Quantum Field Theory: Example Sheet 3

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1. The Weyl representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{1}$$

Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{2}$$

2. Starting with the Dirac equation, show that the Dirac conjugate field $\overline{\psi}(x)$ obeys

$$i\partial_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} = 0 . \tag{3}$$

Derive this equation from the Dirac action as the variational equation with respect to ψ.

Show that $j^\mu = \overline{\psi}\gamma^\mu\psi$ is a Noether current, and hence conserved. Show that the axial currrent $a^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$ is conserved if $m = 0$.

3. Explain why one may split any Dirac spinor uniquely into left- and right-handed spinor parts $\psi = \psi_L + \psi_R$. Show that any gamma matrix γ^μ maps a left-handed into a right-handed spinor, and vice versa, and deduce that any non-trivial solution of the Dirac equation with $m \neq 0$ has both left- and right-handed parts.

Find the plane wave solutions of the massless Dirac equation with purely a left-handed part. For a given value of the 3-momentum, what is the dimension of the space of solutions?

4. With the notation as in the lectures show that

$$\sum_{s=\pm\frac{1}{2}} u_s(\vec{p})\overline{u}_s(\vec{p}) = \not{p} + m \tag{4}$$

$$\sum_{s=\pm\frac{1}{2}} v_s(\vec{p})\overline{v}_s(\vec{p}) = \not{p} - m \tag{5}$$

where the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

5. The Fourier decomposition of the Dirac field operator $\psi(\vec{x})$ and its conjugate momentum $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned}
\psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=\pm\frac{1}{2}}\left[a^s_{\vec{p}}\,u_s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + b^{s\,\dagger}_{\vec{p}}\,v_s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right] \\
\psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=\pm\frac{1}{2}}\left[a^{s\,\dagger}_{\vec{p}}\,u_s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}}\,v_s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}}\right]\,. \qquad (6)
\end{aligned}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned}
\{a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}\} &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \\
\{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\} &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q})\,, \qquad (7)
\end{aligned}$$

with all other anticommutators vanishing; that is,

$$\{a^r_{\vec{p}}, a^s_{\vec{q}}\} = \{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{a^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\} = \{a^r_{\vec{p}}, b^s_{\vec{q}}\} = \ldots = 0\,. \qquad (8)$$

Show that these imply that the field and its conjugate momentum satisfy the equal time anticommutation relations

$$\begin{aligned}
\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0 \\
\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})\,. \qquad (9)
\end{aligned}$$

6. Using the results of Question 5, show that the quantum Hamiltonian

$$H = \int d^3x\;\bar{\psi}(-i\gamma^i\partial_i + m)\psi \qquad (10)$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3}\,E_{\vec{p}}\sum_{s=1}^{2}\left[a^{s\,\dagger}_{\vec{p}}a^s_{\vec{p}} + b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}}\right]\,. \qquad (11)$$

7. The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 Dirac field as a boson. We start with the usual decomposition (6). This time we choose bosonic commutation relations for the annihilation and creation operators,

$$\begin{aligned}
[a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}] &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \\
[b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}] &= -(2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \qquad (12)
\end{aligned}$$

with all other commutators vanishing. Note the strange minus sign for the b operators. Repeat the calculation of Question 5 to show that these are equivalent to the commutation relations

$$\begin{aligned}[\psi_\alpha(\vec{x}),\psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}),\psi^\dagger_\beta(\vec{y})] &= 0\\ [\psi_\alpha(\vec{x}),\psi^\dagger_\beta(\vec{y})] &= \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})\,. \end{aligned} \tag{13}$$

Now repeat the calculation of Question 6, to show that, after normal ordering, the Hamitonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{s=1}^{2} \left[a^{s\,\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\,\dagger}_{\vec{p}} b^s_{\vec{p}} \right] . \tag{14}$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more b particles. This is the reason a theory of bosonic spin 1/2 particles is sick.

8. Using the methods presented in the lectures, find an expression for the Feynman propagator of a Dirac field

$$S_F(x-y) \equiv \langle 0|\mathrm{T}\psi(x)\overline{\psi}(y)|0\rangle \tag{15}$$

in terms of the θ-function and integrals over 3-momentum.

Deduce, by evaluating a suitable contour integral, that

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip.(x-y)} \frac{\gamma.p+m}{p^2-m^2+i\epsilon}\ . \tag{16}$$

Verify that S_F is a Green's function for the Dirac operator.

9. The Hamiltonian H for a quantum system can be split into two parts thus:

$$H = H_0 + H_{\rm int} \tag{17}$$

where H_0 is the Hamiltonian for a known (simple) system and $H_{\rm int}$ is the interaction Hamiltonian. Define the Interaction picture for the system. Show that the evolution operator for states in the Interaction picture between times t_0 and t_1 is $U(t_1,t_0)$ where

$$U(t_1,t_0) = \mathrm{T}\exp\left\{-i\int_{t_0}^{t_1} dt H_I(t)\right\} \tag{18}$$

and

$$H_I(t) = e^{iH_0t}H_{\rm int}e^{-iH_0t}\,. \tag{19}$$

Verify directly from the time-ordered exponential form of $U(t_1, t_0)$ to second order in the interaction and for $t_0 < \tau < t_1$ that

$$U(t_1, t_0) = U(t_1, \tau)U(\tau, t_0)\,. \tag{20}$$

10. Show that the time ordered product $\mathrm{T}\,(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that $D_F(x_1 - x_2)$ has the same symmetry property.

Check Wick's theorem for the case of three scalar fields,

$$\begin{aligned}\mathrm{T}\,(\phi(x_1)\phi(x_2)\phi(x_3)) \;&= \;:\phi(x_1)\phi(x_2)\phi(x_3): + \;\phi(x_1)D_F(x_2 - x_3)\\ &\quad + \;\phi(x_2)D_F(x_3 - x_1) + \phi(x_3)D_F(x_1 - x_2)\,.\end{aligned} \tag{21}$$

11. A quantum field theory with two scalar fields has Lagrangian density $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_{\text{int}}(x)$ where

$$\mathcal{L}_0(x) = \frac{1}{2}(\partial\phi(x))^2 - \frac{1}{2}m^2(\phi(x))^2 + \frac{1}{2}(\partial\psi(x))^2 - \frac{1}{2}\mu^2(\psi(x))^2 \tag{22}$$

and

$$\mathcal{L}_{\text{int}}(x) = -\frac{\lambda}{4}(\phi(x))^2(\psi(x))^2\;. \tag{23}$$

Find the $O(\lambda)$ contribution to the scattering amplitude for the process

$$\phi(p_1) + \psi(p_2) \to \phi(p_1') + \psi(p_2') \tag{24}$$

where $p_1,\; p_2$ and $p_1',\; p_2'$ are the initial and final momenta of the scattering particles.