

### Symmetry and Particle Physics, 3

1. Define  $\sigma_\mu = (1, \sigma)$  and  $\bar{\sigma}_\mu = (1, -\sigma)$ , in terms of the three independent  $2 \times 2$  Pauli matrices  $\sigma$ . Show

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2g_{\mu\nu} 1, \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\beta\dot{\beta}}.$$

Without assuming any explicit expression for the Pauli matrices show that  $\frac{1}{2}\text{tr}(\sigma_\mu x^\mu \bar{\sigma}_\nu x^\nu) = \det(\sigma_\mu x^\mu) = g_{\mu\nu} x^\mu x^\nu$ .

Explain why a matrix  $A = [A_{\alpha\dot{\alpha}}]$  can be written as  $A = \sigma_\mu \frac{1}{2}\text{tr}(\bar{\sigma}^\mu A)$ . Verify that  $B = [B_\alpha^\beta]$  can be written as

$$B = 1 \frac{1}{2}\text{tr}(B) - \sigma_{[\mu} \bar{\sigma}_{\nu]} \frac{1}{8}\text{tr}(\sigma^{[\mu} \bar{\sigma}^{\nu]} B).$$

2. A Lie group has group elements  $g(a)$  depending on group parameters  $a^r$ , with  $g(0) = e$ , the identity, and under group multiplication  $g(a)g(b) = g(\varphi^r(a, b))$  for some  $\varphi^r(a, b)$ . Let  $g(a)^{-1} = g(\bar{a})$  where  $\varphi(\bar{a}, a) = 0$ . Why must  $\varphi^r(a, 0) = a^r$ ,  $\varphi^r(0, b) = b^r$ ? Assume  $\varphi^r(a, b)$  is expanded near the origin according to

$$\varphi^a(a, b) = a^a + b^a + c^a_{bc} a^b b^c + O(a^2 b, a b^2).$$

Use this to find  $\bar{a}(a)$  for  $a$  small. Let  $g(d) = g(a)^{-1}g(b)^{-1}g(a)g(b)$  and show that for  $a, b$  small  $d^a = f^a_{bc} a^b b^c$  where  $f^a_{bc} = c^a_{bc} - c^a_{cb}$ . Using an expansion to one higher order show that the associativity condition  $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$  leads to the Jacobi identity.

Assume the Lie group has generators  $T_a$  satisfying  $[T_a, T_b] = f^c_{ab} T_c$ . For an element of the Lie algebra  $a^a T_a$  there is an associated group element given by  $g(a) = \exp(a^a T_a)$ . Use the Baker-Campbell-Hausdorff formula for  $\exp C = \exp A \exp B$  in the form  $C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) + \dots$  to obtain  $\varphi(a, b)$  for  $a, b$  small and verify that this is compatible with the general expansion of  $\varphi$ .

3. Using the same notation as the previous question, where  $c^r = \varphi^r(a, b)$ , obtain

$$\frac{\partial c^r}{\partial b^s} = \lambda_s^a(b) \mu_a^r(c), \quad \mu_a^r(c) = \left. \frac{\partial}{\partial b^a} \varphi^r(c, b) \right|_{b=0}, \quad \mu_a^r(c) \lambda_r^b(c) = \delta_a^b.$$

Show that the equation for the structure constants  $f^a_{bc}$  may also be expressed as

$$\frac{\partial}{\partial b^r} \lambda_s^a(b) - \frac{\partial}{\partial b^s} \lambda_r^a(b) = -f^a_{bc} \lambda_r^b(b) \lambda_s^c(b).$$

[For those familiar with differential forms:

Defining the differential form  $\omega^a(b) = db^r \lambda_r^a(b)$  verify that the above equation is equivalent, if  $d$  is the exterior derivative, to  $d\omega^a(b) = -\frac{1}{2} f^a_{bc} \omega^b(b) \wedge \omega^c(b)$ . Show that the Jacobi identity for  $f^a_{bc}$  is implied by  $d^2\omega^a(b) = 0$ . If  $g$  are matrices forming a representation,  $g(c) = g(a)g(b)$ , show that  $g(b)^{-1}dg(b) = \omega^a(b)t_a$  where  $\{t_a\}$  are matrix generators satisfying  $[t_a, t_b] = f^c_{ab}t_c$ .]

4. An  $SU(2)$  matrix may be represented by

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1,$$

where  $\bar{\alpha} = \alpha^*$ ,  $\bar{\beta} = \beta^*$ . Let  $\alpha = |\alpha| e^{i\lambda}$  where  $|\alpha|^2 = 1 - |\beta|^2$  so that  $A$  is regarded as depending on the real parameter  $\lambda$  and the complex parameter  $\beta$ . Determine  $d\lambda$ ,  $d\beta$ ,  $d\bar{\beta}$  in terms of infinitesimal  $\phi$ ,  $\theta$ ,  $\bar{\theta}$  where

$$\begin{pmatrix} \alpha + d\alpha & \beta + d\beta \\ -\bar{\beta} - d\bar{\beta} & \bar{\alpha} + d\bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 + i\phi & \theta \\ -\bar{\theta} & 1 - i\phi \end{pmatrix}.$$

Hence show that the invariant measure for  $SU(2)$  with these parameters is  $d\lambda d^2\beta$ . What are the ranges of  $\lambda$  and  $\beta$ ? Calculate  $\int d\lambda d^2\beta$ . (For  $d\beta = d\beta_1 + id\beta_2$ ,  $d^2\beta = d\beta_1 d\beta_2$ .)

5. Consider two  $SU(2)$  algebras,  $I_{\pm}, I_3$  and  $U_{\pm}, U_3$ , where  $[I_3, I_{\pm}] = \pm I_{\pm}$ ,  $[I_+, I_-] = 2I_3$  with  $I_3^{\dagger} = I_3$ ,  $I_+^{\dagger} = I_-$ , and similarly for  $I \rightarrow U$ . Assume

$$[I_-, U_+] = 0, \quad [I_3, U_+] = -\frac{1}{2}U_+, \quad [U_-, I_+] = 0 \quad [U_3, I_+] = -\frac{1}{2}I_+.$$

Define  $V_+ = [I_+, U_+]$  and  $V_- = V_+^{\dagger}$ . Explain why  $[I_+, V_+] = [U_+, V_+] = 0$ . Evaluate  $[V_+, V_-]$  and show that  $V_{\pm}, V_3 = I_3 + U_3$  form a  $SU(2)$  algebra. Let  $U_3 = -\frac{1}{2}I_3 + \frac{3}{2}Y$ . Work out the various commutators involving  $Y$ .

Explain how the **I**-spin and **U**-spin operators with the above relations generate a  $SU(3)$  algebra.

6. A simple Lie algebra  $\mathcal{L}$  has  $\dim \mathcal{L}$  elements  $X_i$  satisfying  $[X_i, X_j] = if_{ijk}X_k$  where  $f_{ijk}$  is totally antisymmetric. Let  $T_i$  be matrices forming a representation  $R$  of  $\mathcal{L}$  of dimension  $\dim R$  and assume  $T_i T_i = C_R 1$ . Define

$$\langle X_i, X_j \rangle = \text{tr}(T_i T_j) \frac{\dim \mathcal{L}}{C_R \dim R}.$$

(a) Evaluate  $\langle J_3, J_3 \rangle$  in the  $j$ -th irreducible representation of  $SU(2)$  and show that the result is independent of  $j$ .

(b) For  $SU(3)$  show that the representation given by  $T_i = \frac{1}{2}\lambda_i$  (where  $\lambda_i$  are the Gell-Mann matrices) gives the same value for  $\langle X_i, X_j \rangle$  as does the adjoint representation  $(T_i^{\text{ad}})_{jk} = -if_{ijk}$ .

7. Let  $\{T^i_j\}$  be  $n \times n$  matrices such that  $T^i_j$  has a 1 in the  $i$ 'th row and  $j$ 'th column and is zero otherwise. Show that they satisfy the Lie algebra

$$[T^i_j, T^k_l] = \delta^k_j T^i_l - \delta^i_l T^k_j.$$

Define  $X = T^i_j X^j_i$  with arbitrary components  $X^j_i$ . Determine the adjoint matrix  $(X^{\text{ad}})^n_{m, k_l}$  by

$$[X, T^k_l] = T^m_n (X^{\text{ad}})^n_{m, k_l},$$

and show that

$$\kappa(X, Y) = \text{tr}(X^{\text{ad}} Y^{\text{ad}}) = 2(n \sum_{i,j} X^j_i Y^i_j - \sum_i X^i_i \sum_j Y^j_j).$$

Show that  $1 + \epsilon X \in U(n)$  for infinitesimal  $\epsilon$  if  $(X^j_i)^* = -X^i_j$ . Hence show that in this case

$$\kappa(X, X) = -2n \sum_{i,j} |\hat{X}^j_i|^2, \quad \hat{X}^j_i = X^j_i - \frac{1}{n} \delta^j_i \sum_k X^k_k,$$

and therefore  $\kappa(X, X) = 0 \Leftrightarrow X^{\text{ad}} = 0$ . What restrictions must be made for  $SU(N)$  and verify that in this case the generators satisfy  $\kappa(X, X) < 0$  so the group is semi-simple.

8. For a group with a Lie algebra with a basis  $\{T_a\}$  such that  $[T_a, T_b] = f^c_{ab} T_c$  let  $g_{ab} = \langle T_a, T_b \rangle$  where  $\langle \cdot, \cdot \rangle$  is an invariant symmetric bilinear form so that  $\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle$ . Show that  $f_{abc} = g_{ad} f^d_{bc}$  is totally antisymmetric. If  $D_{\mu}$  is an appropriate covariant derivative involving a gauge field  $A_{\mu}^a$  verify

$$\partial_{\mu} \langle X(x), Y(x) \rangle = \langle D_{\mu} X(x), Y(x) \rangle + \langle X(x), D_{\mu} Y(x) \rangle.$$

Let  $T^{\mu}_{\nu} = \langle F^{\mu\sigma}, F_{\nu\sigma} \rangle - \frac{1}{4} \delta^{\mu}_{\nu} \langle F^{\sigma\rho}, F_{\sigma\rho} \rangle$ . Show, using the Bianchi identity,  $\partial_{\mu} T^{\mu}_{\nu} = \langle D_{\mu} F^{\mu\sigma}, F_{\nu\sigma} \rangle$ . For a variation  $\delta A_{\mu}^a$  obtain also  $\delta \frac{1}{4} \epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \partial_{\mu} \epsilon^{\mu\nu\sigma\rho} \langle \delta A_{\nu}, F_{\sigma\rho} \rangle$ . By letting  $A_{\mu} \rightarrow t A_{\mu}$  and differentiating with respect to  $t$  and then integrating show that

$$\frac{1}{4} \epsilon^{\mu\nu\sigma\rho} \langle F_{\mu\nu}, F_{\sigma\rho} \rangle = \epsilon^{\mu\nu\sigma\rho} \partial_{\mu} \langle A_{\nu}, \partial_{\sigma} A_{\rho} + \frac{1}{3} [A_{\sigma}, A_{\rho}] \rangle.$$

9. With notation as in the previous question define a three dimensional Lagrangian

$$\mathcal{L} = \epsilon^{\mu\nu\rho} \left( g_{ab} A_{\mu}^a \partial_{\nu} A_{\rho}^b + \frac{1}{3} f_{abc} A_{\mu}^a A_{\nu}^b A_{\rho}^c \right).$$

For a gauge transformation  $\delta A_{\mu}^a = -\partial_{\mu} \lambda^a - f^a_{bc} A_{\mu}^b \lambda^c$  show that  $\delta \mathcal{L} = -\partial_{\mu} (\epsilon^{\mu\nu\rho} g_{ab} \lambda^a \partial_{\nu} A_{\rho}^b)$  so that  $\int d^3x \mathcal{L}$  is invariant.