Quantum Field Theory: Example Sheet 2

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1. A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} p_n^2 + \tfrac{1}{2} \omega_n^2 q_n^2 \right) \tag{1}$$

where ω_n is the frequency of the nth mode. (Compare this Hamiltonian to the Lagrangian (3) in Example Sheet **1**. We have set the mass per unit length in that question to $\sigma = 1$ to simplify some of the formulae a little.) After quantization, q_n and p_n become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm}\,. \tag{2}$$

Introduce creation and annihilation operators a_n and $a_n^\dagger$,

$$a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n\,. \tag{3}$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm}\,. \tag{4}$$

Show that the Hamiltonian of the system can be written in the form

$$H = \sum_{n=1}^{\infty} \tfrac{1}{2}\, \omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n \right)\,. \tag{5}$$

Given the existence of a ground state $|0\rangle$ such that $a_n|0\rangle = 0$, explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n\,. \tag{6}$$

Show further that $[H, a_n^\dagger] = \omega_n\, a_n^\dagger$ and hence calculate the energy of the state

$$|l_1, l_2, \ldots, l_N\rangle = \left(a_1^\dagger\right)^{l_1} \left(a_2^\dagger\right)^{l_2} \ldots \left(a_N^\dagger\right)^{l_N} |0\rangle\,. \tag{7}$$

2. The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right] \tag{8}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right]\,. \tag{9}$$

Show that the commutation relations

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{10}$$

imply that

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^{\dagger}_{\vec{p}}, a^{\dagger}_{\vec{q}}] = 0 \quad \text{and} \quad [a_{\vec{p}}, a^{\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \, . \tag{11}$$

3. Consider a real scalar field with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_{\mu}\phi \, \partial^{\mu}\phi - \tfrac{1}{2}m^2\phi^2 \, . \tag{12}$$

Show that, after normal ordering, the conserved four-momentum $P^{\mu} = \int d^3x \, T^{0\mu}$ takes the operator form

$$P^{\mu} = \int \frac{d^3p}{(2\pi)^3} \, p^{\mu} \, a^{\dagger}_{\vec{p}} a_{\vec{p}} \tag{13}$$

where $p^0 = E_{\vec{p}}$ in this expression. From this expression for P^{μ} verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^{\mu}, \phi(x)] = -i\partial^{\mu}\phi(x) \, . \tag{14}$$

4. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x) \, . \tag{15}$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

5. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalized one-particle states $|p\rangle = \sqrt{2E_{\vec{p}}} \, a^{\dagger}_{\vec{p}} \, |0\rangle$ satisfy

$$\langle 0| \, \phi(x) \, |p\rangle = e^{-ip\cdot x} \, . \tag{16}$$

6. The retarded propagator in Klein-Gordon theory is defined as

$$D_R(x-y) = \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

where $\phi(x)$ and $\phi(y)$ are Heisenberg fields. Show that

$$D_R(x-y) = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip.(x-y)} - e^{ip.(x-y)} \right)$$

where $p^0 = E_{\vec{p}}$ in the exponents.

Show that $D_R(x-y)$ can be reexpressed as a 4-momentum integral

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip.(x-y)}$$

and find the appropriate contour for the p^0 integral.

Describe any similarity or difference between $D_R(x-y)$ and the Feynman propagator $D_F(x-y)$.

7. The purpose of this question is to introduce you to non-relativistic quantum field theory. This is the only place you will encounter such a thing in this course. Consider the Lagrangian density for a complex scalar field ϕ given by

$$\mathcal{L} = i\phi^\star \partial_0 \phi - \frac{1}{2m} \nabla\phi^\star \cdot \nabla\phi \,. \tag{17}$$

Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry $\phi \to e^{i\alpha}\phi$. Show that the momentum conjugate to ϕ is $i\phi^\star$ and compute the classical Hamiltonian.

We now wish to quantize this theory. We will work in the Schrödinger picture. Explain why the correct commutation relations are

$$[\phi(\vec{x}), \phi(\vec{y})] = [\phi^\dagger(\vec{x}), \phi^\dagger(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \phi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,. \tag{18}$$

Expand the fields in a Fourier decomposition as

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}\,, \\ \phi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\,. \end{aligned} \tag{19}$$

Determine the commutation relations obeyed by $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. Why do we have only a single set of creation and annihilation operators $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ even though ϕ is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass m.

8. Show that if $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, then

$$\left[\gamma^\kappa\gamma^\lambda\,,\, \gamma^\mu\gamma^\nu\right] = 2g^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2g^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2g^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2g^{\kappa\nu}\gamma^\mu\gamma^\lambda\,.$$

By expressing $S^{\kappa\lambda} = \frac{1}{4}\left[\gamma^\kappa\,,\, \gamma^\lambda\right]$ as $\frac{1}{2}(\gamma^\kappa\gamma^\lambda - g^{\kappa\lambda})$ etc., evaluate $\left[S^{\kappa\lambda}\,,\, S^{\mu\nu}\right]$.

9. Show using Q.8 that if $S^i = \frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k$, then $[S^i, S^j] = i\epsilon_{ijk}S^k$. Show also that $[\gamma^0, S^i] = [\gamma^5, S^i] = 0$, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that $(S^1)^2 = (S^2)^2 = (S^3)^2 = \frac{1}{4}$.

Verify these results in the particular representation

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

and show that

$$S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$

in this representation. What can you deduce about rotations and spin in the Dirac field theory?

10. Using just the algebra $\{\gamma^\mu,\ \gamma^\nu\} = 2g^{\mu\nu}$ (i.e. without resorting to a particular representation), and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu\gamma^\mu$ and $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\ \gamma^\nu]$, prove the following results:

1. $\mathrm{Tr}\gamma^\mu = 0$
2. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
3. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$
4. $(\gamma^5)^2 = 1$
5. $\mathrm{Tr}\gamma^5 = 0$
6. $\not{p}\,\not{q} = 2p\cdot q - \not{q}\,\not{p} = p\cdot q + 2S^{\mu\nu}p_\mu q_\nu$
7. $\mathrm{Tr}(\not{p}\,\not{q}) = 4p\cdot q$
8. $\mathrm{Tr}(\not{p}_1 \dots \not{p}_n) = 0$ if n is odd
9. $\mathrm{Tr}(\not{p}_1\,\not{p}_2\,\not{p}_3\,\not{p}_4) = 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right]$
10. $\mathrm{Tr}(\gamma^5\,\not{p}_1\,\not{p}_2) = 0$
11. $\gamma_\mu\,\not{p}\,\gamma^\mu = -2\,\not{p}$
12. $\gamma_\mu\,\not{p}_1\,\not{p}_2\gamma^\mu = 4p_1\cdot p_2$
13. $\gamma_\mu\,\not{p}_1\,\not{p}_2\,\not{p}_3\gamma^\mu = -2\,\not{p}_3\,\not{p}_2\,\not{p}_1$
14. $\mathrm{Tr}(\gamma^5\,\not{p}_1\,\not{p}_2\,\not{p}_3\,\not{p}_4) = 4i\,\epsilon_{\mu\nu\rho\sigma}\,p_1^\mu\,p_2^\nu\,p_3^\rho\,p_\sigma^4$

[*Useful tricks to use are* $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$ *and inserting* $(\gamma^5)^2 = 1_4$ *into a trace.*]
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