

Symmetry and Particle Physics, 1

1. Show that representations of the angular momentum algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$ act on finite-dimensional vector spaces, \mathcal{V}_j , of dimension $2j + 1$, where $j = 0, 1/2, 1, \dots$. Show how it is possible to define a set of orthonormal states, $|j, m\rangle$ satisfying

$$J_{\pm}|j, m\rangle = N_{j,m}^{\pm}|j, m \pm 1\rangle,$$

and derive the expression for $N_{j,m}^{\pm}$.

2. Consider the tensor product space $\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2}$, for which the total angular momentum is $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, which can be written in terms of components J_{\pm}, J_3 . Let \mathcal{U}_M be the subspace for which J_3 has the eigenvalue M . Show that if $M \geq |j_1 - j_2|$ there is a one-dimensional subspace of \mathcal{U}_M which is orthogonal to $J_- \mathcal{U}_{M+1}$. Hence show that there is a single normalised state, unique up to an overall phase factor, $|\phi\rangle \in \mathcal{U}_M$ such that $J_+|\phi\rangle = 0$ if $M \geq |j_1 - j_2|$ and that we may identify $|JJ\rangle = |\phi\rangle$ for $J = M$. What happens if $M < |j_1 - j_2|$?

3. Prove that the tensors δ_{ij} and ϵ_{ijk} ($i, j, k = 1, 2, 3$) are invariant under three-dimensional rotations. If \mathbf{u} and \mathbf{v} are vectors in three dimensional Euclidean space, show that

$$T_{ij} = u_i v_j = \tilde{T}_{ij} + \frac{1}{2}\epsilon_{ijk}V_k + \frac{1}{3}\delta_{ij}S$$

separates the components of T_{ij} into subsets of 5, 3, 1 that transform amongst themselves under $SO(3)$ rotations, where

$$\tilde{T}_{ij} = \frac{1}{2}(u_i v_j + u_j v_i) - \frac{1}{3}\delta_{ij}u_k v_k \quad , \quad V_k = (\mathbf{u} \times \mathbf{v})_k \quad , \quad S = \mathbf{u} \cdot \mathbf{v}.$$

Explain the relation to the results that, if \mathcal{V}_j is the vector space for angular momentum j , then $\mathcal{V}_1 \otimes \mathcal{V}_1 \simeq \mathcal{V}_2 \oplus \mathcal{V}_1 \oplus \mathcal{V}_0$.

4. Three 3×3 matrices T_i are defined by $(T_i)_{jk} = -i\epsilon_{ijk}$. Prove the results ($a = |\mathbf{a}|$),

(i) $[T_i, T_j] = i\epsilon_{ijk}T_k,$

(ii) $(\mathbf{a} \cdot \mathbf{T})^3 = a^2 \mathbf{a} \cdot \mathbf{T},$

(iii) $\exp(-i\mathbf{a} \cdot \mathbf{T}) = 1 - i\mathbf{a} \cdot \mathbf{T} \frac{\sin a}{a} + (\mathbf{a} \cdot \mathbf{T})^2 \frac{\cos a - 1}{a^2}.$

What are the possible eigenvalues of $\mathbf{n} \cdot \mathbf{T}$ if \mathbf{n} is a unit vector?

5. Assuming that the Pauli matrices σ_i are traceless and satisfy $\text{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ show that for any 2×2 matrix C can be written as $C = \sigma_i \frac{1}{2}\text{tr}(\sigma_i C) + 1 \frac{1}{2}\text{tr}(C)$.

Explain how three-dimensional rotations may be described either by a matrix $A \in SU(2)$ or an orthogonal matrix, R , which are related by

$$A\sigma_j A^{-1} = \sigma_i R_{ij}.$$

Obtain an explicit formula for R_{ij} in terms of A and show that

$$R_{ii} = (\text{tr}(A))^2 - 1 \quad R_{ij}\sigma_i\sigma_j = 2\text{tr}(A)A - 1.$$

Show that there are two $SU(2)$ matrices associated with a rotation matrix.

If $[R_{ij}]$ describes a rotation about the z -axis through an angle θ show that $R_{ii} = 2\cos\theta + 1$ and obtain the associated $SU(2)$ matrices A .

6. Show that the spin-1/2 representation of the angular momentum operators is given by $J_i^{(\frac{1}{2})} = \sigma_i/2$ ($i = 1, 2, 3$), where σ_i are Pauli matrices.

A general three-dimensional rotation may be defined as a rotation around the 3 axis by an angle ψ followed by a rotation around the 2 axis by an angle θ followed by a further rotation around the 3 axis by an angle ϕ (ϕ, θ, ψ are Euler angles). This rotation is generated by the operator $U(R) = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\psi J_3}$. Show that the spin-1/2 representation matrices for this rotation have the form

$$D_{mm'}^{(\frac{1}{2})}(R) = e^{-i\phi m - i\psi m'} d_{mm'}^{(\frac{1}{2})}(\theta),$$

and determine the matrix $d^{(\frac{1}{2})}(\theta)$.

7. Show that

$$\text{tr}_{\nu_j}(e^{-i\theta J_3}) = \chi_j(\theta) = \frac{\sin(j + \frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

and that the characters $\chi_j(\theta)$ satisfy the orthogonality conditions

$$\int_0^{2\pi} d\theta \sin^2\frac{1}{2}\theta \chi_j(\theta)\chi_{j'}(\theta) = \pi \delta_{jj'}.$$

8. Let $C_{i_1\dots i_l}$ be a symmetrical traceless tensor of rank l . Let $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ be a three-dimensional unit vector giving a point on the unit sphere. Define a tangential derivative such that $\nabla_i \hat{x}_j = \delta_{ij} - \hat{x}_i \hat{x}_j$. For the spherical harmonic $Y_l(\hat{\mathbf{x}}) = C_{i_1\dots i_l} \hat{x}_{i_1} \dots \hat{x}_{i_l}$ show that

$$\nabla^2 Y_l(\hat{\mathbf{x}}) = -l(l+1) Y_l(\hat{\mathbf{x}}).$$

Show also $\partial^2 Y_l(\mathbf{x}) = 0$ is equivalent to the traceless condition on $C_{i_1\dots i_l}$. Verify that $\partial^2(x_i Y_l(\mathbf{x}) - \frac{1}{2l+1} \mathbf{x}^2 \partial_i Y_l(\mathbf{x})) = 0$ so this defines a symmetric traceless rank- $(l+1)$ tensor..

9. Explain the experimental results

$$2 \frac{d\sigma}{d\Omega}(np \rightarrow \pi^0 d) = \frac{d\sigma}{d\Omega}(pp \rightarrow \pi^+ d), \quad \frac{d\sigma}{d\Omega}(pd \rightarrow \pi^+ H_3) = 2 \frac{d\sigma}{d\Omega}(pd \rightarrow \pi^0 He_3),$$

where d is the deuteron and H_3, He_3 are the tritium, helium-3 nuclei, respectively.

10. Show that the six possible charge combinations for the decays $\Delta \rightarrow \pi + N$ separate into pairs whose amplitudes A are equal because of symmetry under $I_3 \rightarrow -I_3$ for all particles. Show that the decay widths, proportional to $|A|^2$, satisfy

$$2\Gamma_{\Delta^{++} \rightarrow \pi^+ p} = 3\Gamma_{\Delta^+ \rightarrow \pi^0 p} = 6\Gamma_{\Delta^0 \rightarrow \pi^- p}.$$