1. Consider the element \( U = \cos \frac{\alpha}{2} I - i \sin \frac{\alpha}{2} \sigma_1 \in SU(2) \). Find \( U^{-1} \). Calculate \( U \sigma_a U^{-1} \) for \( a = 1, 2, 3 \), and deduce that \( \text{Ad} U \), the rotation in \( SO(3) \) corresponding to \( U \), is a rotation by \( \alpha \) about the \( x_1 \)-axis.

2. Let \( \exp iH = U \). Show that if \( H \) is hermitian then \( U \) is unitary. Show also, that if \( H \) is traceless then \( \det U = 1 \). How do these results relate to the theorem that the exponential map \( X \rightarrow \exp X \) sends \( L(G) \), the Lie algebra of \( G \), to \( G \)?

3. Show that \( \exp -\frac{1}{2} i \alpha \cdot \sigma = \cos \frac{\alpha}{2} I - i \sin \frac{\alpha}{2} \hat{\alpha} \cdot \sigma \) where \( \alpha = \alpha \hat{\alpha} \). Deduce that in the region \(-\text{Tr}(X^2) < 2\pi^2\), the exponential map \( X \rightarrow \exp X \) from \( L(SU(2)) \) to \( SU(2) \) is 1-to-1, and onto almost all of \( SU(2) \). Describe the map for \(-\text{Tr}(X^2) = 2\pi^2\).

4. For a matrix Lie group \( G \), consider the action of \( G \) on itself by conjugation, defined by \( g' \rightarrow gg'g^{-1} \). Show that the eigenvalues of \( g' \) and \( gg'g^{-1} \) are the same for all \( g \), so the eigenvalues are invariants of an orbit.

Find the eigenvalues of the \( SU(2) \) matrix \( \cos \frac{\alpha}{2} I - i \sin \frac{\alpha}{2} \hat{\alpha} \cdot \sigma \) where \( \alpha = \alpha \hat{\alpha} \). Deduce the orbit structure of \( SU(2) \) under the action of \( SU(2) \) on itself by conjugation.

5. Consider the two \( SU(2) \) elements \( g = a_0 I + i a \cdot \sigma \) with \( a_0^2 + a \cdot a = 1 \), and \( g' = b_0 I + i b \cdot \sigma \) with \( b_0^2 + b \cdot b = 1 \). Recall that \( \sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k \). Calculate \( gg' \) and deduce that the left action of \( g \) on \( SU(2) \), \( g' \rightarrow gg' \), defines a \( 4 \times 4 \) matrix

\[
g_L = \begin{pmatrix}
    a_0 & -a_1 & -a_2 & -a_3 \\
    a_1 & * & * & * \\
    * & * & * & * \\
    * & * & * & *
\end{pmatrix}
\]

where the entries * are to be determined. Show that \( g_L \) is an \( O(4) \) matrix and that the determinant of \( g_L \) is \((a_0^2 + a \cdot a)^2 = 1\), so \( g_L \) is an \( SO(4) \) matrix.

In this way we have found the subgroup \( SU(2)_L \) of \( SO(4) \). By considering elements close to the identity, determine the Lie algebra \( L(SU(2)_L) \) as a subalgebra of \( L(SO(4)) \). Repeat the above calculations for the right action \( g' \rightarrow g'g^{-1} \), and hence identify the subgroup \( SU(2)_R \) of \( SO(4) \), and also the Lie algebra \( L(SU(2)_R) \) as a subalgebra of \( L(SO(4)) \). Show that \( L(SU(2)_L) \oplus L(SU(2)_R) = L(SO(4)) \), and that elements in the two summands mutually commute. [Hint: Think about the original actions.]

6. Verify that the set of matrices

\[
\begin{pmatrix}
    1 & a & b \\
    0 & 1 & c \\
    0 & 0 & 1
\end{pmatrix}
\]

for \( a, b, c \in \mathbb{R} \)
forms a matrix Lie group, G. What is the underlying manifold of G? Is the group abelian? Find the Lie algebra, L(G), and calculate the bracket of two general elements of it.

7. A useful basis for the Lie algebra of GL(n) consists of the $n^2$ matrices $T^{ij}$ ($1 \leq i, j \leq n$), where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

8. Show that if $D$ is a representation of a Lie group G, acting on a vector space V of dimension N, and $A$ a fixed invertible $N \times N$ matrix, then one may define another (equivalent) representation by the formula

$$\tilde{D}(g) = A D(g) A^{-1}.$$ 

Show that $D$ and $\tilde{D}$ are related by a change of basis in V. Re-express the formula for $\tilde{D}$ in the case that $A = D(g_0)$ for some fixed element $g_0 \in G$.

Show that if $d$ is the representation of L(G) associated to $D$, then $\tilde{d}$, defined by the formula $\tilde{d}(X) = A d(X) A^{-1}$, is the representation associated to $\tilde{D}$. Check that this is a representation. [$\tilde{d}$ is referred to again as a representation of L(G) equivalent to $d$.]

9. Let $D$ be a finite-dimensional representation of G acting on V, and $(\ , \ )$ a positive definite inner product on V invariant under G, i.e.

$$(D(g)u, D(g)v) = (u, v) : u, v \in V , g \in G.$$ 

[\text{D is said to be unitary in this case.}]

Let $W$ be an invariant subspace of V. Show that $W_{\perp}$, the orthogonal complement of $W$ in V, is also invariant.

Deduce that $D$ is totally reducible to irreducible pieces.

10. (a) Let L be a real Lie algebra (i.e. there is a basis $T_i : i = 1, \ldots, n$ with real structure constants $c_{ijk}$). Suppose $d$ is a representation of L. Write down the algebraic equations that the matrices $d(T_i)$ must satisfy. Show that the complex conjugate matrices $d(T_i)^*$ also define a representation of L.

(b) Show that the fundamental representation $d(T_a) = -\frac{1}{2}i\sigma_a$ and its complex conjugate $\tilde{d}(T_a) = \frac{1}{2}i(\sigma_a)^*$ are equivalent representations of L(SU(2)), with the commutation relations (Lie brackets) $[T_a, T_b] = \epsilon_{abc} T_c$. Is your matrix $A$ (as in Q.8) in SU(2)? If not, could it be?

Show that the weights of the representations $d$ and $\tilde{d}$ are the same. [The weights of $d$ are the eigenvalues of $id(T_3)$, and similarly for $\tilde{d}$.]

11. (a) Show using the Jacobi identity that the representation $\text{ad}$ of L(G), defined by $(\text{ad} X)Y = [X, Y]$, is indeed a representation.

(b) Show using the Baker–Campbell–Hausdorff formula that if $d$ is a representation of L(G), then one can sensibly attempt to construct a representation $D$ of G by the formula $D(\exp X) = \exp(d(X))$. Could there be problems with this construction?