Quantum Field Theory: Example Sheet 3

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1. Using the methods presented in the lectures, find an expression for the Feynman propagator of a Dirac field

\[ \Delta_F(x - y) \equiv \langle 0 | T\psi(x)\bar{\psi}(y) | 0 \rangle \]

in terms of the \( \theta \)-function and integrals over 3-momentum.

Deduce, by evaluating a suitable contour integral, that

\[ \Delta_F(x - y) = i \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x - y)} \gamma^a p_a - m \frac{p^2 + m^2 - i\epsilon}{p^2 + m^2 - i\epsilon}. \]

Verify that \( \Delta_F \) is related to the Green's function for the Dirac operator.

2. The Lagrangian density for an electromagnetic (Maxwell) field is

\[ \mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} \]

where \( F_{ab} = \partial_a A_b - \partial_b A_a \), and \( A_a \) is the 4-vector potential. Show that \( \mathcal{L} \) is invariant under gauge transformations

\[ A_a \to A_a + \partial_a \xi \]

where \( \xi = \xi(x) \) is a scalar field with arbitrary (differentiable) dependence on \( x \).

Use Noether's theorem, and the spacetime translational invariance of the action, to construct the energy-momentum tensor \( T^{ab} \) for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

\[ \Theta^{ab} = T^{ab} - F^{ca} \partial_c A^b. \]

Show that this object also defines four conserved currents. Moreover, show that it is symmetric, gauge invariant and traceless.
Comment: $T^{ab}$ and $\Theta^{ab}$ are both equally good definitions of the energy-momentum tensor. However $\Theta^{ab}$ clearly has the nicer properties. Moreover, if you couple an electromagnetic field to general relativity then it is $\Theta^{ab}$ which appears in Einstein’s equations.

3. The Lagrangian density for a massive vector field $C_a$ is given by

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} m^2 C_a C^a$$

where $F_{ab} = \partial_a C_b - \partial_b C_a$. Derive the equations of motion and show that when $m \neq 0$ they imply

$$\partial_a C^a = 0.$$  

Further show that $C_0$ can be eliminated completely in terms of the other fields by solving

$$-\partial_i \partial^i C_0 + m^2 C_0 = \partial_i \dot{C}_i.$$  

Construct the canonical momenta $\Pi_i$ conjugate to $C_i$, $i = 1, 2, 3$ and show that the canonical momentum conjugate to $C_0$ is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of $C_0$, $C_i$ and $\Pi_i$. (Note: Do not be concerned that the canonical momentum for $C_0$ is vanishing. $C_0$ is non-dynamical — it is determined entirely in terms of the other fields using equation (1).)

Find the Feynman Green’s function for the massive spin-one field.

4. The propagator for a massless scalar field is given by

$$\Delta_F(x, y) = i \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} \frac{1}{p^2 - i\epsilon}.$$  

Evaluating this integral to express the propagator directly as a simple function of position in spacetime. Is your expression Lorentz invariant.

5. Using the path integral, find an expression for the probability amplitude of a non-relativistic particle of mass $m$ moving in one dimension and starting at $x_i$ at $t = 0$ being at $x_f$ at time $T$.

6. A non-relativistic one-dimensional simple harmonic oscillator of angular frequency $\omega$ and mass $m$ has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2.$$
where \( p \) is the momentum operator and \( x \) the position operator. Use the path integral method to find the amplitude for the transition from position \( x_i \) and time \( t_i \) to \( x_f \) and time \( t_f \).

7. Suppose that the simple harmonic oscillator of the previous question is in its ground state at \( t \to -\infty \). Instead of being free, it is subject to a force \( f(t) \). Find the probability that it remains in its ground state as \( t \to \infty \).

8. Consider an state \( |A\rangle \) that is an eigenstate of the field operator for a real scalar field such that \( \phi(x,0)|A\rangle = A(x)|A\rangle \) for a real function of position \( A(x) \). For free field theory, use path integral methods to show that

\[
\langle A|0 \rangle \propto \exp \left[ -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} E(k) \tilde{A}(k) \tilde{A}(-k) \right],
\]

where

\[
\tilde{A}(k) = \int d^3x \, e^{-ik \cdot x} A(x)
\]

and \( E(k) = \sqrt{k^2 + m^2} \).

Please address any comments, especially about errors and omissions to: mjp1@cam.ac.uk