# QFT: Cross Sections and Decay Rates

B.C. Allanach



So far, we have considered transition amplitudes between $|i\rangle$ and $|f\rangle$ asymptotic states of definite momententum: the probability is expressed in terms of the transition amplitude part of the $S$-matrix

$$\langle f|(S-1)|i\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_1+p_2-\sum_{i=1}^{n}q_i). \tag{1}$$

The probability of scattering will be proportional to the modulus squared of this quantity. There will be *two* momentum preserving delta functions, which is one too many. Really, this has come about because we have pretended that the external states are *pure momentum eigenstates*. This is an approximation: they are really *a very sharply peaked superposition of momentum eigenstates*. When we take this fact into account, it ends up absorbing the extraneous delta function, and we end up with initial particles that have some localisation in space. We will show now how this works in detail.

So, the $|i\rangle$ states should really be a sharply peaked superposition of pure momentum eigenstates

$$|i\rangle = \int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_1}\frac{d^3\tilde{p}_2}{(2\pi)^3 2E_2} f_1(\tilde{p}_1)f_2(\tilde{p}_2)|\tilde{p}_1\tilde{p}_2\rangle. \tag{2}$$

The $f_i(\tilde{p}_i)$ are then the momentum space wave-packets of incoming particles and are sharply peaked at $\tilde{p}_i = p_i$. We may consider the *outgoing* particles in $|f\rangle$ to be approximately pure momentum eigenstates provided that the experiment does not resolve them at the scale of their de Broglie wavelengths. This is indeed usually the case in collisions at accelerators. We focus now on the example of

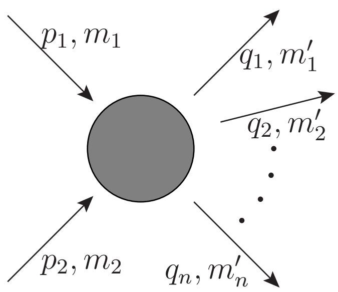


Figure 1: Kinematics of $2 \to n$ scattering.

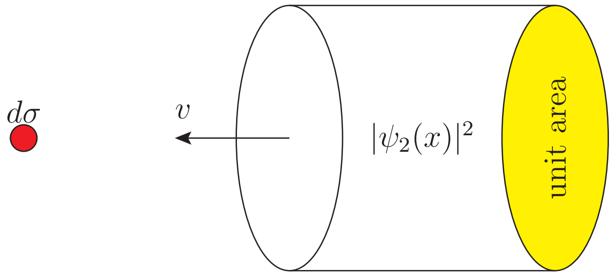


Figure 2: In the rest frame of initial particle 1, whose strength of interaction is as if it presents an effective cross-sectional area ($d\sigma$).

$2 \to n$ scattering where the initial 4-momenta are peaked at $p_1$ and $p_2$ and the final ones at $q_{i=1\ldots n}$. The transition probability is then

$$W = (2\pi)^8 \int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_1} \frac{d^3\tilde{p}_2}{(2\pi)^3 2E_2} \frac{d^3 p_1'}{(2\pi)^3 2E_1'} \frac{d^3 p_2'}{(2\pi)^3 2E_2'} \Big[ |\mathcal{M}|^2 f_1(\tilde{p}_1) f_1^*(p_1') \\ f_2(\tilde{p}_2) f_2^*(p_2') \delta^4(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2) \delta^4(\sum_i q_i - p_1' - p_2') \Big]. \quad (3)$$

Expanding the second $\delta$ function explicitly and using $\sum_i q_i = p_1 + p_2 \approx \tilde{p}_1 + \tilde{p}_2$ (because of the sharply peaked nature of the $f_i$)

$$W = \int d^4x \underbrace{\int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_1} f_1(\tilde{p}_1) e^{i\tilde{p}_1 . x}}_{\sqrt{2E_1}\psi_1(x)} \underbrace{\int \frac{d^3 p_1'}{(2\pi)^3 2E_1'} f_1^*(p_1') e^{-ip_1' . x}}_{\sqrt{2E_1'}\psi_1^*(x)} \\ \int \frac{d^3\tilde{p}_2}{(2\pi)^3 2E_2} f_2(\tilde{p}_2) e^{i\tilde{p}_2 . x} \int \frac{d^3 p_2'}{(2\pi)^3 2E_2'} f_2^*(p_2') e^{-ip_2' . x} \Big[ (2\pi)^4 \\ \delta^4(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2) |\mathcal{M}|^2 \Big]. \quad (4)$$

Using the normalisation for the spacetime wavefunction of particle $i$

$$\psi_i(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} e^{-ip\cdot x} f_i(p) \quad (5)$$

converts back to our version of the wave-function used in the definition of the $S$-matrix.

At this stage, $|\mathcal{M}|^2$ is a function of $\tilde{p}_1, \tilde{p}_2, p_1', p_2', q_i$. We use the notion of the sharp peaks in $f_{1,2}$ to approximate $|\mathcal{M}|^2$ in the integral by the value which depends only upon $\{q_i\}$ and the values of initial momenta at the peaks, i.e. $p_{1,2}$.

Hence we have the transition probability density per unit time

$$\frac{dW}{d^4x} = \frac{|\psi_1(x)|^2}{2E_1} \frac{|\psi_2(x)|^2}{2E_2} (2\pi)^4 \delta^4(\sum_i q_i - p_1 - p_2) |\mathcal{M}|^2. \quad (6)$$

The initial state is depicted in the rest frame of particle 1 in Fig. 2, with the incoming beam of particle 2. We have

$$dW/d^4x = d\sigma\rho\ \phi, \tag{7}$$

where $d\sigma$ is the differential cross-section to scatter into final states of definite momenta, $\rho$ is the probability density of the target particle 1 and $\phi$ being the probability flux incident. We have $\rho = |\psi_1(x)|^2$ and $\phi = |\psi_2(x)|^2 v$, $v$ being the relative velocity between the two particles. Then, substituting all of this into Eq. 6,

$$d\sigma = \frac{(2\pi)^4}{\mathcal{F}}\delta^4(p_1 + p_2 - \sum_{i=1}^{n} q_i)|\mathcal{M}|^2. \tag{8}$$

$\mathcal{F} \equiv 4E_1E_2v$ is known as the flux factor. The masses of the incident particles are $m_1$ and $m_2$, respectively. Going to the rest frame of the second incident particle such that $p_2^\mu = (m_2, \mathbf{0})$, $p_1^\mu = (\sqrt{m_1^2 + p_1^2}, \mathbf{p_1})$. The relative velocity is $v = |\mathbf{p_1}|/E_1$ so

$$\mathcal{F} = 4|\mathbf{p_1}|E_2 = 4m_2\sqrt{E_1^2 - m_1^2} = 4\sqrt{(p_1.p_2)^2 - m_1^2m_2^2}, \tag{9}$$

providing a manifestly Lorentz-invariant definition of the flux factor. Note that in the massless limit $m_{1,2} \ll E_{1,2}$, $\mathcal{F} = 2s$, where $s = (p_1 + p_2)^2$.

In order to find the integrated cross-section, we must sum over the possible momenta of final states in the usual Lorentz invariant manner

$$\sigma = \int \prod_{i=1}^{n}\left(\frac{d^3q_i}{(2\pi)^3 2E_{q_i}}\right)\frac{|\mathcal{M}|^2}{\mathcal{F}}(2\pi)^4\delta^4(p_1 + p_2 - \sum_{i=1}^{n} q_i) \tag{10}$$

However, sometimes we may wish to obtain the *differential cross-section*, in order to examine its behaviour as a function of a kinematical variable.

## 2 to 2 Scattering

2 to 2 scattering is an important example: let us examine the cross section with respect to variations of Mandlestam variable

$$t \equiv (p_1 - q_1)^2 = m_1^2 + {m_1'}^2 - 2E_{p_1}E_{q_1} + 2\mathbf{p_1}.\mathbf{q_1} \Rightarrow \frac{dt}{d\cos\theta} = 2|\mathbf{p_1}||\mathbf{q_1}|, \tag{11}$$

where $\cos\theta$ is the angle between $\mathbf{p_1}$ and $\mathbf{q_1}$. $\cos\theta$ is a frame-dependent quantity, so we must be careful to define the frame (often, the centre of mass frame is used). $s = (p_1 + p_2)^2$ is usually considered to be a constant of the scattering: the centre of mass energy. $u = (p_1 - q_2)^2$ is a dependent variable: it can be phrased in terms of $\cos\theta$ and $\sqrt{s}$, or in terms of $s, t, m_i$ and $m_i'$. We also write

$$\frac{d^3q_2}{2E_{q_2}} = d^4q_2\delta(q_2^2 - {m_2'}^2)\theta(q_2^0) \tag{12}$$

where $\theta(x)$ is the Heaviside theta function (i.e. $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x >= 0$) and

$$\frac{d^3q_1}{2E_{q_1}} = \frac{\mathbf{q_1}^2 d|\mathbf{q_1}| d\cos\theta d\phi}{2E_{q_1}} = \frac{1}{4|\mathbf{p_1}|} dE_{q_1} d\phi dt. \tag{13}$$

Then, performing the $q_2$ and $\phi$ integrals,

$$\frac{d\sigma}{dt} = \frac{1}{8\pi\mathcal{F}|\mathbf{p_1}|} \int dE_{q_1} |\mathcal{M}|^2 \delta(s - {m'_2}^2 + {m'_1}^2 - 2q_1.(p_1 + p_2)). \tag{14}$$

To get a simple expression, we now boost to the *centre of mass* frame, so that if $p_1^\mu = (\sqrt{\mathbf{p_1}^2 + m_1^2}, \mathbf{p_1})$ then $p_2^\mu = (\sqrt{\mathbf{p_1}^2 + m_2^2}, -\mathbf{p_1})$. Considering the Mandlestam variable $s = (\sqrt{\mathbf{p_1}^2 + m_1^2} + \sqrt{\mathbf{p_1}^2 + m_2^2})^2$,

$$|\mathbf{p_1}| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \qquad \mathcal{F} = 2\lambda^{1/2}(s, m_1^2, m_2^2) \tag{15}$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Then

$$\left(\frac{d\sigma}{dt}\right)_{C.o.M} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)} \tag{16}$$

## Decay Rate

For a decay rate, we have the decaying particle as a wave-packet strongly peaked at 4-momentum $p$: $f(p)$, whose Fourier transform is $\psi(x)$ such that

$$\frac{dW}{d^4x} = \frac{|\psi(x)|^2}{2E_p} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p - \sum_{i=1}^n q_i) \tag{17}$$

Eq. 17 must be equal to the probability of finding the decaying parent in unit volume $|\psi(x)|^2$ multiplied by the differential decay rate (or differential "width") $d\Gamma$ into particles of momenta $q_i$, thus

$$\Gamma = \frac{1}{2E_p} \int \prod_{i=1}^n \left( \frac{d^3q_i}{(2\pi)^3 2E_{q_i}} \right) |\mathcal{M}|^2 (2\pi)^4 \delta^4(p - \sum_{i=1}^n q_i). \tag{18}$$

We note that $\Gamma$ is *not* Lorentz invariant, going as $1/E$ of the decaying particle. The standard convention is to define $\Gamma$ in the rest frame of the decaying particle. Putting in the correct units, we have lifetime $\tau = 6.58 \times 10^{-25}/(\Gamma/GeV)$ seconds.