Quantum Field Theory: Example Sheet 4

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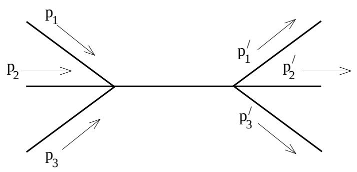
1. A real scalar field with ϕ^4 interaction has the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{1}$$

Use Dyson's formula (the expression for S as a time-ordered exponential of the integral of $-iH_{\rm I}$, and its perturbative expansion) and Wick's theorem to show that the leading order contribution to 3-particle $\rightarrow$ 3-particle scattering includes the amplitude

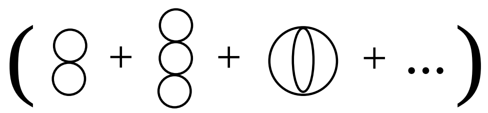


$$= (-i\lambda)^2 \, \frac{i}{(p_1+p_2+p_3)^2 - m^2} \,. \tag{2}$$

Check that this result is consistent with the Feynman rules for the theory. What other diagrams also contribute to this process?

2. Examine $\langle 0|S|0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different diagrams arising from an application of Wick's theorem. Confirm that to order λ^2, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$$\langle 0|\, S\,|0\rangle = \exp \tag{3}$$



3. Consider the Lagrangian density for three scalar fields ϕ_i, $i = 1,2,3$, given by

$$\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2(\sum_{i=1}^{3}\phi_i^2) - \frac{\lambda}{8}(\sum_{i=1}^{3}\phi_i^2)^2 \,. \tag{4}$$

Show that the Feynman propagator for the free field theory (i.e. $\lambda = 0$) is of the form

$$\langle 0|T\phi_i(x)\phi_j(y)|0\rangle = \delta_{ij}D_F(x-y) \tag{5}$$

where $D_F(x-y)$ is the usual scalar propagator. Write down the Feynman rules of the theory. Compute the amplitude for the scattering $\phi_i\phi_j \rightarrow \phi_k\phi_l$ to lowest order in λ.

4. The Lagrangian density for Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\psi\,. \tag{6}$$

a) Consider $\psi\psi \to \psi\psi$ scattering, with the initial and final states given by

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\,a^{s\,\dagger}_{\vec{p}}\,a^{r\,\dagger}_{\vec{q}}\,|0\rangle\,, \\ |f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\,a^{s'\,\dagger}_{\vec{p}'}\,a^{r'\,\dagger}_{\vec{q}'}\,|0\rangle\,. \end{aligned} \tag{7}$$

Show using Dyson's formula and Wick's theorem that the scattering amplitude at order g^2 is given by

$$\mathcal{M} = (-ig)^2\left(\frac{[\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}')\cdot u^r(\vec{q})]}{(p'-p)^2-\mu^2} - \frac{[\bar{u}^{s'}(\vec{p}')\cdot u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}')\cdot u^s(\vec{p})]}{(q'-p)^2-\mu^2}\right)\,.$$

Draw the two Feynman diagrams that correspond to these two terms.

b) Consider now $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering, with initial and final states given by

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\,a^{s\,\dagger}_{\vec{p}}\,b^{r\,\dagger}_{\vec{q}}\,|0\rangle\,, \\ |f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\,a^{s'\,\dagger}_{\vec{p}'}\,b^{r'\,\dagger}_{\vec{q}'}\,|0\rangle\,. \end{aligned} \tag{8}$$

Show that the amplitude is this time given by

$$\mathcal{M} = -(-ig)^2\left(\frac{[\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})]\,[\bar{v}^{r}(\vec{q})\cdot v^{r'}(\vec{q}')]}{(p-p')^2-\mu^2} - \frac{[\bar{v}^{r}(\vec{q})\cdot u^s(\vec{p})]\,[\bar{u}^{s'}(\vec{p}')\cdot v^{r'}(\vec{q}')]}{(p+q)^2-\mu^2}\right)\,.$$

(Be careful with minus signs.) What are the Feynman diagrams that now contribute?

5. The Lagrangian density for a pseudoscalar Yukawa interaction is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\gamma^5\psi\,. \tag{9}$$

Write down the Feynman rules for this theory. Use these to write down the amplitude at order g^2 for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering.

6. Any vector function $\mathbf{f}(\mathbf{x})$ has a decomposition into a sum of transverse (zero divergence) and longitudinal (zero curl) parts, namely

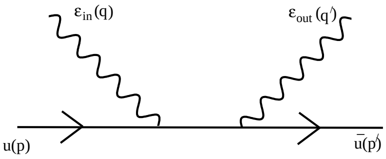
$$\mathbf{f} = \nabla\times\mathbf{g} + \nabla h \equiv \mathbf{f}^T + \mathbf{f}^L \tag{10}$$

where $\mathbf{g}$ and h are unique if one imposes the additional constraint $\nabla\cdot\mathbf{g} = 0$ and certain vanishing conditions at infinity. By taking the divergence and curl of equation (10), determine $\mathbf{g}$ and h in terms of $\mathbf{f}$. Show formally that

$$\mathbf{f}^T = \mathbf{f} - \nabla(\nabla^2)^{-1}\nabla\cdot\mathbf{f}\,. \tag{11}$$

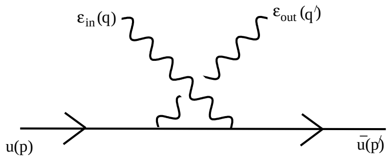
Use this result to comment on the commutation relations of the quantized electromagnetic gauge potential in Coulomb gauge.

7. Consider the Compton scattering process $e^- \gamma \to e^- \gamma$ in QED. Let the incoming photon have polarization vector ϵ_{in}^μ, and the outgoing photon have polarization $\epsilon_{\text{out}}^\mu$. Use the Feynman rules to derive the following amplitude associated to the lowest order diagram,



$$= i(-ie)^2 \bar{u}^{r'}(\vec{p}') \; \not{\epsilon}_{\text{out}} \frac{(\not{p} + \not{q} + m)}{(p+q)^2 - m^2} \; \not{\epsilon}_{\text{in}} u^s(\vec{p}) \, .$$

Compute also the contribution from the diagram



The complete amplitude at order e^2 is the sum of these two contributions. Show that if ϵ_{in} is replaced by the incoming photon momentum q then the total amplitude vanishes. Check that the same holds true if ϵ_{out} is replaced by q' . (Note that it will be helpful to recall the equation $(\not{p} - m)u(\vec{p}) = 0$ satisfied by the spinor).

8. Show that

$$[\bar{u}_{s'}(p')\gamma^\mu u_s(p)]^* = [\bar{u}_s(p)\gamma^\mu u_{s'}(p')] \, . \tag{12}$$

Hence show that

$$\sum_{ss'} [\bar{u}_{s'}(p')\gamma^\nu u_s(p)]^* \, [\bar{u}_{s'}(p')\gamma^\mu u_s(p)] = \text{Tr}\left[(\gamma.p' + m)\gamma^\mu(\gamma.p + m)\gamma^\nu\right] \, . \tag{13}$$

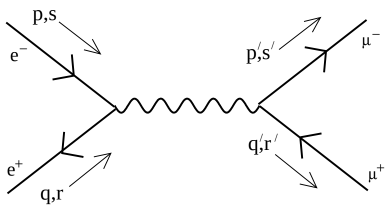
Similarly show that

$$\sum_{sr} [\bar{v}_r(q)\gamma^\nu u_s(p)]^* \, [\bar{v}_r(q)\gamma^\mu u_s(p)] = \text{Tr}\left[(\gamma.q - m)\gamma^\mu(\gamma.p + m)\gamma^\nu\right] \tag{14}$$

$$= \text{Tr}\gamma.q\gamma^\mu\gamma.p\gamma^\nu - m^2\text{Tr}\gamma^\mu\gamma^\nu \tag{15}$$

$$= 4\left[p^\mu q^\nu + p^\nu q^\mu - (p.q + m^2)g^{\mu\nu}\right] \, . \tag{16}$$

9. Use the QED Feynman rules to show that the amplitude for $e^-e^+ \to \mu^-\mu^+$ is given at lowest order in e by,



$$= \mathcal{M} = (-ie)^2 \frac{[\bar{v}^r_{\rm e}(\vec{q})\gamma_\mu u^s_{\rm e}(\vec{p})]\,[\bar{u}^{s'}_{\rm m}(\vec{p}')\gamma^\mu v^{r'}_{\rm m}(\vec{q}')]}{(p+q)^2} \qquad (17)$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons. Let m, M denote the electron and muon masses, respectively. Show that

$$\sum_{srs'r'} |\mathcal{M}|^2 = \frac{e^4}{s^2} \mathrm{Tr}\,[(\gamma.p' + M)\gamma^\mu(\gamma.q' - M)\gamma^\nu]\,\mathrm{Tr}\,[(\gamma.p + m)\gamma_\nu(\gamma.q - m)\gamma_\mu]\ . \qquad (18)$$

In order to simplify this, assume that the momentum components are so large that it is a good approximation to neglect the electron and muon masses. With $m = M = 0$, show that

$$\sum_{srs'r'} |\mathcal{M}|^2 = \frac{e^4}{s^2} 32\,[p.p'q.q' + p.q'q.p'] = 4e^4(1 + \cos^2\theta) \qquad (19)$$

where θ is the scattering angle in the centre-of-mass frame.