Symmetries, Fields and Particles. Examples 1.

1. O(n) consists of $n \times n$ real matrices M satisfying $M^T M = I$. Check that O(n) is a group. U(n) consists of $n \times n$ complex matrices U satisfying $U^\dagger U = I$. Check similarly that U(n) is a group.

Verify that O(n) and SO(n) are the subgroups of real matrices in, respectively, U(n) and SU(n). By considering how U(n) matrices act on vectors in $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that U(n) is a subgroup of SO($2n$).

2. Show that for matrices $M \in$ O(n), the first column of M is an arbitrary unit vector, the second is a unit vector orthogonal to the first, ... , the kth column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of O(n). By similar reasoning, determine the dimension of U(n).

Show that any column of a unitary matrix U is not in the (complex) linear span of the remaining columns.

3. Consider the real 3×3 matrix,

$$R(\mathbf{n}, \theta)_{ij} = \cos\theta\, \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector in $\mathbb{R}^3$. Verify that $\mathbf{n}$ is an eigenvector of $R(\mathbf{n}, \theta)$ with eigenvalue one. Now choose an orthonormal basis for $\mathbb{R}^3$ with basis vectors $\{\mathbf{n}, \mathbf{m}, \tilde{\mathbf{m}}\}$ satisfying,

$$\mathbf{m} \cdot \mathbf{m} = 1, \qquad \mathbf{m} \cdot \mathbf{n} = 0, \qquad \tilde{\mathbf{m}} = \mathbf{n} \times \mathbf{m}.$$

By considering the action of $R(\mathbf{n}, \theta)$ on these basis vectors show that this matrix corresponds to a rotation through an angle θ about an axis parallel to $\mathbf{n}$ and check that it is an element of $SO(3)$.

4. Show that the set of matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ forms a group. How would you check that it is a Lie group? Assuming that it is a Lie group, determine its dimension. By splitting α and β into real and imaginary parts, consider the group manifold as a subset of $\mathbb{R}^4$ and show that it is non-compact. You may use the fact that a compact subset S of $\mathbb{R}^n$ is necessarily bounded; in other words there exists $B > 0$ such that $|\mathbf{x}| < B$ for all $\mathbf{x} \in S$.

5. Show that any SU(2) matrix U can be expressed in the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Deduce that an alternative form for an SU(2) matrix is

$$U = a_0 I + i\mathbf{a} \cdot \boldsymbol{\sigma}$$

with $(a_0, \mathbf{a})$ real, $\boldsymbol{\sigma}$ the Pauli matrices, and $a_0^2 + \mathbf{a} \cdot \mathbf{a} = 1$. Using the second form, calculate the product of two SU(2) matrices.

6. Consider a real vector space V with product $* : V \times V \to V$. The product is bilinear and associative. In other words, for all elements X,Y,$Z \in V$ and scalars α, $\beta \in \mathbb{R}$, we have

$$(\alpha X + \beta Y) * Z = \alpha X * Z + \beta Y * Z, \qquad Z * (\alpha X + \beta Y) = \alpha Z * X + \beta Z * Y$$

and also $(X * Y) * Z = X * (Y * Z)$. Define the bracket of two vectors X and $Y \in V$ as the commutator,

$$[X, Y] = X * Y - Y * X$$

Show that, equipped with this bracket, V becomes a Lie algebra.

7. Verify that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

forms a matrix Lie group, G. What is the underlying manifold of G? Is the group abelian? Find the Lie algebra, L(G), and calculate the bracket of two general elements of it. Is the Lie algebra simple?

8. A useful basis for the Lie algebra of GL(n) consists of the n^2 matrices T^{ij} $(1 \le i, j \le n)$, where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

9. Let $\exp iH = U$. Show that if H is hermitian then U is unitary. Show also, that if H is traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \to \exp X$ sends L(G), the Lie algebra of G, to G?