Symmetries, Fields and Particles. Examples 2.

1. Let X be an element of the Lie algebra of a matrix Lie group G. Consider the curve in G defined by $g(t) = \text{Exp}(tX)$ where t is a real parameter. Show that,

$$g(t_1)g(t_2) = g(t_2)g(t_1) = g(t_1 + t_2)$$

for all values of t_1 and t_2. Assuming there is no non-zero value of t for which $g(t)$ is equal to the identity, show that the curve defines a Lie subgroup of G which is isomorphic to $(\mathbb{R}, +)$ (ie the real line with addition as the group multiplication law).

2. Verify the Baker–Campbell–Hausdorff (BCH) formula

$$\exp X \cdot \exp Y = \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \dots\right)$$

to the order shown.

3. Let $g(t) = \text{Exp}\; it\sigma_1$. By evaluating $g(t)$ as a matrix, show that $\{g(t) : 0 \leq t \leq 2\pi\}$ is a 1-parameter subgroup of SU(2). Describe geometrically how this subgroup sits inside the group manifold of SU(2).

4. For each $A \in SU(2)$, we define a 3×3 matrix with entries,

$$R(A)_{ij} = \tfrac{1}{2}\text{tr}_2\left(\sigma_i A \sigma_j A^\dagger\right) \qquad - (1)$$

for $i, j = 1, 2, 3$. Using the Pauli matrix identity,

$$\sum_{j=1}^{3} (\sigma_j)_{\alpha\beta}(\sigma_j)_{\delta\gamma} = 2\delta_{\alpha\gamma}\delta_{\delta\beta} - \delta_{\alpha\beta}\delta_{\delta\gamma}$$

show that $R(A)$ is an element of $SO(3)$. Hint: You may appeal to the fact that $SU(2) \simeq S^3$ is connected to check that $\text{det}R = 1$.

*(Harder) Check that we may invert Eqn (1) to express $A \in SU(2)$ in terms of $R \in SO(3)$ by setting,

$$A = \pm\frac{(I_2 + \sigma_i R_{ij}\sigma_j)}{2\sqrt{1 + \text{Tr}_3 R}}$$

where I_2 is the 2×2 unit matrix and summation over repeated indices is implied.*

This map provides an isomorphism between $SO(3)$ and $SU(2)/\mathbb{Z}_2$.

5. Let G be a matrix Lie group and d a representation of its Lie algebra $L(G)$. For group elements $g = \text{Exp}\, X$ with $X \in L(G)$ we define $D(g) = \text{Exp}(d(X))$. Using the BCH formula show that

$$D(g_1 g_2) = D(g_1)D(g_2)$$

for all group elements $g_1 = \mathrm{Exp}\, X_1$ and $g_2 = \mathrm{Exp}\, X_2$ with $X_1,\ X_2 \in L(G)$. Can we conclude that D is a representation of G? Explain your answer.

6. (a) Let L be a real Lie algebra (i.e. there is a basis $T^a : a = 1, \ldots, n = \mathrm{Dim}\, L$ with real structure constants f^{ab}_c). Suppose R is a representation of L. Write down the algebraic equations that the matrices $R(T^a)$ must satisfy. Show that the complex conjugate matrices $\bar{R}(T^a) = R(T^a)^*$ also define a representation of L.

(b) Show that the fundamental representation $R(T^a) = -\frac{1}{2} i\sigma_a$, with $a = 1, 2, 3$ and its complex conjugate $\bar{R}(T^a) = \frac{1}{2} i(\sigma_a)^*$ are *equivalent* representations of L(SU(2)).

Show that the *weights* of the $L(SU(2))$ representations R and $\bar{R}$ are the same.

7. Let R_1 and R_2 be two representations of a Lie algebra L with representation spaces V_1 and V_2 respectively. The tensor product of R_1 and R_2 is defined by the formula,

$$R_1 \otimes R_2(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X)$$

where I_1 and I_2 are the identity maps on V_1 and V_2 respectively. Show that $R_1 \otimes R_2$ is a representation of L with representation space $V_1 \otimes V_2$.

8. Find the multiplicity of each weight of the tensor product $R_N \otimes R_M$ where N and M are nonegative integers. Here R_Λ denotes the irreducible representation of $L(SU(2))$ with highest weight Λ defined in the lectures. Deduce the Clebsch-Gordon decomposition,

$$R_N \otimes R_M = R_{|N-M|} \oplus R_{|N-M|+2} \oplus \cdots \oplus R_{N+M}\,.$$

Verify that the dimensions of the reducible representation defined on the two sides of the equation are the same.

9 a) A Lie algebra is semi-simple if it has no abelian ideals. A semi-simple Lie algebra has a non-degenerate Killing form.

Consider the Lie algebra defined in Sheet 1, Question 8. Is it semi-simple? Find its Killing form explicitly and determine whether it is degenerate.

b) A finite-dimensional real Lie algebra is of compact type if it has a basis $\{T^a\}$ in which the Killing form has components, $\kappa^{ab} = -\kappa\,\delta^{ab}$ for some positive constant κ. Let L be a real Lie algebra of compact type and let I be an ideal of L. Let $\mathrm{I}_\perp$ denote the orthogonal complement of I with respect to the Killing form κ. ($Y \in \mathrm{I}_\perp$ if and only if $\kappa(X, Y) = 0$ for all $X \in \mathrm{I}$.) By considering $\kappa(X, [Y, Z])$, where $X \in \mathrm{I}$, $Y \in \mathrm{L}$ and $Z \in \mathrm{I}_\perp$, show that $\mathrm{I}_\perp$ is an ideal and that

$$\mathrm{L} = \mathrm{I} \oplus \mathrm{I}_\perp$$

where the summands mutually commute.

Deduce that any semi-simple complex Lie algebra M of finite dimension is the direct sum of a finite number of simple Lie algebras. You may use the fact stated in the lectures that any such M has a real form of compact type.