Symmetry and Particle Physics, 2

1. Let $T^{\alpha_1\ldots\alpha_{2j}}$ be a symmetric $SU(2)$ tensor for $j=\frac{1}{2},1,\frac{3}{2},\ldots$. Show that the action of the spin operator is given by

$$\mathbf{S}^{\alpha_1\ldots\alpha_{2j}}{}_{\beta_1\ldots\beta_{2j}}T^{\beta_1\ldots\beta_{2j}}=\sum_{i=1}^{2j}\tfrac{1}{2}(\sigma)^{\alpha_i}{}_{\beta}T^{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_{2j}}\ ,$$

where σ are the Pauli matrices.

Define for $m=-j,-j+1,\ldots,j$

$$T^{(jm)}=\big[(j+m)!(j-m)!\big]^{-\frac{1}{2}}T^{\underbrace{1\ldots1}_{j+m}\underbrace{2\ldots2}_{j-m}}.$$

Calculate $S_\pm T^{(jm)}$ and $S_3T^{(jm)}$. Show that $\bar{T}_{\alpha_1\ldots\alpha_{2j}}T^{\alpha_1\ldots\alpha_{2j}}=(2j)!\sum_m T^{(jm)*}T^{(jm)}$ for $\bar{T}_{\alpha_1\ldots\alpha_{2j}}$ the conjugate tensor.

2. A field $\phi(x)$ transforms under the action of a Poincaré transformation (Λ,a) such that $U[\Lambda,a]\phi(x)U[\Lambda,a]^{-1}=\phi(\Lambda x+a)$. For an infinitesimal transformation, $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu$ and correspondingly $U[\Lambda,a]=1-i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}-ia^\mu P_\mu$ show that

$$[M_{\mu\nu},\phi(x)]=-i(x_\mu\partial_\nu-x_\nu\partial_\mu)\phi(x)\,,\qquad [P_\mu,\phi(x)]=i\partial_\mu\phi(x)\,.$$

Verify that $M_{\mu\nu}\to i(x_\mu\partial_\nu-x_\nu\partial_\mu)$ and $P_\mu\to -i\partial_\mu$ satisfy the algebra for $[M_{\mu\nu},M_{\sigma\rho}]$ and $[M_{\mu\nu},P_\sigma]$ expected for the Poincaré group.

3. Define $B(\theta,\mathbf{n})\in Sl(2,\mathbb{C})$ by

$$B(\theta,\mathbf{n})=\cosh\tfrac{1}{2}\theta+\sigma\cdot\mathbf{n}\ \sinh\tfrac{1}{2}\theta\,,\qquad \mathbf{n}^2=1\,,$$

Show that this corresponds to a Lorentz boost with velocity $\mathbf{v}=\tanh\theta\,\mathbf{n}$. Show that

$$\big(1+\tfrac{1}{2}\sigma\cdot\delta\mathbf{v}\big)B(\theta,\mathbf{n})=B(\theta',\mathbf{n}')R\,,$$

where, to first order in $\delta\mathbf{v}$,

$$\theta'=\theta+\delta\mathbf{v}\cdot\mathbf{n}\,,\quad \mathbf{n}'=\mathbf{n}+\coth\theta(\delta\mathbf{v}-\mathbf{n}\,\mathbf{n}\cdot\delta\mathbf{v})\,,$$

and R is an infinitesimal rotation given by

$$R=1+\tanh\tfrac{1}{2}\theta\ \tfrac{1}{2}i\,(\delta\mathbf{v}\times\mathbf{n})\cdot\sigma=1+\tfrac{\gamma}{\gamma+1}\ \tfrac{1}{2}i\,(\delta\mathbf{v}\times\mathbf{v})\cdot\sigma\,,\qquad \gamma=(1-\mathbf{v}^2)^{-\frac{1}{2}}\,.$$

Show that $\mathbf{v}'=\mathbf{v}+\delta\mathbf{v}-\mathbf{v}\,\mathbf{v}\cdot\delta\mathbf{v}$. [Note $\sigma\cdot\mathbf{a}\,\sigma\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}\,1+i\sigma\cdot(\mathbf{a}\times\mathbf{b})$.]

4. The group of four dimensional space-time symmetries may be generalised to conformal transformations $x \to x'$ defined by the requirement

$$\mathrm{d}x'^2 = \Omega(x)^2 \mathrm{d}x^2 \,,$$

where $\mathrm{d}x^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ (recall that Lorentz invariance requires $\Omega = 1$). For an infinitesimal transformation $x'^\mu = x^\mu + f^\mu(x)$, $\Omega(x)^2 = 1 + 2\sigma(x)$. Show that

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 2\sigma\, g_{\mu\nu} \quad \Rightarrow \quad 4\sigma = \partial \cdot f \,.$$

Hence obtain

$$4\,\partial_\sigma \partial_\mu f_\nu = g_{\mu\nu}\,\partial_\sigma \partial \cdot f + g_{\sigma\nu}\,\partial_\mu \partial \cdot f - g_{\sigma\mu}\,\partial_\nu \partial \cdot f \,.$$

From this obtain $2\partial_\sigma \partial_\mu \partial \cdot f = -g_{\sigma\mu}\,\partial^2 \partial \cdot f$ and hence show that $\partial_\sigma \partial_\mu \partial \cdot f = 0$. Why does it then follow that $f_\mu(x)$ can only be quadratic in x? Show that $f^\mu(x)$ must have the general form

$$f^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2b\cdot x\, x^\mu \,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \,.$$

Show also that an inversion $x'^\mu = x^\mu / x^2$ is a conformal transformation. Calculate the finite conformal transformation obtained by an inversion followed by a translation by b^μ followed by another inversion and show that it is compatible with the result for $f^\mu(x)$.

5. A four-dimensional space is defined in terms of 6-vectors $\eta^A = (\eta^\mu, \eta^+, \eta^-)$, $\mu = 0,1,2,3$, subject to the relations

$$\eta \cdot \eta = g_{AB}\eta^A \eta^B = g_{\mu\nu}\eta^\mu \eta^\nu - \eta^+ \eta^- = 0 \,, \qquad \eta^A \sim C\eta^A \,.$$

Using $\eta^\pm = \eta^4 \pm \eta^5$ show that this space is invariant under transformations $\eta^A \to G^A{}_B \eta^B$ where $[G^A{}_B]$ are matrices belonging to the group $SO(4,2)$. For an infinitesimal transformation, $G^A{}_B = \delta^A{}_B + \omega^A{}_B$, show that $\omega^A{}_B$ may be decomposed in the form

$$\left[\omega^A{}_B\right] = \begin{pmatrix} \omega^\mu{}_\nu & a^\mu & b^\mu \\ 2b_\nu & -\lambda & 0 \\ 2a_\nu & 0 & \lambda \end{pmatrix} , \qquad \omega_{\mu\nu} = -\omega_{\nu\mu} \,.$$

Suppose, for $\eta^+ \neq 0$, $\eta^A = \eta^+(x^\mu, 1, x^2)$. Using $\delta\eta^A = \omega^A{}_B \eta^B$ determine the corresponding δx^μ. What transformation corresponds to $\eta^+ \leftrightarrow \eta^-$? For four points x_i, $i = 1,2,3,4$, calculate $\eta_1 \cdot \eta_2\, \eta_3 \cdot \eta_4 / (\eta_1 \cdot \eta_3\, \eta_2 \cdot \eta_4)$. Why is this a conformal invariant?

6. Consider the subgroup of the Galilean group corresponding to translations and boosts where

$$t' = t + b \,, \qquad \mathbf{x}' = \mathbf{x} + \mathbf{a} + \mathbf{v}t \,.$$

Denoting the corresponding group element by $(b, \mathbf{a}, \mathbf{v})$ work out the group multiplication law and show that $(b_2, \mathbf{a}_2, \mathbf{v}_2)^{-1}(b_1, \mathbf{a}_1, \mathbf{v}_1)^{-1}(b_2, \mathbf{a}_2, \mathbf{v}_2)(b_1, \mathbf{a}_1, \mathbf{v}_1) = (0, 0, b_1\mathbf{v}_2 - b_2\mathbf{v}_1)$. Suppose, for infinitesimal $b, \mathbf{a}, \mathbf{v}$, the associated unitary operator has the form

$$U[b, \mathbf{a}, \mathbf{v}] = 1 + ib\,H - i\mathbf{a}\cdot\mathbf{P} + i\mathbf{v}\cdot\mathbf{K} \,,$$

and determine the corresponding commutators. Require $U[0, \mathbf{a}, \mathbf{v}] = T[\mathbf{a}]U_B[\mathbf{v}]$ for general $\mathbf{a}, \mathbf{v}$ and assume $T[\mathbf{a}]U_B[\mathbf{v}] = e^{im\,\mathbf{a}\cdot\mathbf{v}}U_B[\mathbf{v}]T[\mathbf{a}]$ for some positive constant m. Show that this leads to the modified commutation relation $[K_i, P_j] = im\,\delta_{ij}$.

Suppose $\mathbf{P}|\mathbf{0}\rangle = \mathbf{0}$ and define $|m\mathbf{v}\rangle = U_B[\mathbf{v}]|\mathbf{0}\rangle$. Show that is an eigenvector of $\mathbf{P}$ and that $H|\mathbf{p}\rangle = (E_0 + \frac{\mathbf{p}^2}{2m})|\mathbf{p}\rangle$.