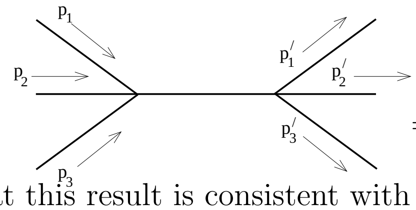
3P1d **Quantum Field Theory: Example Sheet 4** Michaelmas 2017

Corrections and suggestions should be emailed to `B.C.Allanach@damtp.cam.ac.uk`. *Starred questions may be handed in to your supervisor for feedback prior to the class if you wish.*

1. A real scalar field with ϕ^4 interaction has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

Use Dyson's formula and Wick's theorem to show that the leading order contribution to 3-particle→3-particle scattering is given by

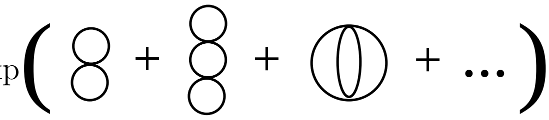


$$= (-i\lambda)^2 \frac{i}{(p_1+p_2+p_3)^2 - m^2}. \tag{2}$$

Check that this result is consistent with the Feynman rules for the theory.

2. Examine $\langle 0|\, S\, |0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different contributions arising from an application of Wick's theorem. Confirm that to order λ^2, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$$\langle 0|\, S\, |0\rangle = \exp \tag{3}$$



3*. Consider the Lagrangian density for three scalar fields ϕ_i, $i = 1, 2, 3$, given by

$$\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2(\sum_{i=1}^{3}\phi_i^2) - \frac{\lambda}{8}(\sum_{i=1}^{3}\phi_i^2)^2\,. \tag{4}$$

Show that the Feynman propagator for the free field theory (i.e. $\lambda = 0$) is of the form

$$\langle 0|T\phi_i(x)\phi_j(y)|0\rangle = \delta_{ij}\Delta_F(x-y) \tag{5}$$

where $\Delta_F(x-y)$ is the usual scalar propagator. Write down the Feynman rules of the theory. Compute the amplitude for the scattering $\phi_i\phi_j \to \phi_k\phi_l$ to lowest nontrivial order in λ.

4*. The Lagrangian density for a Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar\psi(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar\psi\psi. \tag{6}$$

(a) Consider $\psi\psi \to \psi\psi$ scattering, with the initial and final states given by

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec p}E_{\vec q}}\, b^{s\,\dagger}_{\vec p} b^{r\,\dagger}_{\vec q}\, |0\rangle\,, \\ |f\rangle &= \sqrt{4E_{\vec p'}E_{\vec q'}}\, b^{s'\,\dagger}_{\vec p'} b^{r'\,\dagger}_{\vec q'}\, |0\rangle\,. \end{aligned} \tag{7}$$

Show using Dyson's formula and Wick's theorem that the scattering amplitude at order λ^2 is given by

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar u^{s'}(\vec{p'})\cdot u^s(\vec p)][\bar u^{r'}(\vec{q'})\cdot u^r(\vec q)]}{(p'-p)^2 - \mu^2} - \frac{[\bar u^{s'}(\vec{p'})\cdot u^r(\vec q)][\bar u^{r'}(\vec{q'})\cdot u^s(\vec p)]}{(q'-p)^2 - \mu^2} \right). \tag{8}$$

Draw the two Feynman diagrams that correspond to these two terms.

(b) Consider now $\psi\bar\psi \to \psi\bar\psi$ scattering, with initial and final states given by

$$|i\rangle = \sqrt{4E_{\vec p}E_{\vec q}}\, b_{\vec p}^{s\dagger} c_{\vec q}^{r\dagger} |0\rangle \tag{9}$$
$$|f\rangle = \sqrt{4E_{\vec p}E_{\vec q}}\, b_{\vec{p'}}^{s'\dagger} c_{\vec{q'}}^{r'\dagger} |0\rangle\,. \tag{10}$$

Show that the amplitude is given by

$$\mathcal{A} = -(-i\lambda)^2 \left(\frac{[\bar u^{s'}(\vec{p'}) \cdot u^s(\vec p)][\bar v^r(\vec q) \cdot v^{r'}(\vec{q'})]}{(p-p')^2 - \mu^2} - \frac{[\bar v^r(\vec q) \cdot u^s(\vec p)][\bar u^{s'}(\vec{p'}) \cdot v^{r'}(\vec{q'})]}{(q+p)^2 - \mu^2} \right) . \tag{11}$$

Be careful with minus signs! What are the Feynman diagrams that now contribute?

5. The Lagrangian density for a pseudoscalar Yukawa interaction is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar\psi(i\,\not\partial - m)\psi - \lambda\phi\bar\psi\gamma^5\psi\,. \tag{12}$$

Write down the Feynman rules for this theory. Use these to write down the amplitude at order λ^2 for $\psi\psi \to \psi\psi$ scattering and $\psi\bar\psi \to \psi\bar\psi$ scattering.

6. Any vector function $\mathbf{f}(\mathbf{x})$ has a decomposition into a sum of transverse (zero divergence) and longitudinal (zero curl) parts, namely

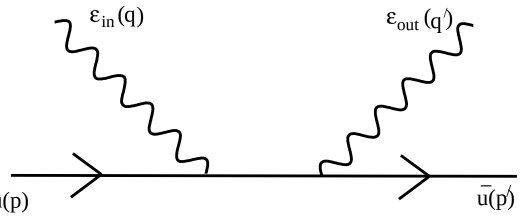
$$\mathbf{f} = \nabla \times \mathbf{g} + \nabla h \equiv \mathbf{f}^T + \mathbf{f}^L, \tag{13}$$

where $\mathbf{g}$ and h are unique if one imposes the additional constraint $\nabla \cdot \mathbf{g} = 0$ and certain vanishing conditions at infinity. By taking the divergence and curl of Eq. 13, determine $\mathbf{d}$ and h in terms of $\mathbf{f}$. Show formally that

$$\mathbf{f}^T = \mathbf{f} - \nabla(\nabla^2)^{-1}\nabla \cdot \mathbf{f}. \tag{14}$$

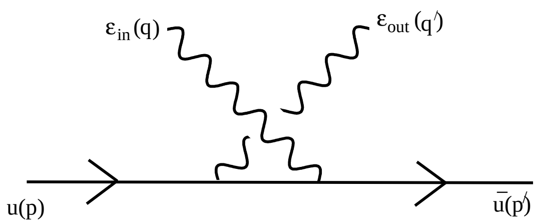
Use this result to comment on the commutation relations of the quantised electromagnetic gauge potential in Coulomb gauge.

7. Consider Compton scattering in which a photon and an electron scatter off each other. Let the incoming photon have polarisation vector ϵ^μ_{in} and the outgoing photon have polarisation $\epsilon^\mu_{\text{out}}$. Use the Feynman rules to derive the following amplitude associated to the lowest order diagram,



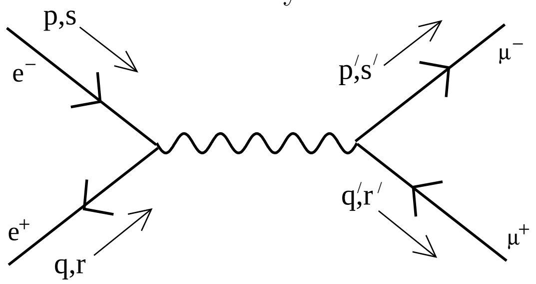
$$= i(-ie)^2 \bar u^{r'}(\vec{p'}) \not\epsilon_{\text{out}} \frac{\not p + \not q + m}{(p+q)^2 - m^2} \not\epsilon_{\text{in}} u^s(\vec p). \tag{15}$$

Also, compute the contribution from the diagram



The complete amplitude at order e^2 is the sum of these two contributions. Show that the total amplitude vanishes if ϵ_{in} is replaced by the incoming photon momentum q then the amplitude vanishes. Check that the same holds true if ϵ_{out} is replaced by q'. *Note: it will be helpful to recall the equation* $\left(\not{p} - m \right) u(\vec{p}) = 0$ *satisfied by the spinor.*

8. Use the Feynman rules to show that the QED amplitude for $e^+e^- \to \mu^+\mu^-$ is given at lowest order in e by



$$= (-ie)^2 \frac{[\bar{v}_e^r(\vec{q})\gamma_\mu u_e^s(\vec{p})][\bar{u}_m^{s'}(\vec{p'})\gamma^\mu v_m^{r'}(\vec{q'})]}{(p+q)^2}, \quad (16)$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons, respectively.

9. Viki Weisskopf is one of the more charming characters from the history of quantum field theory. This is from his his biography

> "Pauli asked me to calculate the amplitude for pair creation of scalar particles by photons. It was only a short time after Bethe and Heitler had solved the same problem for electrons and positrons. I met Bethe in Cophenhagen at a conference and asked him to tell me how he did the calculations. I also inquired how long it would take to perform this task; he answered "It would take me three days, but you will need about three weeks." He was right, as usual; furthermore, the published result was wrong by a factor of two."

Can you do better?