacuum and $T^1{}_1$ vanishes.

To work out the force, we need to be more precise about what our initial configuration is. In this theory, we can obtain it just by adding a kink to an anti-kink. The obvious guess is

$$\phi(x) \stackrel{?}{=} \tanh(x+a) - \tanh(x-a),$$

but this has the wrong boundary condition. It vanishes on both the left and the right. We need to subtract 1, obtaining

$$\phi(x) = \tanh(x+a) - \tanh(x-a) - 1.$$

Note that since our equation of motion is not linear, this is not a genuine static solution! It is approximately a solution, because the kink and anti-kink are well-separated. It is also a useful initial configuration which evolves dynamically as the kinks start to move.

Now we make some simplifications and approximations. By restricting attention to fields that are initially at rest, we have $\dot{\phi} = 0$ at $t = 0$. That gets rid of one term. Also, we only care about the force expression evaluated at b. Here, $\phi(x) = 1 + \eta(x)$ where η is small, so we can expand the expression for F to quadratic order in η, and this gives

$$F = \left(-\frac{1}{2}\eta'^2 + 2\eta^2 \right)_b .$$

Near $x = b$ we can approximate the kink and anti-kink by their exponentially small tail fields

$$\tanh(x+a) \sim 1 - 2e^{-2(x+a)}, \quad \tanh(x-a) \sim -1 + 2e^{2(x-a)}.$$

Substituting into $\phi(x)$ this gives

$$\begin{aligned} \eta &= -2e^{-2(x+a)} - 2e^{2(x-a)}, \\ \eta' &= 4e^{-2(x+a)} - 4e^{2(x-a)}. \end{aligned}$$

In F only the cross terms between the kink and anti-kink tails contribute, because a kink exerts no force on itself, and similarly for the anti-kink. (You can check this explicitly by keeping all terms in F.) So

$$\begin{aligned} F &= 2\left\{ -\frac{1}{2}(4e^{-2(x+a)})(-4e^{2(x-a)}) + 2(-2e^{-2(x+a)})(-2e^{2(x-a)}) \right\}_b \\ &= \left(32e^{-4a}\right)_b \\ &= 32e^{-4a}. \end{aligned}$$

The force is independent of the precise location of b, as expected. Expressed in terms of the separation of the kink and anti-kink, $s = 2a$, the force is $F = 32e^{-2s}$. The force is positive (towards the right), so the kink and anti-kink attract.

What is the interpretation of the factor of 2 in $2s$? Recall that our potential was given by

$$U(\phi) = \frac{1}{2}(1 - \phi^2)^2.$$

Setting $\phi = 1 + \eta$, and then expanding gives

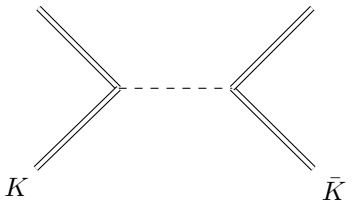
$$U(\eta) \approx \frac{1}{2}(-2\eta)^2 = \frac{1}{2}m^2\eta^2,$$

where $m = 2$. This is the same "2" that appears in the exponent in the force. What about the overall strength 32? Recall that when we expanded the kink solution, the amplitude of the tail was $A = 2$. It turns out if we re-did our theory and put back the different possible parameters, the force would be given by

$$F = 2m^2A^2e^{-ms}.$$

This is an interesting and important phenomenon. The mass m was the *perturbative* mass of the field. It is the meson mass, something we obtain by perturbation theory. However, the same mass appears in the force between the solitons, which is a non-perturbative phenomenon!

We can interpret the force between the kink and anti-kink diagrammatically. From the quantum field theory point of view, we can think of this force as being due to meson exchange, and we can try to invent a Feynman diagram calculus that involves solitons. This is a bit controversial, but at least heuristically, we can introduce new propagators representing solitons, using double lines, and draw the interaction as



So what happens to this soliton? The force we derived was positive. So the kink is made to move to the right. By symmetry, we will expect the anti-kink to move towards the left. They will collide!

What happens when they collide? All our analysis so far assumed the kinks were well-separated, so everything breaks down. We can only understand this numerically. Numerical simulations show that there are two regimes:

- If the kinks are moving slowly, then they will annihilate into *meson radiation.*
- If the kinks are moving very quickly, then they often bounce off each other.

1.4 Quantization of kink motion

We now briefly discuss how to quantize kinks. The most naive way of doing so is pretty straightforward. We use the moduli space approximation, and then we have a very simple kink Lagrangian.

$$L = \frac{1}{2}M\dot{a}^2.$$

This is just a free particle moving in $\mathbb{R}$ with mass M. This a is known as the *collective coordinate* of the kink. Quantizing a free particle is very straightforward. It is just IB Quantum Mechanics. For completeness, we will briefly outline this procedure.

We first put the system in Hamiltonian form. The conjugate momentum to a is given by

$$P = M\dot{a}.$$

Then the Hamiltonian is given by

$$H = P\dot{a} - L = \frac{1}{2M}P^2.$$

Then to quantize, we replace P by the operator $-i\hbar\frac{\partial}{\partial a}$. In this case, the quantum Hamiltonian is

$$H = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial a^2}.$$

A wavefunction is a function of a and t, as in ordinary QM for a single particle.

As usual, the stationary states are given by

$$\psi(a) = e^{i\kappa a},$$

and the momentum and energy (eigenvalues) are

$$P = \hbar\kappa, \quad H = E = \frac{\hbar^2\kappa^2}{2M} = \frac{P^2}{2M}.$$

Is this actually "correct"? Morally speaking, we really should quantize the complete $1 + 1$ dimensional field theory. What would this look like?

In normal quantum field theory, we consider perturbations around a vacuum solution, say $\phi \equiv 1$, and we obtain mesons. Here if we want to quantize the kink solution, we should consider field oscillations around the kink. Then the solution contains both a kink and the meson field. The mesons give rise to quantum corrections to the kink mass M.

Should we be worried about these quantum corrections? Unsurprisingly, it turns out these quantum corrections are of the order of the meson mass. So we should not be worried when the meson mass is small.

Meson-kink scattering can also be studied in the full quantum theory. To first approximation, since the kink is heavy, mesons are reflected or transmitted with some probabilities, while the momentum of the kink is unchanged. But when we work to higher orders, then of course the kink will move as a result. This is all quite complicated.

For more details, see Rajaraman's *Solitons and Instantons*, or E. Weinberg's *Classical Solutions in Quantum Field Theory*.

The thing that is really hard to understand in the quantum field theory is kink-antikink pair production. This happens when the mesons collide at high energy, and the phenomena are highly relativistic. What we have done so far is perturbative and makes the non-relativistic approximation to get the adiabatic picture.

1.5 Sine-Gordon kinks

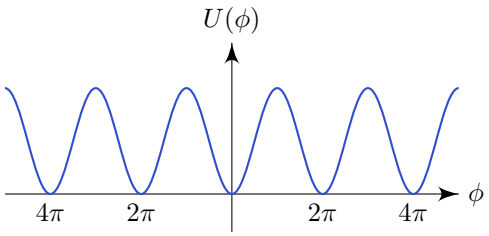
We end the section by briefly talking about kinks in a different theory, namely the *sine-Gordon theory*. In this theory, kinks are often known as *solitons* instead.

The sine-Gordon theory is given by the potential

$$U(\phi) = 1 - \cos\phi.$$

Again, we suppress coupling constants, but it is possible to add them back.

The potential looks like



Now there are *infinitely many* distinct vacua. In this case, we find we need to pick W such that

$$\frac{\mathrm{d}W}{\mathrm{d}\phi} = 2\sin\frac{1}{2}\phi.$$

Static sine-Gordon kinks

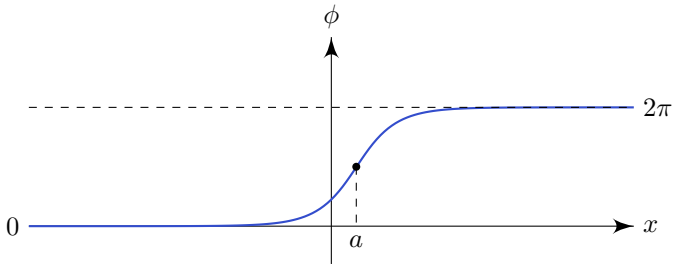
To find the static kinks in the sine-Gordon theory, we again look at the Bogomolny equation. We have to solve

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = 2\sin\frac{1}{2}\phi.$$

This can be solved by integrating a cosec, and ultimately gives us the solution

$$\phi(x) = 4\tan^{-1} e^{x-a},$$

which interpolates between 0 and 2π.



Unlike the ϕ^4 theory, dynamical multi-kink solutions can be derived *exactly*. Ultimately, this is due to the sine-Gordon equation being *integrable*. For more details, refer to the Part II Integrable Systems course. Dynamical multi-kink solutions can be derived exactly. One of the earlier ways to do so was via Bäcklund transforms, but that was very complicated. People later invented better methods, but they are still not very straightforward.

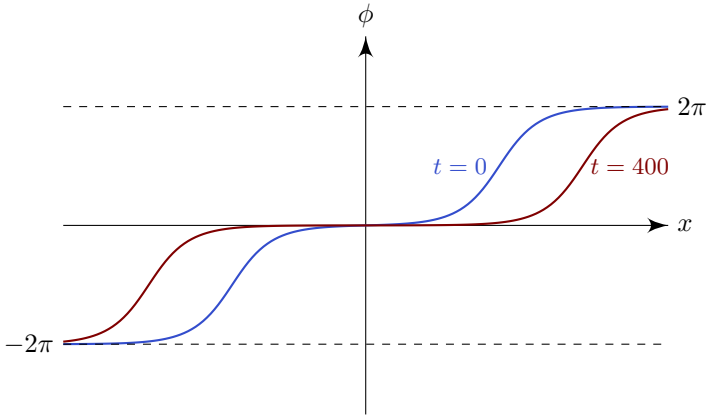
Example. There is a two-kink solution

$$\phi(x,t) = 4\tan^{-1}\left(\frac{v\sinh\gamma x}{\cosh\gamma v t}\right),$$

where, as usual, we have

$$\gamma = (1-v^2)^{-1/2}.$$

For $v = 0.01$, this looks like



Note that since $\phi(x,t) = \phi(x,-t)$, we see that this solution involves two solitons at first approaching each other, each with speed v, and then later bouncing off. Thus, the two kinks *repel* each other. In ϕ^4 theory, we saw that a kink and an anti-kink attracted, but here there are two kinks, which is qualitatively different.

We can again compute the force just like in the ϕ^4 theory, but alternatively, since we have a full, exact solution, we can work it out directly from the solution! The answers, fortunately, agree. If we do the computations, we find that the point of closest approach is $\sim 2\log\left(\frac{2}{v}\right)$ if v is small.

There are some important comments to make. In the sine-Gordon theory, we can have very complicated interactions between kinks and anti-kinks, and these can connect vastly different vacua. However, *static* solutions must join $2n\pi$ and $2(n\pm 1)\pi$ for some n, because if we want to join vacua further apart, we will have more than one kink, and they necessarily interact.

If we have multiple kinks and anti-kinks, then each of them can have their own velocity, and we might expect some very complicated interaction between them, such as annihilation and pair production. But remarkably, in sine-Gordon theory the interactions are *not* complicated. If we try to do numerical simulations, or use the exact solutions, we see that we do not have energy loss due to "radiation". Instead, the solitons remain very well-structured and retain their identities. This, again, is due to the theory being integrable.

Topology of the sine-Gordon equation

There are also a lot of interesting things to note without going into details about what the solutions look like.

The important realization is that our potential is periodic in ϕ. For the sine-Gordon theory, it is much better to think of ϕ as a field modulo 2π, i.e. as a function

$$\phi : \mathbb{R} \to S^1.$$

In this language, the boundary condition is that $\phi(x) = 0 \bmod 2\pi$ as $x \to \pm\infty$. Thus, instead of thinking of the kink as joining two vacua, we can think of it as "winding around the circle" instead.

We can go further. Since the boundary conditions of ϕ are now the same on two sides, we can join the ends of the domain $\mathbb{R}$ together, and we can think of ϕ as a map

$$\phi : S^1 \to S^1$$

instead. This is a *compactification* of the space.

Topologically, such maps are classified by their *winding number*, or the *degree*, which we denote Q. This is a topological (homotopy) invariant of a map, and is preserved under continuous deformations of the field. Thus, it is preserved under time evolution of the field.

Intuitively, the winding number is just how many times the field goes around the target circle. There are multiple (equivalent) ways of making this precise.

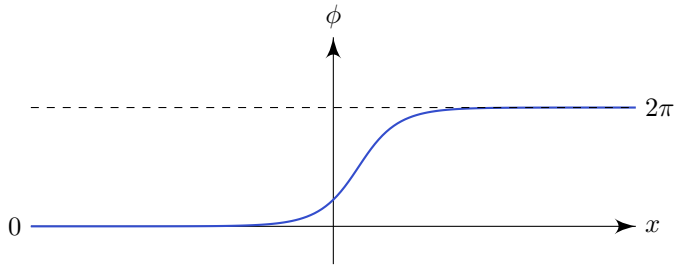
The first way, which is the naive way, is purely topological. We simply have to go back to the first picture, where we regard ϕ as a real value. Suppose the boundary values are

$$\phi(-\infty) = 2n_-\pi, \quad \phi(\infty) = 2n_+\pi.$$

Then we set the winding number to be $Q = n_+ - n_-$.

Topologically, we are using the fact that $\mathbb{R}$ is the *universal covering space* of the circle, and thus we are really looking at the induced map on the fundamental group of the circle.

Example. As we saw, a single kink has $Q = 1$.



Thus, we can think of Q as the *net soliton number*. But this construction is rather specific to maps from S^1 to S^1. We want something more general that can be used for more complicated systems. We can do this in a more "physics" way. We note that there is a *topological* current

$$j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\phi,$$

where $\varepsilon^{\mu\nu}$ is the anti-symmetric tensor in $1+1$ dimensions, chosen so that $\varepsilon^{01} = 1$.

In components, this is just

$$j^\mu = \frac{1}{2\pi}(\partial_x \phi, -\partial_t \phi).$$

This is conserved because of the symmetry of mixed partial derivatives, so that

$$\partial_\mu j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\mu\partial_\nu\phi = 0.$$

As usual, a current induces a conserved charge

$$Q = \int j^0 \,\mathrm{d}x = \frac{1}{2\pi}\int \partial_x \phi \,\mathrm{d}x = \frac{1}{2\pi}(\phi(\infty) - \phi(-\infty)) = n_+ - n_-,$$

which is the formula we had earlier.

Note that all these properties do not depend on ϕ satisfying any field equation! It is completely topological. Finally, there is also a differential geometry way of defining Q. We note that the target space S^1 has a normalized volume form ω so that

$$\int_{S^1} \omega = 1.$$

For example, we can take

$$\omega = \frac{1}{2\pi}\,\mathrm{d}\phi.$$

Now, given a mapping $\phi : \mathbb{R} \to S^1$, we can pull back the volume form to obtain

$$\phi^*\omega = \frac{1}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x.$$

We can then define the degree of the map to be

$$Q = \int \phi^*\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x.$$

This is exactly the same as the formula we obtained using the current!

Note that even though the volume form is normalized on S^1 and has integral 1, the integral when pulled back is not 1. We can imagine this as saying if we wind around the circle n times, then after pulling back, we would have pulled back n "copies" of the volume form, and so the integral will be n times that of the integral on S^1.

We saw that these three definitions gave the same result, and different definitions have different benefits. For example, in the last two formulations, it is not *a priori* clear that the winding number has to be an integer, while this is clear in the first formulation.

2 Vortices

We are now going to start studying *vortices*. These are topological solitons in two space dimensions. While we mostly studied ϕ^4 kinks before, what we are going to do here is closer to sine-Gordon theory than ϕ^4 theory, as it is largely topological in nature.

A lot of the computations we perform in the section are much cleaner when presented using the language of differential forms. However, we shall try our best to provide alternative versions in coordinates.

2.1 Topological background

Sine-Gordon kinks

We now review what we just did for sine-Gordon kinks, and then try to develop some analogous ideas in higher dimension. The sine-Gordon equation is given by

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{\partial^2 \theta}{\partial x^2} + \sin\theta = 0.$$

We want to choose boundary conditions so that the energy has a chance to be finite. The first part is, of course, to figure out what the energy is. The energy-momentum conservation equation given by Noether's theorem is

$$\partial_t \left(\frac{\theta_t^2 + \theta_x^2}{2} + (1 - \cos\theta) \right) + \partial_x(-\theta_t \theta_x) = \partial_\mu P^\mu = 0.$$

The energy we will be considering is thus

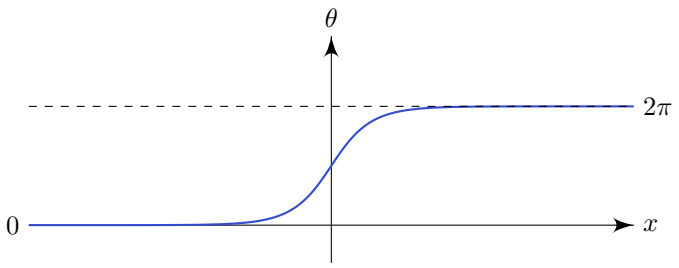
$$E = \int_{\mathbb{R}} P^0 \, \mathrm{d}x = \int_{\mathbb{R}} \left(\frac{\theta_t^2 + \theta_x^2}{2} + (1 - \cos\theta) \right) \, \mathrm{d}x.$$

Thus, to obtain finite energy, we will want $\theta(x) \to 2n_\pm \pi$ for some integers $n_\pm$ as $x \to \pm\infty$. What is the significance of this $n_\pm$?

Example. Consider the basic kink

$$\theta_K(x) = 4 \tan^{-1} e^x.$$

Picking the standard branch of $\tan^{-1}$, this kink looks like



This goes from $\theta = 0$ to $\theta = 2\pi$.

To better understand this, we can think of θ as an angular variable, i.e. we identify $\theta \sim \theta + 2n\pi$ for any $n \in \mathbb{Z}$. This is a sensible thing because the energy density and the equation etc. are unchanged when we shift everything by $2n\pi$. Thus, θ is not taking values in $\mathbb{R}$, but in $\mathbb{R}/2\pi\mathbb{Z} \cong S^1$.

Thus, for each fixed t, our field θ is a map

$$\theta : \mathbb{R} \to S^1.$$

The number $Q = n_+ - n_-$ equals the number of times θ covers the circle S^1 on going from $x = -\infty$ to $x = +\infty$. This is the winding number, which is interpreted as the topological charge.

As we previously discussed, we can express this topological charge as the integral of some current. We can write

$$Q = \frac{1}{2\pi}\int_{\theta(\mathbb{R})} \mathrm{d}\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x.$$

Note that this formula automatically takes into account the orientation. This is the form that will lead to generalization in higher dimensions.

This function $\frac{\mathrm{d}\theta}{\mathrm{d}x}$ appearing in the integral has the interpretation as a topological charge density. Note that there is a topological conservation law

$$\partial_\mu j^\mu = \frac{\partial j^0}{\partial t} + \frac{\partial j^1}{\partial x} = 0,$$

where

$$j^0 = \theta_x, \quad j^1 = -\theta_t.$$

This conservation law is not a consequence of the field equations, but merely a mathematical identity, namely the commutation of partial derivatives.

Two dimensions

In the sine-Gordon kink, the target space was a circle S^1. This time, we are concerned with the *unit disk*

$$D = \{(x^1, x^2) : |\mathbf{x}|^2 < 1\} \subseteq \mathbb{R}^2.$$

We will first consider fields

$$\Phi : D \to D.$$

In the case of a sine-Gordon kink, we still cared about moving solitons. However, here we will mostly work with static solutions, and study fields at a fixed time. Thus, there is no time variable appearing.

Using the canonical isomorphism $\mathbb{R}^2 \cong \mathbb{C}$, we can think of the target space as the unit disk in the complex plane, and write the field as

$$\Phi = \Phi^1 + i\Phi^2.$$

However, we will usually view the domain D as a real space instead.

We will impose some boundary conditions. We pick any function $\chi : S^1 \to \mathbb{R}$, and consider

$$g = e^{i\chi} : S^1 \to S^1 \subseteq D.$$

Here g is a genuine function, and has to be single-valued. So χ must be single-valued modulo 2π. We then require

$$\Phi_{\partial D} = g = e^{i\chi}.$$

In particular, Φ must send the boundary to the boundary, and it has an integer winding number N.

The target space D has a canonical choice of measure $\mathrm{d}x^1 \wedge \mathrm{d}x^2$, so we can expect the new topological charge to be given by

$$Q = \frac{1}{\pi}\int_D \mathrm{d}\Phi^1 \wedge \mathrm{d}\Phi^2 = \frac{1}{\pi}\int_D \det\begin{pmatrix} \frac{\partial\Phi^1}{\partial x^1} & \frac{\partial\Phi^1}{\partial x^2} \\ \frac{\partial\Phi^2}{\partial x^1} & \frac{\partial\Phi^2}{\partial x^2}\end{pmatrix}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Thus, the charge density is given by

$$j^0 = \frac{1}{2}\varepsilon_{ab}\varepsilon_{ij}\frac{\partial\Phi^a}{\partial x^i}\frac{\partial\Phi^b}{\partial x^j}.$$

Crucially, it turns out this charge density is a total derivative, i.e. we have

$$j^0 = \frac{\partial V^i}{\partial x^i}$$

for some function V. It is not immediately obvious this is the case. However, we can in fact pick

$$V^i = \frac{1}{2}\varepsilon_{ab}\varepsilon^{ij}\Phi^a\frac{\partial\Phi^b}{\partial x^j}.$$

To see this actually works, we need to use the anti-symmetry of ε^{ij} and observe that

$$\varepsilon^{ij}\frac{\partial^2\Phi^b}{\partial x^i\partial x^j} = 0.$$

Equivalently, using the language of differential forms, we view the charge density j^0 as the 2-form

$$j^0 = \mathrm{d}\Phi^1 \wedge \mathrm{d}\Phi^2 = \mathrm{d}(\Phi^1\,\mathrm{d}\Phi^2) = \frac{1}{2}\mathrm{d}(\Phi^1\,\mathrm{d}\Phi^2 - \Phi^2\,\mathrm{d}\Phi^1).$$

By the divergence theorem, we find that

$$Q = \frac{1}{2\pi}\oint_{\partial D}\Phi^1\,\mathrm{d}\Phi^2 - \Phi^2\,\mathrm{d}\Phi^1.$$

We then use that on the boundary,

$$\Phi^1 = \cos\chi, \quad \Phi^2 = \sin\chi,$$

so

$$Q = \frac{1}{2\pi}\oint_{\partial D}(\cos^2\chi + \sin^2\chi)\,\mathrm{d}\chi = \frac{1}{2\pi}\oint_{\partial D}\mathrm{d}\chi = N.$$

Thus, the charge is just the winding number of g! Now notice that our derivation didn't really depend on our domain being D. It could have been any region bounded by a simple closed curve in $\mathbb{R}^2$. In particular, we can take it to be a disk D_R of arbitrary radius R.

What we are *actually* interested in is a field

$$\Phi : \mathbb{R}^2 \to D.$$

We then impose asymptotic boundary conditions

$$\Phi \sim g = e^{i\chi}$$

as $|x| \to \infty$. We can still define the charge or degree by

$$Q = \frac{1}{\pi}\int_{\mathbb{R}^2} j^0 \,\mathrm{d}^2x = \frac{1}{\pi}\lim_{R\to\infty}\int_{D_R} j^0 \,\mathrm{d}^2x.$$

This is then again the winding number of g. It is convenient to rewrite this in terms of an inner product. $\mathbb{R}^2$ itself has an inner product, and under the identification $\mathbb{R}^2 \cong \mathbb{C}$, the inner product can be written as

$$(a, b) = \frac{\bar{a}b + a\bar{b}}{2}.$$

Usually, this expression isn't that useful, and we will just use the fact that for real numbers a and b, we have

$$(a, b) = (ai, bi) = ab, \quad (ai, b) = (a, bi) = 0.$$

In particular, we can evaluate

$$(i\Phi, \mathrm{d}\Phi) = (i\Phi^1 - \Phi^2, \mathrm{d}\Phi^1 + i\mathrm{d}\Phi^2) = \Phi^1\,\mathrm{d}\Phi^2 - \Phi^2\,\mathrm{d}\Phi^1.$$

This is just (twice) the current V we found earlier! So we can write our charge as

$$Q = \frac{1}{2\pi}\lim_{R\to\infty}\oint_{|x|=R}(i\Phi, \mathrm{d}\Phi).$$

We will refer to $(i\Phi, \mathrm{d}\Phi)$ as the current, and the charge density is $j^0 = \frac{1}{2}\mathrm{d}(i\Phi, \mathrm{d}\Phi)$.

We have actually seen this current before. In quantum mechanics, we have the Schrödinger equation

$$i\frac{\partial\Phi}{\partial t} = -\frac{1}{2m}\Delta\Phi + V(x)\Phi, \quad \Delta = \nabla^2.$$

Probability is conserved in quantum mechanics, and the probability conservation law is given by

$$\frac{1}{2}\partial_t(\Phi, \Phi) + \frac{1}{2m}\nabla\cdot(i\Phi, \nabla\Phi) = 0.$$

What appears here is just our topological current!

2.2 Global $U(1)$ Ginzburg–Landau vortices

We now put the theory to use. We are going to study *Ginzburg–Landau vortices*. Our previous discussion involved a function taking values in the unit disk D. We will not impose such a restriction on our vortices, a priori. However, we will later see that any solution must take values in D.

The potential energy of the Ginzburg–Landau theory is given by

$$V(\Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left((\nabla\Phi, \nabla\Phi) + \frac{\lambda}{4}(1 - (\Phi, \Phi))^2 \right) \mathrm{d}^2x.$$

where $\lambda > 0$ is some constant.

Note that the inner product is invariant under phase rotation, i.e.

$$(e^{i\chi}a, e^{i\chi}b) = (a, b)$$

for $\chi \in \mathbb{R}$. So in particular, the potential satisfies

$$V(e^{i\chi}\Phi) = V(\Phi).$$

Thus, our theory has a *global* U(1) *symmetry*.

The Euler–Lagrange equation of this theory says

$$-\Delta\Phi = \frac{\lambda}{2}(1 - |\Phi|^2)\Phi.$$

This is the *ungauged Ginzburg–Landau equation*.

To justify the fact that Φ takes values in D, we use the following lemma:

Lemma. Assume Φ satisfies the Ginzburg–Landau equation. Then at any maximum point of $|\Phi|$, we have $|\Phi| \leq 1$.

Proof. Consider the function

$$W(x) = 1 - |\Phi(x)|^2.$$

Then we want to show that $W \geq 0$ when W is minimized. We note that if W is at a minimum, then the Hessian matrix (of second partial derivatives) must have non-negative eigenvalues. So, taking the trace, we must have $\Delta W \geq 0$. Now we can compute ΔW directly. We have

$$\begin{aligned}
\nabla W &= -2(\Phi, \nabla\Phi)\\
\Delta W &= \nabla^2 W\\
&= -2(\Phi, \Delta\Phi) - 2(\nabla\Phi, \nabla\Phi)\\
&= \lambda|\Phi|^2 W - 2|\nabla\Phi|^2.
\end{aligned}$$

Rearranged, this says

$$2|\nabla\Phi|^2 + \Delta W = \lambda|\Phi|^2 W.$$

But we know $2|\nabla\Phi|^2 \geq 0$, and $\Delta W \geq 0$. So we must have $W \geq 0$. □

By itself, this doesn't force $|\Phi| \in [0, 1]$, since we could imagine $|\Phi|$ having no maximum. However, if we prescribe boundary conditions such that $|\Phi| = 1$ on the boundary, then this would indeed require $|\Phi| \leq 1$ everywhere. Often, we can think of Φ as some "complex order parameter". Then the condition $|\Phi| \leq 1$ is very natural.

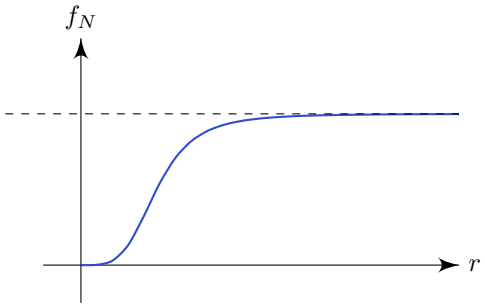
The quantities we are interested in are *vortices*.

Definition (Ginzburg–Landau vortex)**.** A basic *Ginzburg–Landau vortex* of charge $N > 0$ is a (smooth) solution of the form

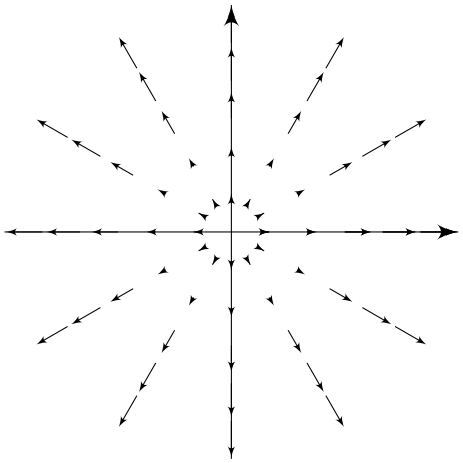
$$\Phi = f_N(r)e^{iN\theta}$$

in polar coordinates (r, θ). Moreover, we require that $f_N(r) \to 1$ as $r \to \infty$.

Note that for Φ to be a smooth solution, we must have $f_N(0) = 0$. In fact, a bit more analysis shows that we must have $f_N = O(r^N)$ as $r \to 0$. Solutions for f_N do exist, and they look roughly like this:



In the case of $N = 1$, we can visualize the field Φ as a vector field on $\mathbb{C}$. Then it looks like



This is known as a 2-*dimensional hedgehog*.

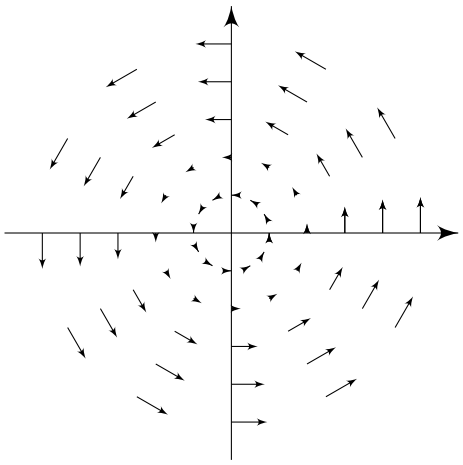
For general N, it might be more instructive to look at how the current looks like. Recall that the current is defined by $(i\Phi, \mathrm{d}\Phi)$. We can write this more explicitly as

$$\begin{aligned}(i\Phi, \mathrm{d}\Phi) &= (if_N e^{iN\theta}, (\mathrm{d}f_N)e^{iN\theta} + if_N N\ \mathrm{d}\theta e^{iN\theta}) \\ &= (if_N, \mathrm{d}f_N + if_N N\ \mathrm{d}\theta).\end{aligned}$$

We see that if_N and $\mathrm{d}f_N$ are orthogonal, while if_N and $if_N N\mathrm{d}\theta$ are parallel. So the final result is

$$(i\Phi, \mathrm{d}\Phi) = f_N^2 N\ \mathrm{d}\theta.$$

So the current just looks like this:



As $|x| \to \infty$, we have $f_N \to 1$. So the winding number is given as before, and we can compute the winding number to be

$$\frac{1}{2\pi} \lim_{R\to\infty} \oint (i\Phi, \mathrm{d}\Phi) = \frac{1}{2\pi} \lim_{R\to\infty} \oint f_N^2 N \, \mathrm{d}\theta = N.$$

Since the winding number of these systems is a discrete quantity, it makes the vortex system stable.

This theory looks good so far. However, it turns out this model has a problem — the energy is infinite! We can expand out $V(f_N e^{iN\theta})$, and see it is a sum of a few non-negative terms. We will focus on the $\frac{\partial}{\partial\theta}$ term. We obtain

$$\begin{aligned} V(f_N e^{iN\theta}) &\geq \int \frac{1}{r^2} \left|\frac{\partial \Phi}{\partial \theta}\right|^2 r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= N^2 \int \frac{1}{r^2} f_N^2 r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= 2\pi N \int_0^\infty \frac{f_N^2}{r} \, \mathrm{d}r. \end{aligned}$$

Since $f_N \to 1$ as $r \to \infty$, we see that the integral diverges logarithmically.

This is bad. We can heuristically understand this by decomposing $\mathrm{d}\Phi$ into two components. At each fixed point, we can decompose $\mathrm{d}\Phi$ into a mode parallel to Φ and a mode perpendicular to Φ. These correspond to the radial and angular modes respectively. Moreover, we will see that the parallel mode is massive, while the perpendicular mode is massless.

We saw that the energy divergence is due to the angular part of the energy, so we see that it is the massless mode that leads to problems.

We can also see the difference between massless and massive modes in a different setting. Consider the equation

$$-\Delta u + M^2 u = f.$$

Working in three dimensions, the solution is given by

$$u(x) = \frac{1}{4\pi} \int \frac{e^{-M|x-y|}}{|x-y|} f(y) \, \mathrm{d}^3 y.$$

Thus, the Green's function is

$$G(x) = \frac{e^{-M|x|}}{|x|}.$$

If the system is massless, i.e. $M = 0$, then G decays as $\frac{1}{|x|}$. However, if the system is massive with $M > 0$, then G decays exponentially as $|x| \to \infty$.

So how do we figure out the massive and massless modes? We do not have a genuine decomposition of Φ itself into "parallel" and "perpendicular" modes, because what is parallel and what is perpendicular depends on the local value of Φ.

Thus, to make sense of this, we have to consider small fluctuations around a fixed configuration Φ. We suppose Φ is a solution to the field equations. Then $\frac{\delta V}{\delta \Phi} = 0$. Thus, for small variations $\Phi \mapsto \Phi + \varepsilon\varphi$, we have

$$V(\Phi + \varepsilon\varphi) = V(\Phi) + \varepsilon^2 \int \left(|\nabla\varphi|^2 + \lambda(\varphi, \Phi)^2 - \frac{2\lambda}{4}(1 - |\Phi|^2)|\varphi|^2 \right) \, \mathrm{d}^2x + O(\varepsilon^3).$$

Ultimately, we are interested in the asymptotic behaviour $|x| \to \infty$, in which case $1 - |\Phi|^2 \to 0$. Moreover, $|\Phi| \to 1$ implies (φ, Φ) becomes a projection into the direction parallel to Φ. Then the potential energy becomes

$$V(\Phi + \varepsilon\varphi) = V(\Phi) + \int \left(|\nabla\varphi|^2 + \lambda|\varphi^{\text{parallel}}|^2 \right) \, \mathrm{d}^2x + O(\varepsilon^3).$$

Thus, for $\lambda > 0$, the parallel mode is massive, while the perpendicular mode is massless. When we study vortices that are gauged, i.e. coupled to the electromagnetic field, we will see that all components are massive, and we don't have this problem. This is the Higgs mechanism.

This is all slightly heuristic, but is a good way to think about the issues.

2.3 Abelian Higgs/Gauged Ginzburg–Landau vortices

We now consider a theory where the complex scalar field Φ is coupled to a magnetic field. This is a U(1) gauge theory, with gauge potential given by a 1-form

$$A = A_1 \, \mathrm{d}x^1 + A_2 \, \mathrm{d}x^2.$$

The coupling between Φ and A is given by *minimal coupling*. To do minimal coupling, we define the *covariant derivative*

$$\mathrm{D}\Phi = \mathrm{D}_A\Phi = \mathrm{d}\Phi - iA\Phi = \sum_{j=1}^{2} \mathrm{D}_j\Phi \, \mathrm{d}x^j.$$

To proceed, it is convenient to have a list of identities about the covariant derivative.

Proposition. If $f(x) \in \mathbb{R}$, then

$$\mathrm{D}(f\Phi) = (\mathrm{d}f) \, \Phi + f \, \mathrm{D}\Phi.$$

If Φ and Ψ are complex scalar fields, then

$$\mathrm{d}(\Phi, \Psi) = (\mathrm{D}\Phi, \Psi) + (\Phi, \mathrm{D}\Psi).$$

In coordinates, these say

$$\mathrm{D}_j(f\Phi) = (\partial_j f)\ \Phi + f\ \mathrm{D}_j\Phi$$
$$\partial_j(\Phi, \Psi) = (\mathrm{D}_j\Phi, \Psi) + (\Phi, \mathrm{D}_j\Psi).$$

The proofs just involve writing all terms out. The second rule is analogous to the fact that

$$\partial_k g(V, W) = g(\nabla_k V, W) + g(V, \nabla_k W)$$

for the Levi-Civita connection.

The curvature term is given by the magnetic field.

Definition (Magnetic field/curvature)**.** The *magnetic field*, or *curvature* is given by

$$B = \partial_1 A_2 - \partial_2 A_1.$$

We can alternatively think of it as the 2-form

$$\mathrm{d}A = B\ \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

The formulation in terms of differential forms is convenient for computations, because we don't have to constrain ourselves to working in Cartesian coordinates — often polar coordinates are more convenient.

Proposition. We have

$$(\mathrm{D}_1\mathrm{D}_2 - \mathrm{D}_2\mathrm{D}_1)\Phi = -iB\Phi.$$

The proof is again by writing it out. Alternatively, we can express this without coordinates. We can extend D to act on p-forms by letting A act as $A\wedge$. Then this result says

Proposition.

$$\mathrm{DD}\Phi = -iB\Phi.$$

Proof.

$$\begin{aligned}
\mathrm{DD}\Phi &= (\mathrm{d} - iA)(\mathrm{d}\Phi - iA\Phi)\\
&= \mathrm{d}^2\Phi - i\mathrm{d}(A\Phi) - iA\ \mathrm{d}\Phi - A \wedge A\ \Phi\\
&= -i\mathrm{d}(A\Phi) - iA\ \mathrm{d}\Phi\\
&= -i\mathrm{d}A\ \Phi + iA\ \mathrm{d}\Phi - iA\ \mathrm{d}\Phi\\
&= -i(\mathrm{d}A)\ \Phi\\
&= -iB\Phi.
\end{aligned}$$

□

The point of introducing the covariant derivative is that we can turn the global U(1) invariance into a local one. Previously, we had a global U(1) symmetry, where our theory is unchanged when we replace $\Phi \mapsto \Phi e^{i\chi}$ for some constant $\chi \in \mathbb{R}$. With the covariant derivative, we can promote this to a *gauge* symmetry.

Consider the simultaneous *gauge transformations*

$$\Phi(x) \mapsto e^{i\chi(x)}\Phi(x)$$
$$A(x) \mapsto A(x) + \mathrm{d}\chi(x).$$

Then the covariant derivative of Φ transforms as

$$\begin{aligned}(\mathrm{d} - iA)\Phi &\mapsto (\mathrm{d} - i(A + \mathrm{d}\chi))(\Phi e^{i\chi})\\ &= (\mathrm{d}\Phi + i\Phi \mathrm{d}\chi - i(A + \mathrm{d}\chi)\Phi)e^{i\chi}\\ &= e^{i\chi}(\mathrm{d} - iA)\Phi.\end{aligned}$$

Similarly, the magnetic field is also invariant under gauge transformations.

As a consequence, we can write down energy functionals that are invariant under these gauge transformations. In particular, we have

$$(\mathrm{D}\Phi, \mathrm{D}\Phi) \mapsto (e^{i\chi}\mathrm{D}\Phi, e^{i\chi}\mathrm{D}\Phi) = (\mathrm{D}\Phi, \mathrm{D}\Phi).$$

So we can now write down the *gauged Ginzburg–Landau energy*

$$V_\lambda(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(B^2 + |\mathrm{D}\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2\right)\, \mathrm{d}^2x.$$

This is then manifestly gauge invariant.

As before, the field equations are the Euler–Lagrange equations. Varying Φ, we obtain

$$-(\mathrm{D}_1^2 + \mathrm{D}_2^2)\Phi - \frac{\lambda}{2}(1 - |\Phi|^2)\Phi = 0.$$

This is just like the previous vortex equation in the ungauged case, but since we have the covariant derivative, this is a coupled equation.

The other equations, obtained by varying A, are

$$\begin{aligned}\partial_2 B &= (i\Phi, \mathrm{D}_1\Phi)\\ -\partial_1 B &= (i\Phi, \mathrm{D}_2\Phi).\end{aligned}$$

These are similar to Maxwell's equation, saying the curl of the magnetic field is given by the current.

It is again an exercise to derive these. It is helpful to use the previous identities such as

$$\partial_j(\Phi, \Psi) = (\mathrm{D}_j\Phi, \Psi) + (\Phi, \mathrm{D}_j\Psi).$$

So we get the integration by parts formula

$$\int_{\mathbb{R}^2} (\mathrm{D}_j\Phi, \Psi)\, \mathrm{d}^2x = -\int_{\mathbb{R}^2} (\Phi, \mathrm{D}_j\Psi)\, \mathrm{d}^2x$$

under suitable boundary conditions.

Recall that in the ungauged Ginzburg–Landau theory, we proved that we always had $|\Phi| \leq 1$. This is again true. The same proof works, with ∇ replaced with D. (It is literally the same proof. Just copy-and-paste.)

Lemma. At any maximum point of $|\Phi|$, we have $|\Phi| \leq 1$.

Proof. Consider the function

$$W(x) = 1 - |\Phi(x)|^2.$$

Then we want to show that $W \geq 0$ when W is minimized. We note that if W is at a minimum, then the Hessian matrix must have non-negative eigenvalues. So,

taking the trace, we must have $\Delta W \geq 0$. Now we can compute ΔW directly. We have

$$\begin{aligned}\partial_j W &= -2(\Phi, \mathrm{D}_j\Phi)\\ \Delta W &= \partial_j^2 W\\ &= -2(\Phi, \mathrm{D}_j^2\Phi) - 2(\mathrm{D}_j\Phi, \mathrm{D}_j\Phi)\\ &= \lambda|\Phi|^2 W - 2|\mathrm{D}\Phi|^2.\end{aligned}$$

Rearranged, this says

$$2|\mathrm{D}\Phi|^2 + \Delta W = \lambda|\Phi|^2 W.$$

But we know $2|\mathrm{D}\Phi|^2 \geq 0$, and $\Delta W \geq 0$. So we must have $W \geq 0$. □

As before, this suggests we interpret $|\Phi|$ as an order parameter. This model was first used to describe superconductors. The matter can either be in a "normal" phase or a superconducting phase, and $|\Phi|$ measures how much we are in the superconducting phase.

Thus, in our model, far away from the vortices, we have $|\Phi| \approx 1$, and so we are mostly in the superconducting phase. The vortices represent a breakdown of the superconductivity. At the core of the vortices, we have $|\Phi| = 0$, and we are left with completely normal matter. Usually, this happens when we have a strong magnetic field. In general, a magnetic field cannot penetrate the superconductor, but if it is strong enough, it will cause such breakdown in the superconductivity.

Similar to the ungauged case, for $\lambda > 0$, there exists basic vortex solutions of the form

$$\begin{aligned}\Phi &= f_N(r)e^{iN\theta}\\ A &= N\alpha_N(r)\,\mathrm{d}\theta.\end{aligned}$$

The boundary conditions are $f_N, \alpha_N \to 1$ as $r \to \infty$ and $f_N, \alpha_N \to 0$ as $r \to 0$.

Let's say a few words about why these are sensible boundary conditions from the point of view of energy. We want

$$\lambda \int_{\mathbb{R}^2} (1 - |\Phi|)^2 < \infty,$$

and this is possible only for $f_N \to 1$. What is less obvious is that we also need $\alpha_N \to 1$. We note that we have

$$\mathrm{D}_\theta\Phi = \frac{\partial\Phi}{\partial\theta} - iA_\theta\Phi = (iNf_N - iN\alpha_N f_N)e^{iN\theta}.$$

We want this to approach 0 as $r \to \infty$. Since $f_N \to 1$, we also need $\alpha_N \to 1$.

The boundary conditions at 0 can be argued as before, so that the functions are regular at 0.

Let's calculate the topological charge. We have

$$\begin{aligned} Q &= \frac{1}{\pi}\int_{\mathbb{R}^2} j^0(\Phi)\,\mathrm{d}^2x \\ &= \lim_{R\to\infty}\frac{1}{2\pi}\oint_{|x|=R}(i\Phi,\mathrm{d}\Phi) \\ &= \lim_{R\to\infty}\frac{1}{2\pi}\oint(if_Ne^{iN\theta}, iNf_Ne^{iN\theta})\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\cdot N\lim_{R\to\infty}\int_0^{2\pi} f_N^2\,\mathrm{d}\theta \\ &= N. \end{aligned}$$

Previously, we understood N as the "winding number", a measure of how "twisted" our field was. However, we shall see shortly that there is an alternative interpretation of this N. Previously, in the sine-Gordon theory, we thought of N as the number of kinks we've got. Similarly, here we can think of an N-vortex as a superposition of N vortices at the origin. In the case of $\lambda = 1$, we will see that there are static solutions involving multiple vortices placed at different points in space.

We can compute the magnetic field and total flux as well. It is convenient to use the $\mathrm{d}A$ definition, as we are not working in Cartesian coordinates. We have

$$\mathrm{d}A = N\alpha_N'(r)\,\mathrm{d}r\wedge\mathrm{d}\theta = \frac{1}{r}N\alpha_N'\,\mathrm{d}x^1\wedge\mathrm{d}x^2.$$

Thus it follows that

$$B = \partial_1A_2 - \partial_2A_1 = \frac{N}{r}\alpha_N'.$$

Now suppose that $|\Phi|^2\to 1$ as $r\to\infty$, and also that $\mathrm{D}_\theta\Phi = o(r^{-1})$. Then we find that

$$(\mathrm{d}\Phi)_\theta = iA_\theta\Phi + o(r^{-1}).$$

Then if we integrate around a circular contour, since only the angular part contributes, we obtain

$$\oint_{|x|=R}(i\Phi,\mathrm{d}\Phi) = \oint_{|x|=R}(i\Phi, iA_\theta\Phi)\,\mathrm{d}\theta + o(1) = \oint_{|x|=R}A + o(1).$$

Note that here we are explicitly viewing A as a differential form so that we can integrate it. We can then note that $|x| = R$ is the boundary of the disk $|x|\le R$. So we can apply Stokes' theorem and obtain

$$\oint_{|x|=R}(i\Phi,\mathrm{d}\Phi) = \int_{|x|\le R}\mathrm{d}A = \int_{|x|\le R}B\,\mathrm{d}x^1\wedge\mathrm{d}x^2,$$

where $\mathrm{d}x^1\wedge\mathrm{d}x^2$ is the usual area element d^2x Now we let $R\to\infty$. Then we get

$$\int_{\mathbb{R}^2}B\,\mathrm{d}x^1\wedge\mathrm{d}x^2 = \lim_{R\to\infty}\oint_{|x|=R}A = \lim_{R\to\infty}\oint(i\Phi,\mathrm{d}\Phi) = 2\pi Q.$$

Physically, what this tells us then is that there is a relation between the topological winding number and the magnetic flux. This is a common property of topological gauge theories. In mathematics, this is already well known — it is the fact that we can compute characteristic classes of vector bundles by integrating the curvature, as discovered by Chern.

Behaviour as $r \to 0$

We saw earlier that for reasons of regularity, it was necessary that $f_N, \alpha_N \to 0$ as $r \to 0$.

But actually, we must have $f_N \sim r^N$ as $r \to 0$ (up to a constant multiple). This has as a consequence that $\Phi \sim (re^{i\theta})^N = z^N$. So the local appearance of the vortex is as the zero of an analytic function with multiplicity N.

To see this, we need to compute that the Euler–Lagrange equations are

$$-f_N''(r) - \frac{1}{r} f_N'(r) + \frac{(N-\alpha_N)^2}{r^2} f_N = \frac{\lambda}{2} f_N (1 - f_N^2).$$

With the boundary condition that f_N and α_N vanish at $r = 0$, we can approximate this locally as

$$-f_N'' - \frac{1}{r} f_N' + \frac{N^2}{r^2} f_N = 0$$

since we have a $\frac{1}{r^2}$ on the left hand side. This equation is homogeneous, and the solutions are just

$$f_N = r^{\pm N}.$$

For regularity, we want the one that $\to 0$. So we take $f_N \sim r^N$.

2.4 Bogomolny/self-dual vortices and Taubes' theorem

As mentioned, we can think of our previous solution as a collection of N vortices all superposed at the origin. Is it possible to have separated vortices all over the plane? Naively, we would expect that the vortices exert forces on each other, and so we don't get a static solution. However, it turns out that in the $\lambda = 1$ case, we do get static, separated solutions.

This is not obvious, and the proof requires some serious analysis. We will not do the analysis, which requires use of Sobolev spaces and PDE theory. However, we will do all the non-hard-analysis part. In particular, we will obtain Bogomolny bounds as we did in the sine-Gordon case, and reduce the problem to finding solutions of a single scalar PDE, which we can let analysts tackle with their tools.

Recall that for the sine-Gordon kinks, we needed to solve

$$\theta'' = \sin\theta,$$

with boundary conditions $\theta(x) \to 0$ or 2π as $x \to \pm\infty$. The only solutions we found were

$$\theta_K(x - X)$$

for any $X \in \mathbb{R}$. This X is interpreted as the location of the kink. So the moduli space of solutions is $\mathcal{M} = \mathbb{R}$.

We shall get a similar but more interesting description for the $\lambda = 1$ vortices. This time, the moduli space will be $\mathbb{C}^N$, given by N complex parameters describing the vortex locations.

Theorem (Taubes' theorem)**.** For $\lambda = 1$, the space of (gauge equivalence classes of) solutions of the Euler–Lagrange equations $\delta V_1 = 0$ with winding number N is $\mathcal{M} \cong \mathbb{C}^N$.

To be precise, given $N \in \mathbb{N}$ and an unordered set of points $\{Z_1, \cdots, Z_N\}$, there exists a solution $A(x; Z_1, \cdots, Z_N)$ and $\Phi(x; Z_1, \cdots, Z_N)$ which solves the Euler-Lagrange equations $\delta V_1 = 0$, and also

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = 0, \quad B = \frac{1}{2}(1 - |\Phi|^2).$$

These are known as the *Bogomolny equations* for vortices. Moreover, Φ has exactly N zeroes $Z_1, \cdots, Z_N$ counted with multiplicity, where

$$\Phi(x; Z_1, \cdots, Z_N) \sim c_j(z - Z_j)^{n_j}$$

as $z \to Z_j$, where $n_j = |\{k : Z_k = Z_j\}|$ is the multiplicity.

This solution is unique up to gauge equivalence. Furthermore,

$$V_1(A(\,\cdot\,, Z_1, \cdots, Z_N), \Phi(\,\cdot\,; Z_1, \cdots, Z_N)) = \pi N \qquad (*)$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} B \, \mathrm{d}^2x = N = \text{winding number}.$$

Note that it is not immediately clear from our description that the moduli space is $\mathbb{C}^N$. It looks more like $\mathbb{C}^N$ quotiented out by the action of the permutation group S_N. However, we will see later that the resulting quotient is still isomorphic to $\mathbb{C}^N$.

There is a lot to be said about this theorem. The equation $(*)$ tells us the energy is just proportional to the number of particles (vortices), and does not depend on where they are. This means there is no force between the vortices. In situations like this, it is said that the Bogomolny bound is saturated. The following equation tells us topology is what is driving the vortices, as we have already seen.

Note that the Euler–Lagrange equations themselves are second-order equations. However, the Bogomolny equations are *first order*. In general, this is a mathematical signature suggesting that interesting structures are present.

We'll discuss three crucial ingredients in this theorem, but we will not complete the proof, because it is too much analysis. The proof can be found in the book by *Jaffe and Taubes*: Vortices and Monopoles (and in papers by Taubes (1980)). One result is that when $\lambda = 1$, there are no finite-energy solutions of the Euler-Lagrange equations except those satisfying the Bogomolny equations. The only vortex solutions are minima of the energy (in each topological sector), and there are no saddle-point solutions.

Holomorphic structure

When there are Bogomolny equations, there is often some complex analysis lurking behind. We can explicitly write the first Bogomolny equation as

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = \frac{\partial\Phi}{\partial x^1} + i\frac{\partial\Phi}{\partial x^2} - i(A_1 + iA_2)\Phi = 0.$$

Recall that in complex analysis, holomorphic functions are characterized by the (complex) Cauchy–Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x^1} + i\frac{\partial f}{\partial x^2}\right) = 0.$$

So we think of the first Bogomolny equation as the covariant Cauchy–Riemann equation. It is possible to convert this into the standard Cauchy–Riemann equation to deduce the local behaviour of Φ.

To do so, we write

$$\Phi = e^{\omega} f.$$

Then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= e^{-\omega}\left(\frac{\partial \Phi}{\partial \bar{z}} - \frac{\partial \omega}{\partial \bar{z}}\Phi\right) \\ &= e^{-\omega}\left(\frac{i(A_i + iA_2)}{2} - \frac{\partial \omega}{\partial \bar{z}}\right)\Phi.\end{aligned}$$

This is equal to 0 if ω satisfies

$$\frac{\partial \omega}{\partial \bar{z}} = i\frac{A_1 + iA_2}{2}.$$

So the question is — can we solve this? It turns out we can always solve this equation, and there is an explicit formula for the solution. In general, for the equation

$$\frac{\partial w}{\partial \bar{z}} = \beta,$$

the solution is given by

$$\omega(z) = \frac{1}{2\pi i}\int_{|w|<r} \frac{\beta(w)}{w - z}\, \mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

A proof can be found in a book on algebraic geometry, namely *Griffiths and Harris*. The proof of this result is on page 5, and in particular does not require learning any algebraic geometry!

So we can write

$$\Phi = e^{\omega} f$$

where f is holomorphic. Since e^{ω} is never zero, we can apply all our knowledge of holomorphic functions to f, and deduce that Φ has isolated zeroes $\{Z_j\}$, where $\Phi \sim (z - Z_j)^{n_j}$ for some integer power n_j.

The Bogomolny equations

We'll now show that (A, Φ) satisfies $V_1(A, \Phi) = \pi N$ iff it satisfies the Bogomolny equations, i.e.

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = 0, \quad B = \frac{1}{2}(1 - |\Phi|^2).$$

We first consider the simpler case of the sine-Gordon equation. As in the ϕ^4 theory, to find soliton solutions, we write the energy as

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} \left(\frac{1}{2}\theta_x^2 + (1-\cos\theta)\right)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\theta_x^2 + 4\sin^2\frac{\theta}{2}\right)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\left(\theta_x - 2\sin\frac{\theta}{2}\right)^2 + 4\theta_x \sin\frac{\theta}{2}\right)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\theta_x - 2\sin\frac{\theta}{2}\right)^2\,\mathrm{d}x + \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(-4\cos\frac{\theta}{2}\right)\,\mathrm{d}x\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\theta_x - 2\sin\frac{\theta}{2}\right)^2\,\mathrm{d}x + \left(-4\cos\frac{\theta(+\infty)}{2} + 4\cos\frac{\theta(-\infty)}{2}\right).
\end{aligned}$$

We then use the kink asymptotic boundary conditions to obtain, say, $\theta(+\infty) = 2\pi$ and $\theta(-\infty) = 0$. So the boundary terms gives 8. Thus, subject to these boundary conditions, we can write the sine-Gordon energy as

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \left(\theta_x - 2\sin\frac{\theta}{2}\right)^2\,\mathrm{d}x + 8.$$

Thus, if we try to minimize the energy, then we know the minimum is at least 8, and if we could solve the first-order equation $\theta_x = 2\sin\frac{\theta}{2}$, then the minimum would be exactly 8. The solution we found does satisfy this first-order equation. Moreover, the solutions are all of the form

$$\theta(x) = \theta_K(x - X), \quad \theta_K(x) = 4\arctan e^x.$$

Thus, we have shown that the minimum energy is 8, and the minimizers are all of this form, parametrized by $X \in \mathbb{R}$.

We want to the do something similar for the Ginzburg–Landau theory¿

Lemma. We have

$$V_1(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\left(B - \frac{1}{2}(1-|\Phi|^2)\right)^2 + 4|\bar\partial_A\Phi|^2\right)\,\mathrm{d}^2x + \pi N,$$

where

$$\bar\partial_A\Phi = \frac{1}{2}(\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi).$$

It is clear that the desired result follows from this.

Proof. We complete the square and obtain

$$V_1(A,\Phi) = \frac{1}{2}\int \left(\left(B - \frac{1}{2}(1-|\Phi|^2)\right)^2 + B(1-|\Phi|^2) + |\mathrm{D}_1\Phi|^2 + |\mathrm{D}_2\Phi|^2\right)\,\mathrm{d}^2x.$$

We now dissect the terms one by one. We first use the definition of $B\,\mathrm{d}x^1\wedge\mathrm{d}x^2 = \mathrm{d}A$ and integration by parts to obtain

$$\int_{\mathbb{R}^2}(1-|\Phi|^2)\,\mathrm{d}A = -\int_{\mathbb{R}^2}\mathrm{d}(1-|\Phi|^2)\wedge A = 2\int_{\mathbb{R}^2}(\Phi,\mathrm{D}\Phi)\wedge A.$$

Alternatively, we can explicitly write

$$\begin{aligned}\int_{\mathbb{R}^2} B(1-|\Phi|^2)\,\mathrm{d}^2x &= \int_{\mathbb{R}^2} (\partial_1 A_2 - \partial_2 A_1)(1-|\Phi|^2)\,\mathrm{d}^2x\\ &= \int_{\mathbb{R}^2} (A_2\partial_1|\Phi|^2 - A_1\partial_2|\Phi|^2)\,\mathrm{d}^2x\\ &= 2\int_{\mathbb{R}^2} (A_2(\Phi, \mathrm{D}_1\Phi) - A_1(\Phi, \mathrm{D}_2\Phi))\,\mathrm{d}^2x.\end{aligned}$$

Ultimately, we want to obtain something that looks like $|\bar{\partial}_A\Phi|^2$. We can write this out as

$$(\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi, \mathrm{D}_1\Phi + i\mathrm{D}_2\Phi) = |\mathrm{D}_1\Phi|^2 + |\mathrm{D}_2\Phi|^2 + 2(\mathrm{D}_1\Phi, i\mathrm{D}_2\Phi).$$

We note that $i\Phi$ and Φ are always orthogonal, and A_i is always a real coefficient. So we can write

$$\begin{aligned}(\mathrm{D}_1\Phi, i\mathrm{D}_2\Phi) &= (\partial_1\Phi - iA_1\Phi, i\partial_2\Phi + A_2\Phi)\\ &= (\partial_1\Phi, i\partial_2\Phi) + A_2(\Phi, \partial_1\Phi) - A_1(\Phi, \partial_2\Phi).\end{aligned}$$

We now use again the fact that $(\Phi, i\Phi) = 0$ to replace the usual derivatives with the covariant derivatives. So we have

$$(\mathrm{D}_1\Phi, i\mathrm{D}_2\Phi) = (\partial_1\Phi, i\partial_2\Phi) + A_2(\Phi, \mathrm{D}_1\Phi) - A_1(\Phi, \mathrm{D}_2\Phi).$$

This tells us we have

$$\int \left(B(1-|\Phi|^2) + |\mathrm{D}_1\Phi|^2 + |\mathrm{D}_2\Phi|^2\right)\,\mathrm{d}^2x = \int \left(4|\bar{\partial}_A\Phi|^2 + 2(\partial_1\Phi, i\partial_2\Phi)\right)\,\mathrm{d}^2x.$$

It then remains to show that $(\partial_1\Phi, i\partial_2\Phi) = j^0(\Phi)$. But we just write

$$\begin{aligned}(\partial_1\Phi_1 + i\partial_1\Phi_2, -\partial_2\Phi_2 + i\partial_2\Phi_1) &= -(\partial_1\Phi_1, \partial_2\Phi_2) + (\partial_1\Phi_2, \partial_2\Phi_1)\\ &= -j^0(\Phi)\\ &= -\det\begin{pmatrix}\partial_1\Phi_1 & \partial_2\Phi_1\\ \partial_1\Phi_2 & \partial_2\Phi_2\end{pmatrix}\end{aligned}$$

Then we are done. □

Corollary. For any (A, Φ) with winding number N, we always have $V_1(A, \Phi) \geq \pi N$, and those (A, Φ) that achieve this bound are exactly those that satisfy

$$\bar{\partial}_A\Phi = 0, \quad B = \frac{1}{2}(1-|\Phi|^2).$$

Reduction to scalar equation

The remaining part of Taubes' theorem is to prove the existence of solutions to these equations, and that they are classified by N unordered complex numbers. This is the main analytic content of the theorem.

To do so, we reduce the two Bogomolny equations into a scalar equation. Note that we have, from the first Bogomolny equation,

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = (\partial_1\Phi + i\partial_2\Phi) - i(A_1 + iA_2)\Phi = 0.$$

So we can write

$$A_1 + iA_2 = -i(\partial_1 + i\partial_2)\log\Phi.$$

Thus, once we've got Φ, we can get A_1 and A_2.

The next step is to use gauge invariance. Under gauge invariance, we can fix the phase of Φ to anything we want. We write (with θ real)

$$\Phi = e^{\frac{1}{2}(u+i\theta)}.$$

Then $|\Phi|^2 = e^u$.

We might think we can get rid of θ completely. However, this is not quite true, since the argument is not well-defined at a zero of Φ, and in general we cannot get rid of θ by a smooth gauge transformation. But since

$$\Phi \sim c_j(z - Z_j)^{n_j}$$

near Z_j, we can expect we can make θ look like

$$\theta = 2\sum_{j=1}^{N} \arg(z - Z_j).$$

We will assume we can indeed do so. Then we have

$$A_1 = \frac{1}{2}(\partial_2 u + \partial_1\theta), \quad A_2 = -\frac{1}{2}(\partial_1 u - \partial_2\theta).$$

We have now solved for A using the first Bogomolny equation. We then use this to work out B and obtain a scalar equation for u by the second Bogomolny equation.

Theorem. The Bogomolny equation $B = \frac{1}{2}(1 - |\Phi|^2)$ is equivalent to the scalar equation for u

$$-\Delta u + (e^u - 1) = -4\pi\sum_{j=1}^{N}\delta_{Z_j}.$$

This is known as *Taubes' equation*.

Proof. We have

$$\begin{aligned} B &= \partial_1 A_2 - \partial_2 A_1 \\ &= -\frac{1}{2}\partial_1^2 u - \frac{1}{2}\partial_2^2 u + \frac{1}{2}(\partial_1\partial_2 - \partial_2\partial_1)\theta \\ &= -\frac{1}{2}\Delta u + \frac{1}{2}(\partial_1\partial_2 - \partial_2\partial_1)\theta. \end{aligned}$$

We might think the second term vanishes identically, but that is not true. Our θ has some singularities, and so that expression is not going to vanish at the singularities. The precise statement is that $(\partial_1\partial_2 - \partial_2\partial_1)\theta$ is a distribution supported at the points Z_j.

To figure out what it is, we have to integrate:

$$\begin{aligned} \int_{|z-Z_j|\leq\varepsilon} (\partial_1\partial_2 - \partial_2\partial_1)\theta \,\mathrm{d}^2x &= \int_{|z-Z_j|=\varepsilon} \partial_1\theta \,\mathrm{d}x^1 + \partial_2\theta \,\mathrm{d}x^2 \\ &= \oint_{|z-Z_j|=\varepsilon} \mathrm{d}\theta = 4\pi n_j, \end{aligned}$$

where n_j is the multiplicity of the zero. Thus, we deduce that

$$(\partial_1\partial_2 - \partial_2\partial_1)\theta = 2\pi \sum \delta_{Z_j}.$$

But then we are done! □

We can think of this u as a non-linear combination of fundamental solutions to the Laplacian. Near the δ functions, the $e^u - 1$ term doesn't contribute much, and the solution looks like the familiar fundamental solutions to the Laplacian with logarithmic singularities. However, far away from the singularities, $e^u - 1$ forces u to tend to 0, instead of growing to infinity.

Taubes proved that this equation has a unique solution, which is smooth on $\mathbb{R}^2 \setminus \{Z_j\}$, with logarithmic singularities at Z_j, and such that $u \to 0$ as $|z| \to \infty$. Also, $u < 0$, so $|\Phi| < 1$.

It is an exercise to check that the Bogomolny equations imply the second-order Euler–Lagrange field equations.

For example, differentiating the second Bogomolny equation and using the first gives

$$\partial_1 B = -(\Phi, \mathrm{D}_1\Phi) = (\Phi, i\mathrm{D}_2\Phi).$$

2.5 Physics of vortices

Recall we began with the ungauged Ginzburg–Landau theory, and realized the solitons didn't have finite energy. We then added a gauge field, and the problem went away by itself. When we did this, we heuristically justified it by saying it "gave mass" to the angular component. However, there is another way of understanding it. And this doesn't require *any* detailed information about the theory at all.

Suppose we work in d space dimensions. Then a general scalar field $\Phi : \mathbb{R}^d \to \mathbb{R}^\ell$ has energy functional given by

$$E[\Phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\Phi|^2 + U(\Phi)\right) \mathrm{d}^d x.$$

for some U.

Theorem (Derrick's scaling argument)**.** Consider a field theory in d-dimensions with energy functional

$$E[\Phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\Phi|^2 + U(\Phi)\right) \mathrm{d}^d x = T + W,$$

with T the integral of the gradient term and W the integral of the term involving U.

- If $d = 1$, then any stationary point must satisfy
$$T = W.$$
- If $d = 2$, then all stationary points satisfy $W = 0$.
- If $d \geq 3$, then all stationary points have $T = W = 0$, i.e. Φ is constant.

This forbids the existence of solitons of finite energy in high dimensions for this type of energy functional.

Proof. Suppose Φ were such a stationary point. Then for any variation Φ_λ of Φ such that $\Phi = \Phi_1$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=1} E[\Phi_\lambda] = 0.$$

Consider the particular variation given by

$$\Phi_\lambda(\mathbf{x}) = \Phi(\lambda \mathbf{x}).$$

Then we have

$$W[\Phi_\lambda] = \int_{\mathbb{R}^d} U(\Phi_\lambda(\mathbf{x}))\, \mathrm{d}^d x = \lambda^{-d} \int_{\mathbb{R}^d} U(\Phi(\lambda\mathbf{x}))\, \mathrm{d}^d(\lambda x) = \lambda^{-d} W[\Phi].$$

On the other hand, since T contains two derivatives, scaling the derivatives as well gives us

$$T[\Phi_\lambda] = \lambda^{2-d} T[\Phi].$$

Thus, we find

$$E[\Phi_\lambda] = \lambda^{2-d} T[\Phi] + \lambda^{-d} W[\Phi].$$

Differentiating and setting $\lambda = 1$, we see that we need

$$(2-d)\, T[\Phi] - d\, W[\Phi] = 0.$$

Then the results in different dimensions follow. □

The $d = 2$ case is rather interesting. We can still get interesting soliton theories if we have sufficiently degenerate classical vacuum fields.

Example. In $d = 2$, we can take $\ell = 3$ and

$$W(\Phi) = (1 - |\Phi|^2)^2.$$

Then the set $W = 0$ is given by the unit sphere $S^2 \subseteq \mathbb{R}^3$. With Φ constrained to this 2-sphere, this is a *σ-model*, and there is a large class of such maps $\Phi(x)$ generated by rational functions.

Derrick's scaling argument is not only a mathematical trick. We can also interpret it physically. Increasing λ corresponds to "collapsing" down the field. Then we see that in $d \geq 3$, both the gradient and potential terms favour collapsing of the field. However, in $d = 1$, the gradient term wants the field to expand, and the potential term wants the field to collapse. If these two energies agree, then these forces perfectly balance, and stationary solitons exist.

Now let's consider the case of a gauge theory. To understand this, we need to know how we want the gauge field to transform. There as several ways of figuring this out.

For example, we can insist the covariant derivative $\mathrm{D}_j \Phi_\lambda$ to scale as a whole. Since

$$\partial_j \Phi_\lambda = \lambda (\partial_j \Phi)(\lambda x),$$

we would want A_j to scale as

$$(A_j)_\lambda(x) = \lambda A_j(\lambda x).$$

We can also do this more geometrically. Consider the function

$$\begin{aligned}\chi_\lambda : \mathbb{R}^d &\to \mathbb{R}^d\\ x &\mapsto \lambda x.\end{aligned}$$

Then our previous transformations were just given by pulling back along χ_λ. Since A is a 1-form, it pulls back as

$$\chi_\lambda^*(A_j\ \mathrm{d}x^j) = \lambda A_j(\lambda x)\ \mathrm{d}x^j.$$

Then since $B = \mathrm{d}A$, the curvature must scale as

$$B_\lambda(x) = \lambda^2 B(\lambda x).$$

Thus, we can obtain a gauged version of Derrick's scaling argument.

Since we don't want to develop gauge theory in higher dimensions, we will restrict our attention to the gauged Ginzburg–Landau model in two dimensions. Since we already used the letter λ, we will denote the scaling parameter by μ. We have

$$\begin{aligned}V_\lambda(A_\mu, \Phi_\mu) &= \frac{1}{2}\int \left(\mu^4 B(\mu x)^2 + \mu^2 |\mathrm{D}\Phi(\mu x)|^2 + \frac{\lambda}{4}(1 - |\Phi(\mu x)|^2)^2\right)\frac{1}{\mu^2}\ \mathrm{d}^2(\mu x)\\ &= \frac{1}{2}\int \left(\mu^2 B^2(y) + |\mathrm{D}\Phi(y)|^2 + \frac{\lambda}{4\mu^2}(1 - |\Phi(y)|^2)^2\right)\ \mathrm{d}^2 y.\end{aligned}$$

Again, the gradient term is scale invariant, but the magnetic field term counteracts the potential term. We can find the derivative to be

$$\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_{\mu=1} V_\lambda(A_\mu, \Phi_\mu) = \int \left(B^2 - \frac{\lambda}{4}(1 - |\Phi|^2)^2\right)\ \mathrm{d}^2 y.$$

Thus, for a vortex soliton, we must have

$$\int B^2\ \mathrm{d}^2 x = \frac{\lambda}{4}\int_{\mathbb{R}^2} (1 - |\Phi|^2)^2\ \mathrm{d}^2 x.$$

Such solutions exist, and we see that this is because they are stabilized by the magnetic field energy. Note that in the case $\lambda = 1$, this equation is just the integral form of one of the Bogomolny equations!

We will eventually want to work with theories in higher dimensions, and Derrick's scaling argument shows that this isn't going to be very successful in more than three dimensions, even if we introduce gauge fields. There are different ways to get around Derrick's argument. In the Skyrme model, which is what we are going to study in the next chapter, there are no gauge fields, but instead we will have some higher powers of derivative terms. In particular, by introducing fourth powers of derivatives, we will have a term that scales as λ^{4-d}, and this allows us to stay safe in three dimensions,

Scattering of vortices

Derrick's scaling argument suggests that vortices can exist if $\lambda > 0$. However, as we previously discussed, for $\lambda \neq 1$, there are forces between vortices in general, and we don't get static, separated vortices. By doing numerical simulations, we find that when $\lambda < 1$, the vortices attract. When $\lambda > 1$, the vortices repel. Thus, when $\lambda > 1$, the symmetric vortices with $N > 1$ are unstable, as they want to break up into multiple single vortices.

We can say a bit more about the $\lambda = 1$ case, where we have static multi-vortices. For example, for $N = 2$, the solutions are parametrized by pairs of points in $\mathbb{C}$, up to equivalence

$$(Z_1, Z_2) \sim (Z_2, Z_1).$$

We said the moduli space is $\mathbb{C}^2$, and this is indeed true. However, Z_1 and Z_2 are not good coordinates for this space. Instead, good coordinates on the moduli space $\mathcal{M} = \mathcal{M}_2$ are given by some functions symmetric in Z_1 and Z_2. One valid choice is

$$Q = Z_1 + Z_2, \quad P = Z_1 Z_2.$$

In general, for vortex number N, we should use the *elementary symmetric polynomials* in $Z_1, \cdots, Z_N$ as coordinates.

Now suppose we set two vortices in motion. For convenience, we fix the center of mass so that $Q(t) = 0$. We can then write P as

$$P = -Z_1^2.$$

We find that in a head-on collision, then after the vortices collide they scatter at $90°$. This is the $90°$ *scattering phenomenon*, and holds for other λ as well. In terms of our coordinates, if Z_1 is initially on the positive real axis and decreasing, then P smoothly evolves from a negative to a positive value, going through 0. This corresponds to $Z_1 \mapsto \pm i Z_1$, $Z_2 = -Z_1$. Note that at the point when they collide, we lose track of which vortex is which.

Similar to the ϕ^4 kinks, we can obtain effective Lagrangians for the dynamics of these vortices. However, this is more complicated.

2.6 Jackiw–Pi vortices

So far, we have been thinking about electromagnetism, using the abelian Higgs model. There is a different system that is useful in condensed matter physics. We look at vortices in Chern–Simons–Higgs theory. This has a different Lagrangian with no relativity any more. Thus, instead of having the curvature term (quadratic in the electromagnetic field), we have the Chern–Simons Lagrangian term. We again work in two space dimensions, with the Lagrangian density given by

$$\mathcal{L} = \frac{\kappa}{4}\varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - (i\Phi, \mathrm{D}_0\Phi) - \frac{1}{2}|\mathrm{D}\Phi|^2 + \frac{1}{2\kappa}|\Phi|^4,$$

where

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$$

is the electromagnetic field and, as before,

$$\mathrm{D}_0\Phi = \frac{\partial\Phi}{\partial t} - iA_0\Phi.$$

The first term is the *Chern–Simons term*, while the rest is the Schrödinger Lagrangian density with a $|\Phi|^4$ potential term. The first term is not obviously gauge invariant, but under a gauge transformation it transforms by a total derivative, so the field equations are gauge invariant.

Varying with respect to Φ, the Euler–Lagrange equation gives the (classical, nonlinear) Schrödinger equation

$$iD_0\Phi + \frac{1}{2}\mathrm{D}_j^2\Phi + \frac{1}{\kappa}|\Phi|^2\Phi = 0.$$

If we vary with respect to A_0, then we find

$$\kappa B + |\Phi|^2 = 0.$$

This is a bit funny, because it looks more like a constraint than an evolution equation. This is characteristic of Chern–Simons theories.

Varying with respect to the spatial components of A gives

$$\begin{aligned}\partial_1 A_0 &= \partial_0 A_1 + \frac{1}{\kappa}(i\Phi, \mathrm{D}_2\Phi)\\ \partial_2 A_0 &= \partial_0 A_2 - \frac{1}{\kappa}(i\Phi, \mathrm{D}_1\Phi).\end{aligned}$$

This is a bit odd, compared to Maxwell theory, because this is directly relating the current to the electromagnetic field, rather than to its derivative.

For static solutions, we need

$$\partial_1 A_0 = \frac{1}{\kappa}(i\Phi, \mathrm{D}_2\Phi).$$

To solve this, we assume that the field again satisfies the holomorphicity condition

$$\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi = 0.$$

Then we can write

$$\partial_1 A_0 = -\frac{1}{\kappa}(i\Phi, \mathrm{D}_2\Phi) = -\frac{1}{\kappa}\partial_1\frac{|\Phi|^2}{2}.$$

We can then integrate this to obtain

$$A_0 = -\frac{|\Phi|^2}{2\kappa}.$$

We can then look at the other two equations, and see how we can solve those. For static fields, the Schrödinger equation becomes

$$A_0\Phi + \frac{1}{2}\mathrm{D}_j^2\Phi + \frac{1}{\kappa}|\Phi|^2\Phi = 0.$$

Substituting in A_0, we obtain

$$\mathrm{D}_j^2\Phi = \frac{|\Phi|^2}{\kappa}$$

Let's then see if this makes sense. We need to see whether this can be consistent with the holomorphicity condition. The answer is yes, if we have the equation $\kappa B + |\Phi|^2 = 0$. We calculate

$$\mathrm{D}_j^2\Phi = \mathrm{D}_1^2\Phi + \mathrm{D}_2^2\Phi = -i(\mathrm{D}_1\mathrm{D}_2 - \mathrm{D}_2\mathrm{D}_1)\Phi = -B\Phi = \frac{1}{\kappa}|\Phi|^2\Phi,$$

exactly what we wanted.

So the conclusion is that we can generate vortex solutions by solving

$$\begin{aligned}\mathrm{D}_1\Phi + i\mathrm{D}_2\Phi &= 0\\ \kappa B + |\Phi|^2 &= 0.\end{aligned}$$

We can do exactly the same reduction as in Taubes' theorem, and find we need to solve

$$\Delta \log|\Phi| = \frac{1}{\kappa}|\Phi|^2.$$

Setting $\rho = |\Phi|^2$, this becomes Liouville's equation

$$\Delta \log \rho = \frac{2}{\kappa}\rho,$$

which can in fact be solved explicitly.

3 Skyrmions

We now move to one dimension higher, and study *Skyrmions*. In recent years, there has been a lot of interest in what many people call "Skyrmions", but what they are studying is a 2-dimensional variant of the original idea that occurs in certain exotic magnetic systems. Instead, we are going to study the original Skyrmions as discovered by Skyrme, which have applications to nuclear physics.

With details to be filled in soon, hadronic physics exhibits (approximate) spontaneously broken *chiral symmetry* $\frac{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}{\mathbb{Z}_2} \cong \mathrm{SO}(4)$, where the unbroken group is (diagonal) SO(3) *isospin*, and the elements of SO(3) are $(g, g) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$, with g and $-g$ identified.

This symmetry is captured in various effective field theories of pions (which are the approximate Goldstone bosons) and heavier mesons. It is also a feature of QCD with very light u and d quarks, with $\mathrm{SU}(2)_L$ acting on the doublet (u_L, d_L) of left-handed Dirac fields, and similarly for $\mathrm{SU}(2)_R$.

The special feature of the Skyrme model is that we describe nucleons as solitons in the effective field theory. Skyrme's original idea was that nucleons and bigger nuclei can be modelled by classical approximations to some "condensates" of pion fields. To explain the conservation of baryon number, the classical field equations have soliton solutions (Skyrmions) with an integer topological charge. This topological charge is then identified with what is known, physically, as the baryon number.

3.1 Skyrme field and its topology

Before we begin talking about the Skyrme field, we first discuss the symmetry group this theory enjoys. Before symmetry breaking, our theory has a symmetry group

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\{\pm(\mathbf{1}, \mathbf{1})\}} = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2}.$$

This might look like a rather odd symmetry group to work with. We can begin by understanding the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ part of the symmetry group. This group acts naturally on SU(2) again, by

$$(A, B) \cdot U = AUB^{-1}.$$

However, notice that the pair $(-\mathbf{1}, -\mathbf{1}) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts trivially. So the true symmetry group is the quotient by the subgroup generated by this element. One can check that after this quotienting, the action is faithful.

In the Skyrme model, the field will be valued in SU(2). It is convenient to introduce coordinates for our Skyrme field. As usual, we let $\boldsymbol{\tau}$ be the vector of three Pauli matrices, and $\mathbf{1}$ be the unit matrix. Then we can write the Skyrme field as

$$U(\mathbf{x}, t) = \sigma(\mathbf{x}, t)\mathbf{1} + i\boldsymbol{\pi}(x, t) \cdot \boldsymbol{\tau}.$$

However, the values of σ and $\boldsymbol{\pi}$ cannot be arbitrary. For U to actually lie in SU(2), we need the coefficients to satisfy

$$\sigma, \pi_i \in \mathbb{R}, \quad \sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1.$$

This is a *non-linear* constraint, and defines what known as a non-linear σ*-model*.

From this constraint, we see that geometrically SU(2) can be identified with S^3. We can also see this directly, by writing

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 \right\},$$

and this gives a natural embedding of SU(2) into $\mathbb{C}^2 \cong \mathbb{R}^4$ as the unit sphere, by sending the matrix to (α, β).

One can check that the action we wrote down acts by isometries of the standard, round S^3 metric. Thus, we obtain an inclusion

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2} \hookrightarrow \mathrm{SO}(4),$$

which happens to be a surjection.

Our theory undergoes spontaneous symmetry breaking, and the canonical choice of vacuum is $U = \mathbf{1}$. Equivalently, this is when $\sigma = 1$ and $\boldsymbol{\pi} = \mathbf{0}$. We see that the stabilizer of $\mathbf{1}$ is given by the diagonal

$$\Delta : \mathrm{SU}(2) \to \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2},$$

since $A\mathbf{1}B^{-1} = \mathbf{1}$ if and only if $A = B$.

Geometrically, if we view $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2} \cong \mathrm{SO}(4)$, then it is obvious that the stabilizer of $\mathbf{1}$ is the copy of $\mathrm{SO}(3) \subseteq \mathrm{SO}(4)$ that fixes the $\mathbf{1}$ axis in $\mathbb{R}^4$. Indeed, the image of the diagonal Δ is $\mathrm{SU}(2)/\{\pm\mathbf{1}\} \cong \mathrm{SO}(3)$.

Note that in our theory, for any choice of $\boldsymbol{\pi}$, there are at most two possible choices of σ. Thus, despite there being four variables, there are only three degrees of freedom. Geometrically, this is saying that $\mathrm{SU}(2) \cong S^3$ is a three-dimensional manifold.

This has some physical significance. We are using the $\boldsymbol{\pi}$ fields to model pions. We have detected and manipulated pions, into pion beams for example, and know they exist. However, as far as we can tell, there is no "σ meson", and this can be explained by the fact that σ isn't a genuine degree of freedom in our theory.

Let's now try to build a Lagrangian for our field U. We need to introduce derivative terms. From a mathematical point of view, the quantity $\partial_\mu U$ isn't a very nice thing to work with. It is a quantity that lives in the tangent space $T_U\mathrm{SU}(2)$, and in particular, the space it lives in depends on the value of U.

What we want to do is to pull this back to $T_{\mathbf{1}}\mathrm{SU}(2) = \mathfrak{su}(2)$, where $\mathfrak{su}(2)$ is the Lie algebra of traceless, antihermitian 2×2 matrices. To do so, we multiply by U^{-1}. We write

$$R_\mu = (\partial_\mu U)U^{-1},$$

which is known as the *right current*. For practical, computational purposes, it is convenient to note that

$$U^{-1} = \sigma\mathbf{1} - i\boldsymbol{\pi} \cdot \boldsymbol{\tau}.$$

Using the $(+,-,-,-)$ metric signature, we can now write the Skyrme Lagrangian density as

$$\mathcal{L} = -\frac{F_\pi^2}{16} \operatorname{Tr}(R_\mu R^\mu) + \frac{1}{32e^2} \operatorname{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) - \frac{1}{8} F_\pi^2 m_\pi^2 \operatorname{Tr}(\mathbf{1} - U).$$

The three terms are referred to as the σ-model term, Skyrme term and pion mass term respectively.

The first term is the obvious kinetic term. The second term might seem a bit mysterious, but we *must* have it (or some variant of it). By Derrick's scaling argument, we cannot have solitons if we just have the first and third terms. We need a higher multiple of the derivative term to make solitons feasible.

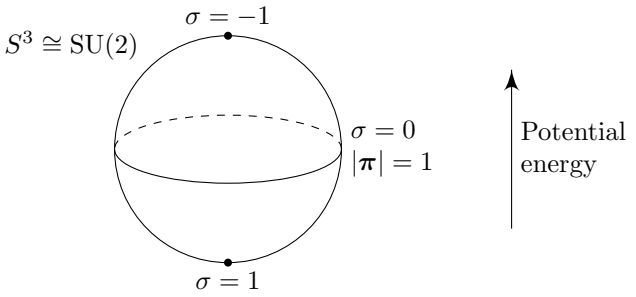
There are really only two possible terms with four derivatives. The alternative is to have the square of the first term. However, Skyrme rejected that object, because the Lagrangian will then have fourth powers of time derivatives. From a classical point of view, this is unpleasant, but is avoided in our theory, because the commutator vanishes when $\mu = \nu$.

Now note that the first two terms have an exact chiral symmetry, i.e. they are invariant under the SO(4) action previously described. In the absence of the final term, this symmetry would be spontaneously broken by a choice of vacuum. As described before, there is a conventional choice $U = \mathbf{1}$. After this spontaneous symmetry breaking, we are left with an isospin SU(2) symmetry. This isospin symmetry rotates the $\boldsymbol{\pi}$ fields among themselves.

The role of the extra term is that now the vacuum *has* to be the identity matrix. The symmetry is now *explicitly broken*. This might not be immediately obvious, but this is because the pion mass term is linear in σ and is minimized when $\sigma = 1$. If σ is eliminated, the pion mass term becomes quadratic in $\boldsymbol{\pi}$. Note that this theory is still invariant under the isospin SU(2) symmetry. Since the isospin symmetry is not broken, all pions have the same mass. In reality, the pion masses are very close, but not exactly equal, and we can attribute the difference in mass as due to electromagnetic effects. In terms of the $\boldsymbol{\pi}$ fields we defined, the physical pions are given by

$$\pi^{\pm} = \pi_1 \pm i\pi_2, \quad \pi^0 = \pi_3.$$

It is convenient to draw the target space SU(2) as



$\sigma = -1$ is the *anti-vacuum*. We will see that in the core of the Skyrmion, σ will take the value $\sigma = -1$.

In the Skyrme Lagrangian, we have three free parameters. This is rather few for an effective field theory, but we can reduce the number further by picking appropriate coefficients. We introduce an energy unit $\frac{F_\pi}{4e}$ and length unit $\frac{2}{eF_\pi}$. Setting these to 1, there is one parameter left, which is the dimensionless pion mass. In these units, we have

$$L = \int \left(-\frac{1}{2} \operatorname{Tr}(R_\mu R^\mu) + \frac{1}{16} \operatorname{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) - m^2 \operatorname{Tr}(\mathbf{1} - U) \right) \mathrm{d}^3 x.$$

In this notation,

$$m = \frac{2m_\pi}{eF_\pi}.$$

In general, we will think of m as being "small".

From here on we set $m = 0$. This has little effect on many Skyrmions, and simplifies the theory. The lack of a mass term means we no longer have the boundary condition that $U \to 1$ at infinity. Hence, we need to manually impose this condition.

Deriving the Euler–Lagrange equation is slightly messy, since we have to vary U while staying inside the group $\mathrm{SU}(2)$. Thus, we vary U multiplicatively,

$$U \mapsto U(\mathbf{1} + \varepsilon V)$$

for some $V \in \mathfrak{su}(2)$. We then have to figure out how R varies, do some differentiation, and then the Euler–Lagrange equation turns out to be

$$\partial_\mu \left(R^\mu + \frac{1}{4}[R^\nu, [R_\nu, R^\mu]] \right) = 0.$$

For static fields, the energy is given by

$$E = \int \left(-\frac{1}{2}\operatorname{Tr}(R_i R_i) - \frac{1}{16}\operatorname{Tr}([R_i, R_j][R_i, R_j]) \right) \mathrm{d}^3x \equiv E_2 + E_4.$$

where we sum i and j from 1 to 3. This is a sum of two terms — the first is quadratic in derivatives while the second is quartic.

Note that the trace functional on $\mathfrak{su}(2)$ is negative definite. So the energy is positive.

We can again run Derrick's theorem.

Theorem (Derrick's theorem)**.** We have $E_2 = E_4$ for any finite-energy static solution for $m = 0$ Skyrmions.

Proof. Suppose $U(\mathbf{x})$ minimizes $E = E_2 + E_4$. We rescale this solution, and define

$$\tilde{U}(\mathbf{x}) = U(\lambda \mathbf{x}).$$

Since U is a solution, the energy is stationary with respect to λ at $\lambda = 1$.

We can take the derivative of this and obtain

$$\partial_i \tilde{U}(\mathbf{x}) = \lambda \tilde{U}(\lambda \mathbf{x}).$$

Therefore we find

$$\tilde{R}_i(\mathbf{x}) = \lambda R_i(\lambda \mathbf{x}),$$

and therefore

$$\begin{aligned}
\tilde{E}_2 &= -\frac{1}{2}\int \operatorname{Tr}(\tilde{R}_i \tilde{R}_i)\, \mathrm{d}^3x \\
&= -\frac{1}{2}\lambda^2 \int \operatorname{Tr}(R_i R_i)(\lambda \mathbf{x})\, \mathrm{d}^3x \\
&= -\frac{1}{2}\frac{1}{\lambda}\int \operatorname{Tr}(R_i R_i)(\lambda \mathbf{x})\, \mathrm{d}^3(\lambda x) \\
&= \frac{1}{\lambda} E_2.
\end{aligned}$$

Similarly,

$$\tilde{E}_4 = \lambda E_4.$$

So we find that

$$\tilde{E} = \frac{1}{\lambda} E_2 + \lambda E_4.$$

But this function has to have a minimum at $\lambda = 1$. So the derivative with respect to λ must vanish at 1. Therefore

$$0 = \frac{\mathrm{d}\tilde{E}}{\mathrm{d}\lambda} = -\frac{1}{\lambda^2} E_2 + E_4 = 0.$$

So $E_4 = E_2$. □

We see that we must have a term quartic in derivatives in order to stabilize the Skyrmion. If we have the mass term as well, then the argument is slightly more complicated, and we get a more complicated relation between the energy contributions.

Baryon number

Recall that our field is a function $U : \mathbb{R}^3 \to \mathrm{SU}(2)$. Since we have a boundary condition $U = \mathbf{1}$ at infinity, we can imagine compactifying $\mathbb{R}^3$ into S^3, where the point at infinity is sent to $\mathbf{1}$.

On the other hand, we know that SU(2) is isomorphic to S^3. Geometrically, we think of the space and SU(2) as "different" S^3. We should think of SU(2) as the "unit sphere", and write it as S^3_1. However, we can think of the compactification of $\mathbb{R}^3$ as a sphere with "infinite radius", so we denote it as S^3_∞. Of course, topologically, they are the same.

The field is then a map

$$U : S^3_\infty \to S^3_1.$$

This map has a degree. There are many ways we can think about the degree. For example, we can think of it as the homotopy class of this map, which is an element of $\pi_3(S^3) \cong \mathbb{Z}$. Equivalently, we can think of it as the map induced on the homology or cohomology of S^3.

Although U evolves with time, the degree is a discrete quantity, so it has to be independent of time (alternatively, the degree of the map is homotopy invariant).

There is an explicit integral formula for the degree,

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \operatorname{Tr}(R_i R_j R_k)\, \mathrm{d}^3 x.$$

The factor of $2\pi^2$ comes from the volume of the 3-sphere, and there is also a factor of 6 coming from how we anti-symmetrize indices. We identify B with the conserved, physical baryon number.

If we were to derive this, then we would have to pull back a normalized volume form from S^3_1 and then integrate over all space. In this formula, we chose to use the SO(4)-invariant volume form, but in general, we can pull back any normalized volume form. Locally, near $\sigma = 1$, this volume element is actually

$$\frac{1}{2\pi^2}\, \mathrm{d}\pi^1 \wedge \mathrm{d}\pi^2 \wedge \mathrm{d}\pi^3.$$

Faddeev–Bogomolny bound

There is a nice result analogous to the Bogomolny energy bound we saw for kinks and vortices, known as the *Faddeev–Bogomolny bound*. We can write the static energy as

$$E = \int -\frac{1}{2}\operatorname{Tr}\left(\left(R_i \mp \frac{1}{4}\varepsilon_{ijk}[R_j, R_k]\right)^2\right) \mathrm{d}^3x \pm 12\pi^2 B.$$

This equality is true for both sign choices. However, to get the strongest result, we should choose the sign such that $\pm 12\pi^2 B > 0$. Then we find the energy lower bound

$$E \geq 12\pi^2|B|.$$

By symmetry, it suffices to consider the case $B > 0$, which is what we will do.

The Bogomolny equation for $B > 0$ should be

$$R_i - \frac{1}{4}\varepsilon_{ijk}[R_j, R_k] = 0.$$

However, it turns out this equation has *no* non-vacuum solution.

Roughly, the argument goes as follows — by careful inspection, for this to vanish, whenever R_i is non-zero, the three vectors R_1, R_2, R_3 must form an orthonormal frame in $\mathfrak{su}(2)$. So U must be an isometry. But this isn't possible, because the spheres have "different radii".

Therefore, true Skyrmions with $B > 0$ satisfy the strict inequality

$$E > 12\pi^2 B.$$

We get a useful lower bound, but the actual energy is always greater than this. It is quite interesting to look at the energies of true solutions numerically, and their energy is indeed somewhat bigger.

3.2 Skyrmion solutions

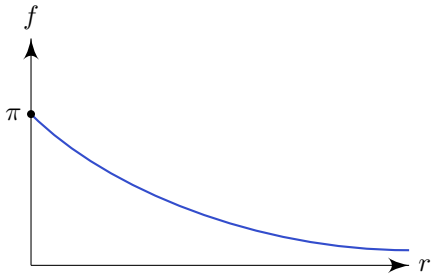
The simplest Skyrmion solution has baryon number $B = 1$. We will continue to set $m = 0$.

$B = 1$ hedgehog Skyrmion

Consider the spherically symmetric function

$$U(\mathbf{x}) = \cos f(r)\mathbf{1} + i \sin f(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}.$$

This is manifestly in SU(2), because $\cos^2 f + \sin^2 f = 1$. This is known as a hedgehog, because the unit pion field is $\hat{\mathbf{x}}$, which points radially outwards. We need some boundary conditions. We need $U \to \mathbf{1}$ at ∞. On the other hand, we will see that we need $U \to -\mathbf{1}$ at the origin to get baryon number 1. So $f \to \pi$ as $r \to 0$, and $f \to 0$ as $r \to \infty$. So f looks roughly like this:



After some hard work, we find that the energy is given by

$$E = 4\pi \int_0^\infty \left(f'^2 + \frac{2\sin^2 f}{r^2}(1 + f'^2) + \frac{\sin^4 f}{r^4} \right) r^2 \, \mathrm{d}r.$$

From this, we can obtain a second-order equation in f, which is not simple. Solutions have to be found numerically. This is a sad truth about Skyrmions. Even in the simplest $B = 1, m = 0$ case, we don't have an analytic expression for what f looks like. Numerically, the energy is

$$E = 1.232 \times 12\pi^2.$$

To compute the baryon number of this solution, we plug our solution into the integral formula, and obtain

$$B = -\frac{1}{2\pi^2} \int_0^\infty \frac{\sin^2 f}{r^2} \frac{\mathrm{d}f}{\mathrm{d}r} \cdot 4\pi r^2 \, \mathrm{d}r.$$

We can interpret $\frac{\mathrm{d}f}{\mathrm{d}r}$ as the radial contribution to B, while there are two factors of $\frac{\sin f}{r}$ coming from the angular contribution due to the $i \sin f(\mathbf{r}) \hat{\mathbf{x}} \cdot \boldsymbol{\tau}$ term.

But this integral is an exact differential. It simplifies to

$$B = \frac{1}{\pi} \int_0^\pi 2\sin^2 f \, \mathrm{d}f.$$

Note that we have reversed a sign, because of the change of limits. We can integrate this directly, and get

$$B = \frac{1}{\pi} \left(f - \frac{1}{2} \sin 2f \right)_0^\pi = 1,$$

as promised.

Intuitively, we see that in this integral, f goes from 0 to π, and we can think of the field wrapping around the sphere S^3 once.

More hedgehogs

We can consider a more general hedgehog with the same ansatz, but with the boundary conditions

$$f(0) = n\pi, \quad f(\infty) = 0.$$

In this case, the same computations gives us $B = n$. So in principle, this gives a Skyrmion of any baryon number. The solutions do exist. However, it turns out

they have extremely high energy, and are nowhere near minimizing the energy. In fact, the energy increases much faster than n itself, because the Skyrmion "onion" structure highly distorts each $B = 1$ Skyrmion. Unsurprisingly, these solutions are unstable.

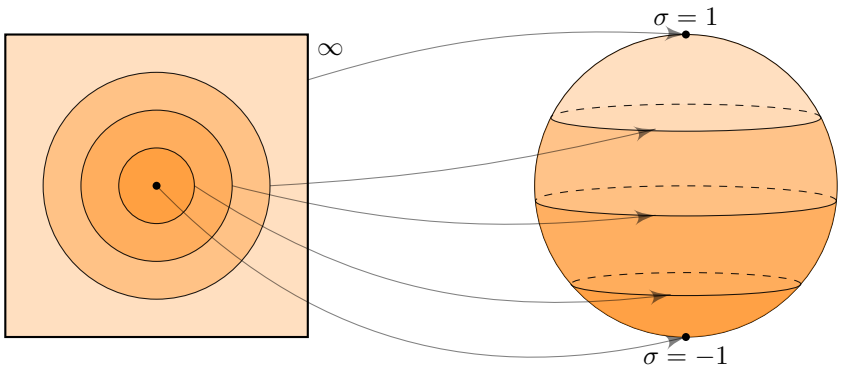
This is not what we want in hadronic physics, where we expect the energy to scale approximately linearly with n. In fact, since baryons attract, we expect the solution for $B = n$ to have less energy than n times the $B = 1$ energy.

We can easily get energies approximately n times the $B = 1$ energy simply by having very separated $B = 1$ Skyrmions, and since they attract, when they move towards each other, we get even lower energies.

A better strategy — rational map approximation

So far, we have been looking at solutions that depend very simply on angle. Most of the "winding" happens in the radial direction. In fact, it is a better idea to wind more in the *angular* direction.

In the case of the $B = 1$ hedgehog, the field looks roughly like this:



In the spherically symmetric hedgehogs for $B > 1$, we wrapped radially around the target sphere many times, and that was not a good idea.

Better is to introduce some angular twists. We can think of the above $B = 1$ solution as follows — we slice up our domain $\mathbb{R}^3$ (or rather, S^3 since we include the point at infinity) into 2-spheres of constant radius, say

$$S^3 = \bigcup_{r\in[0,\infty]} S^2_r.$$

We also slice up the S^3 target into constant σ levels, which are 2-spheres:

$$S^3 = \bigcup_{\sigma} S^2_\sigma.$$

Then the function $f(r)$ maps the 2-sphere S^2_r into the 2-sphere $S^2_{\cos f(r)}$, and we earlier chose the identity map from S^2_r to $S^2_{\cos f(r)}$. This gave a spherically symmetric hedgehog solution.

But we don't have to do this! Pick any function $R : S^2 \to S^2$. Then we can use this as the map that sends S^2_r to $S^2_{\cos f(r)}$. For simplicity, we will use the

same R for all r. If we do this, then we obtain a non-trivial map $U : S^3 \to S^3$, i.e. a Skyrme field.

Since R is a map from a sphere to a sphere (but one dimension lower), R has a degree. It turns out this degree is the same as the degree of the induced map U! So to produce higher baryon number hedgehogs, we simply have to find maps $R : S^2 \to S^2$ of higher degree.

Fortunately, this is easier than for maps between 3-spheres, because a 2-sphere is a Riemann surface. We can use complex coordinates to work on 2-spheres. By complex analysis, any complex, globally holomorphic map between 2-spheres is given by a *rational map*, a ratio of polynomials.

Pick any rational function $R_k(z)$ of degree k. This is a map $S^2 \to S^2$. We use coordinates r, z, where $r \in \mathbb{R}^+$ and $z \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \cong S^2$. Then we can consider generalized hedgehogs

$$U(\mathbf{x}) = \cos f(r)\mathbf{1} + i \sin f(r)\hat{\mathbf{n}}_{R_k(z)} \cdot \boldsymbol{\tau},$$

with $f(0) = \pi$, $f(\infty) = 0$, and $\hat{\mathbf{n}}_{R_k}$ is the normalized pion field $\hat{\boldsymbol{\pi}}$, given by the unit vector obtained from $R_k(z)$ when we view S^2 as a subset of $\mathbb{R}^3$ in the usual way. This formula is called the *rational map ansatz*. Explicitly,

$$\hat{\mathbf{n}}_R = \frac{1}{1 + |R|^2}(\bar{R} + R, i(\bar{R} - R), 1 - |R|^2).$$

This ansatz is in some sense a separation of variables, where we separate the radial and angular dependence of the field. The baryon number is $B = k$.

Note that even if we find a minimum among this class of fields, it is not a true minimal energy Skyrmion. However, it gets quite close, and is much better than our previous attempt.

There is quite a lot of freedom in this construction, since we are free to pick $f(r)$, as well as the rational function $R_k(z)$. The topological degree k of $R_k(z)$ we care about is the same as the algebraic degree of $R_k(z)$. Precisely, if we write

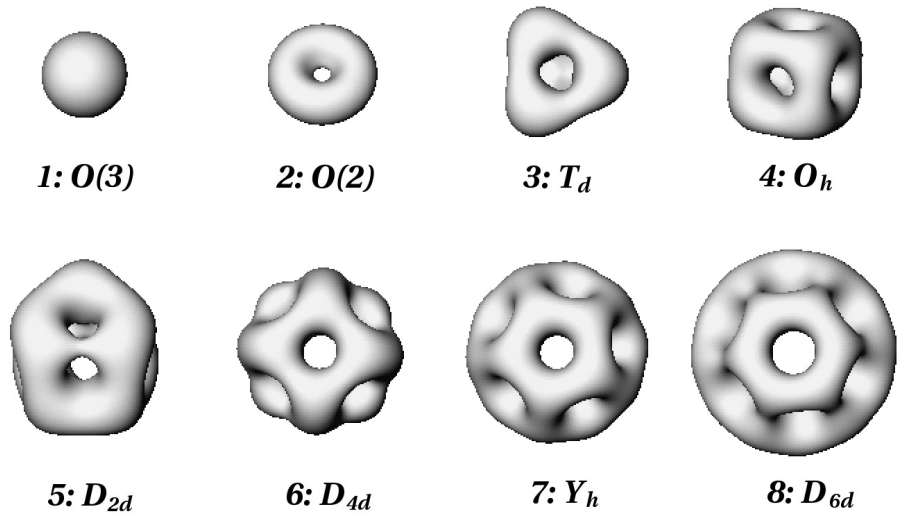
$$R_k(z) = \frac{p_k(z)}{q_k(z)},$$

where p_k and q_k are coprime, then the algebraic degree of R_k is the maximum of the degrees of p_k and q_k as polynomials. Since there are finitely many coefficients for p_k and q_k, this is a finite-dimensional problem, which is much easier than solving for arbitrary functions. We will talk more about the degree later.

Numerically, we find that minimal energy fields for $k \leq 4$ are obtained with

$$R_1(z) = z \qquad R_2(z) = z^2$$
$$R_3(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz} \qquad R_4(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}.$$

The true minimal energy Skyrmions have also been found numerically, and are very similar to the optimal rational map ansatz fields. In fact, the search for the true minima often starts from a rational map ansatz.



Constant energy density surfaces of Skyrmions up to baryon number $B = 8$ (for $m = 0$), by R. A. Battye and P. M. Sutcliffe

We observe that

- for $B = 1$, we recover the hedgehog solution.
- for $B = 2$, our solution has an axial symmetry.
- for $B = 3$, the solution might seem rather strange, but has tetrahedral symmetry.
- for $B = 4$, the solution has cubic symmetry.

In each case, these are the unique rational maps with such high symmetry. The function f can be found numerically, and depends on B. Even though the rational map ansatz fields are not the exact Skyrmion solutions, the exact solutions enjoy the same symmetries.

Geometrically, what we are doing is that we are viewing S^3 as the *suspension* ΣS^2, and our construction of U from R is just the suspension of maps. The fact that degree is preserved by suspension is an example of the fact that homology is *stable*.

More on rational maps

Why does the algebraic degree of R_k agree with the topological degree? One way of characterizing the topological degree is by counting pre-images. Consider a generic point c in the target 2-sphere, and consider the equation

$$R_k(z) = \frac{p_k(z)}{q_k(z)} = c.$$

We can rearrange this to say

$$p_k - cq_k = 0.$$

For a *generic* c, the z^k terms do not cancel, so this is a polynomial equation of degree k. Also, generically, this equation doesn't have repeated roots, so has exactly k solutions. So the number of points in the pre-image of a generic c is k.

Because R_k is a holomorphic map, it is automatically orientation preserving (and in fact conformal). So each of these k points contributes $+1$ to the degree.

In the pictures above, we saw that the Skyrmions have some "hollow polyhedral" structures. How can we understand these?

The holes turn out to be zeroes of the baryon density, and are where the energy density is small. At the center of the holes, the angular derivatives of the Skyrme field U is zero, but the radial derivative is not.

We can find these holes precisely in the rational map approximation. This allows us to find the symmetry of the system. They occur where the derivative $\frac{\mathrm{d}R_k}{\mathrm{d}z} = 0$. Since $R = \frac{p}{q}$, we can rewrite this requirement as

$$W(z) = p'(z)q(z) - q'(z)p(z) = 0.$$

W is known as the *Wronskian*.

A quick algebraic manipulation shows that W has degree at most $2k - 2$, and generically, it is indeed $2k - 2 (= 2B - 2)$. This is the number of holes in the Skyrmion.

We can look at our examples and the pictures to check this number.

Example. For

$$R_4(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1},$$

the Wronskian is

$$W(z) = (4z^3 + 4\sqrt{3}iz)(z^4 - 2\sqrt{3}iz^2 + 1) - (4z^3 - 4\sqrt{3}iz)(z^4 + 2\sqrt{3}iz^2 + 1).$$

The highest degree z^7 terms cancel. But surprisingly there isn't any z^6 term either. Thus W turns out to be the degree 5 polynomial

$$W(z) = -8\sqrt{3}i(z^5 - z).$$

We can easily list the roots — they are $z = 0, 1, i, -1, -i$.

Generically, we expect there to be 6 roots. It turns out the Wronksian has a zero at ∞ as well. To see this more rigorously, we can rotate the Riemann sphere a bit by a Möbius map, and then see there are 6 finite roots. Looking back at our previous picture, there are indeed 6 holes when $B = 4$.

Although the rational map ansatz is a good way to find approximate and then exact solutions for B up to ≈ 20, they are hollow with $U = -1$ at the center. This is not a good model for larger nuclei, especially when we introduce non-zero pion mass.

For $m \approx 1$, there are better, less hollow Skyrmions when $B \geq 8$.

3.3 Other approaches to Skyrmions

There are other ways to get Skyrmion solutions.

Product Ansatz

Suppose $U_1(\mathbf{x})$ and $U_2(\mathbf{x})$ are Skyrmions with baryon numbers B_1 and B_2. Since the target space is a group SU(2), we can take the product

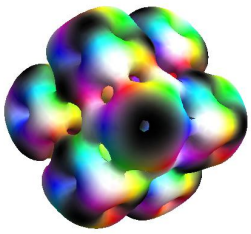
$$U(\mathbf{x}) = U_1(\mathbf{x})U_2(\mathbf{x}).$$

Then the baryon number is $B = B_1 + B_2$. To see this, we can consider the product when U_1 and U_2 are well-separated, i.e. consider $U(\mathbf{x} - \mathbf{a})U_2(\mathbf{x})$ with $|\mathbf{a}|$ large. Here, the baryon number can be calculated easily. We can then vary $\mathbf{a}$ continuously to 0, and B doesn't change as we make this deformation. So we are done. Alternatively, this follows from an Eckmann–Hilton argument.

This can help us find Skyrmions with baryon number B starting with B well-separated $B = 1$ hedgehogs. Of course, this will not be energy-minimizing, but we can numerically improve the field by letting the separation vary.

It turns out this is not a good way to find Skyrmions. In general, it doesn't give good approximations to the Skyrmion solutions. They tend to lack the desired symmetry, and this boils down to the problem that the product is not commutative, i.e. $U_1U_2 \neq U_2U_1$. Thus, we cannot expect to be able to approximate symmetric things with a product ansatz.

The product ansatz can also be used for several $B = 4$ subunits to construct configurations with baryon number $4n$ for $n \in \mathbb{Z}$. For example, the following is a $B = 31$ Skyrmion:

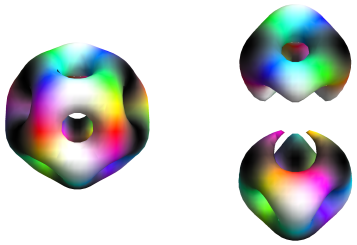


B = 31 Skyrmion by P. H. C. Lau and N. S. Manton

This is obtained by putting eight $B = 4$ Skyrmions side by side, and then cutting off a corner.

This strategy tends to work quite well. With this idea, we can in fact find Skyrmion solutions with baryon number infinity! We can form an infinite cubic crystal out of $B = 4$ subunits. For $m = 0$, the energy per baryon is $\approx 1.038 \times 12\pi^2$. This is a very close to the lower bound!

We can also do other interesting things. In the picture below, on the left, we have a usual $B = 7$ Skyrmion. On the right, we have deformed the Skyrmion into what looks like a $B = 4$ Skyrmion and a $B = 3$ Skyrmion. This is a cluster structure, and it turns out this deformation doesn't cost a lot of energy. This two-cluster system can be used as a model of the lithium-7 nucleus.



B = 7 Skyrmions by C. J. Halcrow

3.4 Asymptotic field and forces for $B = 1$ hedgehogs

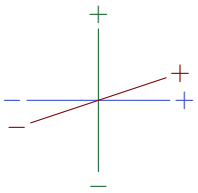
We now consider what happens when we put different $B = 1$ hedgehogs next to each other. To understand this, we look at the profile function f, for $m = 0$. For large r, this has the asymptotic form

$$f(r) \sim \frac{C}{r^2}.$$

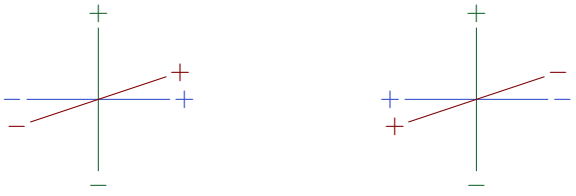
To obtain this, we linearize the differential equation for f and see how it behaves as $r \to \infty$ and $f \to 0$. The linearized equation doesn't determine the coefficient C, but the full equation and boundary condition at $r = 0$ does. This has to be worked out numerically, and we find that $C \approx 2.16$.

Thus, as $\sigma \sim 1$, we find $\boldsymbol{\pi} \sim C\frac{\mathbf{x}}{r^3}$. So the $B = 1$ hedgehog asymptotically looks like three pion dipoles. Each pion field itself has an axis, but because we have three of them, the whole solution is spherically symmetric.

We can roughly sketch the Skyrmion as

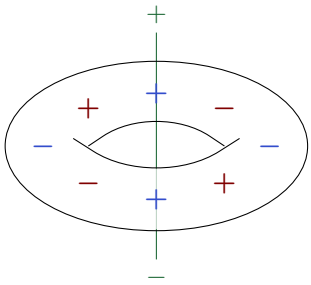


Note that unlike in electromagnetism, scalar dipoles attract if they are facing each other and oppositely oriented. This is because the fields have low gradient. So the lowest energy arrangement of two $B = 1$ Skyrmions while they are separated is



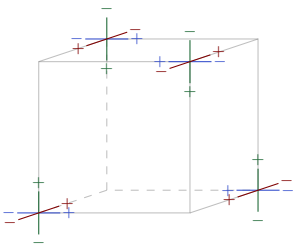
Compared to the left-hand Skyrmion, the right-hand Skyrmion is rotated by 180° about a line (the green axis) perpendicular to the line separating the Skyrmions.

These two Skyrmions attract! So two Skyrmions in this "attractive channel" can merge to form the $B = 2$ torus, which is the true minimal energy solution.



The blue and the red fields have no net dipole, even before they merge. There is only a quadrupole. However, the field has a strong net green dipole. The whole field has toroidal symmetry, and these symmetries are important if we want to think about quantum states and the possible spin this Skyrmion could have.

For $B = 4$ fields, we can begin with the arrangement



To obtain the orientations, we begin with the bottom-left, and then obtain the others by rotating by 180° about the axes perpendicular to the faces of the cube shown here.

This configuration only has a tetrahedral structure. However, the Skyrmions can merge to form a cubic $B = 4$ Skyrmion. All dipole moments cancel.

3.5 Fermionic quantization of the $B = 1$ hedgehog

We begin by quantizing the $B = 1$ Skyrmion. The naive way to do this is to view the Skyrmion as a rigid body, and quantize it. However, if we do this, and assume the wavefunction is unchanged after a 2π rotation, then we will find that the Skyrmion must have integer spin. However, this is not what we want, since we want Skyrmions to model protons and nucleons, which have half-integer spin. In other words, the naive quantization makes the Skyrmion bosonic.

In general, if we want to take the Skyrme model as a low energy effective field theory of QCD with an odd number of colours, then we must require the $B = 1$ Skyrmion to be in a fermionic quantum state, with half-integer spin.

As a field theory, the configuration space for a baryon number B Skyrme field is $\text{Maps}_B(\mathbb{R}^3 \to \text{SU}(2))$, with appropriate boundary conditions. These are all topologically the same as $\text{Maps}_0(\mathbb{R}^3 \to \text{SU}(2))$, because if we fix a single element $U_0 \in \text{Maps}_B(\mathbb{R}^3 \to \text{SU}(2))$, then multiplication by U_0 gives us a homeomorphism between the two spaces. Since we imposed the vacuum boundary condition, this space is also the same as $\text{Maps}_0(S^3 \to S^3)$.

This space is *not* simply connected. In fact, it has a first homotopy group

$$\pi_1(\mathrm{Maps}_0(S^3 \to S^3)) = \pi_1(\Omega^3 S^3) = \pi_4(S^3) = \mathbb{Z}_2.$$

Thus, $\mathrm{Maps}_0(S^3 \to S^3)$ has a universal double cover. In our theory, the wavefunctions on $\mathrm{Maps}(S^3 \to S^3)$ should not be single-valued, but they are well-defined functions on the double cover. The wavefunction Ψ changes sign after going around a non-contractible loop in the configuration space. It can be shown that this is not just a choice, but required in a low-energy version of QCD.

This has some basic consequences:

(i) If we rotate a 1-Skyrmion by 2π, then Ψ changes sign.

(ii) Ψ also changes sign when one exchanges two 1-Skyrmions (without rotating them in the process). This was shown by Finkelstein and Rubinstein.

This links spin with statistics. If we quantized Skyrmions as bosons, then both (i) and (ii) do not happen. Thus, in this theory, we obtain the spin-statistics theorem *from topology*.

(iii) In general, if B is odd, then a rigid rotation by 2π is a non-contractible loop, while if B is even, it is contractible. Thus, spin is half-integer if B is odd, and integer if B is even.

(iv) There is another feature of the Skyrme model. So far, our rotations are spatial rotations. We can also rotate the value of the pion field, i.e. rotate the target 3-sphere. This is an *isospin rotation*. This behaves similarly to above. Thus, isospin is half-integer if B is odd, and integer if B is even.

3.6 Rigid body quantization for higher B

We now make the *rigid body approximation*. We allow a Skyrmion to translate, rotate and isorotate rigidly, but do not allow it to deform. This is a low-energy approximation, and we have reduced the infinite dimensional space of configurations to a finite-dimensional space. The group acting is

$$(\text{translations}) \times (\text{rotations}) \times (\text{isorotations}).$$

The translation part is trivial. The Skyrmion just gets the ability to move, and thus gains momentum. So we are going to ignore it. The remaining group acting is $\mathrm{SO}(3) \times \mathrm{SO}(3)$. This is a bit subtle. We have $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, so we would expect $\pi_1(\mathrm{SO}(3) \times \mathrm{SO}(3)) = (\mathbb{Z}_2)^2$. However, in the full theory, we only have a single $\mathbb{Z}_2$. So we need to identify a loop in the first $\mathrm{SO}(3)$ with a loop in the second $\mathrm{SO}(3)$. Our wavefunction is thus a function on the cover of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ that is $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$.

While $\mathrm{SO}(3) \times \mathrm{SO}(3)$ is the symmetry group of the full theory, for a particular classical Skyrmion solution the orbit is smaller than the whole group. This is because the Skyrmion is often invariant under some subgroup of $\mathrm{SO}(3) \times \mathrm{SO}(3)$. For example, the $B = 3$ solution has a tetrahedral symmetry. We then require the wavefunction to be invariant under this group up to a sign.

For a single $\mathrm{SO}(3)$, we have a rigid-body wavefunction $|J\ L_3\ J_3\rangle$, where J is the spin, L_3 is the third component of spin relative to body axes, and J_3 is

the third component of spin relative to space axes. Since we have two copies of SO(3), the wavefunction can be represented by

$$|J\ L_3\ J_3\rangle \otimes |I\ K_3\ I_3\rangle\,,$$

or a linear combination of these states. I and I_3 is the isospin and its projection we are physically familiar with. The values of J_3 and I_3 are not constrained, i.e. they take all the standard $(2J+1)(2I+1)$ values. Thus, we are going to suppress these labels.

However, the symmetry of the Skyrmion places constraints on the body-fixed projections, and not all values of J and I are allowed.

Example. For the $B = 1$ hedgehog, there is a lot of symmetry. Given an axis $\hat{\mathbf{n}}$ and an angle α, we know that classically, if we rotate and isorotate by the same axis and angle, then the wavefunction is unchanged. So we must have

$$e^{i\alpha\hat{\mathbf{n}}\cdot\mathbf{L}}e^{i\alpha\hat{\mathbf{n}}\cdot\mathbf{K}}\,|\Psi\rangle = |\Psi\rangle\,,$$

where $\mathbf{L}$ and $\mathbf{K}$ are the body-fixed spin and isospin operators. It follows, by considering α small, and all $\hat{\mathbf{n}}$, that

$$(\mathbf{L}+\mathbf{K})\,|\Psi\rangle = 0.$$

So the "grand spin" $\mathbf{L}+\mathbf{K}$ must vanish. So $\mathbf{L}\cdot\mathbf{L} = \mathbf{K}\cdot\mathbf{K}$. Recall that $\mathbf{L}\cdot\mathbf{L} = \mathbf{J}\cdot\mathbf{J}$ and $\mathbf{K}\cdot\mathbf{K} = \mathbf{I}\cdot\mathbf{I}$, where $\mathbf{J}$ and $\mathbf{I}$ are the space-fixed spin and isospin operators. It follows that we must have $J = I$.

Since 1 is an odd number, for any axis $\hat{\mathbf{n}}$, we must also have

$$e^{i2\pi\hat{\mathbf{n}}\cdot\mathbf{L}}\,|\Psi\rangle = -\,|\Psi\rangle\,.$$

So I and J must be half-integer.

Thus, the allowed states are

$$J = I = \frac{n}{2}$$

for some odd n. If we work out the formula for the energy in terms of the spin, we see that it increases with J. It turns out the system is highly unstable if $J \geq \frac{5}{2}$, so $\frac{1}{2}$ and $\frac{3}{2}$ are the only physically allowed values.

The $J = I = \frac{1}{2}$ states corresponds to p and n, with spin $\frac{1}{2}$. The $J = I = \frac{3}{2}$ states correspond to the $\Delta^{++}, \Delta^{+}, \Delta^{0}$ and Δ^{-} delta baryon resonances, with spin $\frac{3}{2}$.

Example. The $B = 2$ Skyrmion has toroidal symmetry, including a continuous SO(2) symmetry. The first constraint is therefore

$$e^{i\alpha L_3}e^{i2\alpha K_3}\,|\Psi\rangle = |\Psi\rangle\,.$$

Note the 2α instead of α in the isospatial part. The picture above shows that when the $B = 2$ Skyrmion is rotated by 2π, the pion field rotates by 4π.

There is another discrete symmetry, given by turning the torus upside down. This gives

$$e^{i\pi L_1}e^{i\pi K_1}\,|\Psi\rangle = -\,|\Psi\rangle\,.$$

The sign is not obvious from what we've said so far, but it is correct. Since we have an even baryon number, the allowed states have integer spin and isospin.

States with isospin 0 are most interesting, and have lowest energy. Then the **K** operators act trivially, and we have

$$e^{i\alpha L_3}\left|\Psi\right\rangle = \left|\Psi\right\rangle, \quad e^{i\pi L_1}\left|\Psi\right\rangle = -\left|\Psi\right\rangle.$$

We have reduced the problem to one involving only body-fixed spin operators. The first constraint tells us

$$L_3\left|\Psi\right\rangle = 0.$$

Thus the allowed states are $|J, L_3 = 0\rangle$. The second constraint requires J to be odd. So the lowest energy states with zero isospin are $|1, 0\rangle$ and $|3, 0\rangle$. In particular, there are no spin 0 states. The state $|1, 0\rangle$ represents the deuteron. This is a spin 1, isospin 0 bound state of p and n. This is a success.

The $|3, 0\rangle$ states have too high energy to be stable, but there is some evidence for a spin 3 dibaryon resonance that decays into two Δ's.

There is also a 2-nucleon resonance with $I = 1$ and $J = 0$, but this is not a bound state. This is also allowed by the Skyrme model.

Example. For $B = 4$, we have cubic symmetry. The symmetry group is rather complicated, and imposes various constraints. But these constraints always involve + signs on the right-hand side. The lowest allowed state is $|0, 0\rangle \otimes |0, 0\rangle$, which agrees with the α-particle. The next $I = 0$ state is

$$\left(|4, 4\rangle + \sqrt{\frac{14}{5}}\,|4, 0\rangle + |4, -4\rangle\right) \otimes |0, 0\rangle$$

involving a combination of L_3 values. This is an excited state with spin 4, and would have rather high energy. Unfortunately, it hasn't been seen experimentally. It is, however, a success of the Skyrme model that there are no rigid-body $J = 1, 2, 3$ states with isospin 0.

Further quantum states of Skyrmions are obtained by quantizing the vibrational modes of Skyrmions and coupling these to rotations. The energy spectra are quite complicated but match nuclear data quite well in several cases (α-particle, Carbon-12, Oxygen-16).

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