3P1b **Quantum Field Theory: Example Sheet 2** Michaelmas 2023

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1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \ . \tag{1}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x \ T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \ , \tag{2}$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x) \ .$$

2*. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x) \, .$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

3. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle$ satisfy

$$\langle 0| \phi(x) |p\rangle = e^{-ip\cdot x} \ .$$

4*. In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2}\epsilon_{ijk} \int d^3x \ \left(x^j T^{0k} - x^k T^{0j}\right) \ .$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = \frac{i}{2}\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \left(p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j}\right) a_{\vec{p}} \ .$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum in its rest frame).

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5. Show that the time ordered product $\mathrm{T}(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

6. The Schwinger-Dyson equation states that

$$\begin{aligned}\left(\Box_x + m^2\right)\langle\phi_x\phi_1\dots\phi_n\rangle =& \langle\mathcal{L}'_{\text{int}}[\phi_x]\phi_1\dots\phi_n\rangle \\ &- i\sum_{j=1}^{n}\delta^{(4)}(x-x_j)\langle\phi_1\dots\phi_{j-1}\phi_{j+1}\dots\phi_n\rangle\ ,\end{aligned} \tag{3}$$

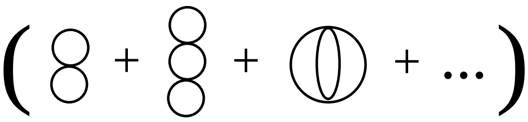
where $\phi_j \equiv \phi(x_j)$ and $\phi_x \equiv \phi(x)$. Recall that the brackets stand for a shorthand of time-ordering, i.e.,

$$\langle\phi_1\dots\phi_n\rangle \equiv \langle\Omega|\,\mathrm{T}\left(\phi_1\dots\phi_n\right)|\Omega\rangle\ ,$$

and for simplicity we are assuming that the interacting part of the Lagrangian ($\mathcal{L}_{\text{int}}$) does not include derivatives of the fields.

In lectures we showed (3) for two fields for a QFT that is local and causal. Derive explicitly (3) for three fields by using the same assumptions.

7. Examine $\langle 0|\,\hat{S}\,|0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different contributions arising from an application of Wick's theorem and derive Feynman rules representing these contributions as diagrams. Confirm that to order λ^2, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the following diagramatic expression,

$\langle 0|\,\hat{S}\,|0\rangle = \exp$  ,

corresponding to the exponential of the sum of distinct vacuum bubble diagrams.

8. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi\ .$$

Calculate the amplitude for meson decay $\phi \to \psi\bar{\psi}$ to leading order in g. Show that the amplitude is only non-zero for $m > 2M$ and explain the physical interpretation of this condition using conservation laws.