Examples Sheet 3

1. Schur's Lemma is "Let D be an irreducible representation of a (Lie) group G acting on a complex vector space V. Let A be an operator acting on V which commutes with the action of G, that is, $AD(g) = D(g)A$ for all $g \in G$. Then $A = \lambda I_V$, where λ is a constant and I_V is the unit operator."

   Prove this by showing that any eigenspace of A is an invariant subspace of V, and that there is therefore precisely one eigenspace of A which is the whole of V, and that this gives the desired result.

2. The following multiplication rule will be useful in this question (cf. Sheet 1, 4c):

$$(aI + \mathbf{b}\cdot\boldsymbol{\sigma})(a'I + \mathbf{b}'\cdot\boldsymbol{\sigma}) = (aa' + \mathbf{b}\cdot\mathbf{b}')I + (a\mathbf{b}' + a'\mathbf{b} + i\mathbf{b}\times\mathbf{b}')\cdot\boldsymbol{\sigma}\,.$$

   (a) Show how $B(\psi, \mathbf{n}) \in SL(2, \mathbb{C})$, where

$$B(\psi, \mathbf{n}) = I\cosh\frac{\psi}{2} + \sigma\cdot\mathbf{n}\ \sinh\frac{\psi}{2}\,, \qquad \mathbf{n}^2 = 1\,,$$

   corresponds to a Lorentz boost with velocity $\mathbf{v} = \tanh\psi\ \mathbf{n}$.

   (b) Show that

$$\big(1 + \frac{1}{2}\sigma\cdot\delta\mathbf{v}\big)B(\psi, \mathbf{n}) = B(\psi', \mathbf{n}')R,$$

   where, to first order in $\delta\mathbf{v}$,

$$\psi' = \psi + \delta\mathbf{v}\cdot\mathbf{n}, \quad \mathbf{n}' = \mathbf{n} + \coth\psi(\delta\mathbf{v} - \mathbf{n}\,\mathbf{n}\cdot\delta\mathbf{v}),$$

   and R is an infinitesimal rotation given by

$$R = 1 + \frac{i}{2}\tanh\frac{\psi}{2}(\delta\mathbf{v}\times\mathbf{n})\cdot\sigma = 1 + \frac{i}{2}\frac{\gamma}{\gamma+1}(\delta\mathbf{v}\times\mathbf{v})\cdot\sigma, \qquad \gamma = (1-\mathbf{v}^2)^{-\frac{1}{2}}.$$

   (c) Show that we must have $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v} - \mathbf{v}\,\mathbf{v}\cdot\delta\mathbf{v}$.

   (d) By considering boosts by velocities $\mathbf{v}, \mathbf{w}$ followed by boosts by $-\mathbf{w}$, $-\mathbf{v}$, find a physical interpetation of this question.

3. A field $\phi(x)$ transforms under the action of a Poincaré transformation (Λ, a) such that $U[\Lambda, a]\phi(x)U[\Lambda, a]^{-1} = \phi(\Lambda x + a)$. For an infinitesimal transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and correspondingly (in physics conventions) $U[\Lambda, a] = 1 - i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} - ia^\mu P_\mu$.

   (a) Show that

$$[M_{\mu\nu}, \phi(x)] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi(x)\,, \qquad [P_\mu, \phi(x)] = i\partial_\mu\phi(x)\,.$$

(b) Verify that $M_{\mu\nu} \to i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ and $P_\mu \to -i\partial_\mu$ satisfy the algebra for $[M_{\mu\nu}, M_{\sigma\rho}]$ and $[M_{\mu\nu}, P_\sigma]$ expected for the Poincaré group.

4. Consider the little group with standard momentum $k^\mu = (\ell, 0, 0, \ell)$, for some fixed $\ell > 0$, that is, the subgroup of proper, orthochronous Lorentz transformations which leaves k^μ invariant.

   (a) Show how the generators of the little group are related to the generators of $SO(1,3)^\uparrow$. [Hint: It will be convenient to define $E_1 := K_1 - J_2$ and $E_2 := K_2 + J_1$.] Find the structure constants of the corresponding Lie algebra and determine whether it is semisimple. [Note: this group is $ISO(2)$, the isometry group of the plane, or the 2-dimensional Euclidean group.]

   (b) Prove that, for appropriately normalized generators,
   $$e^{\theta J_3}(a_1 E_1 + a_2 E_2)e^{-\theta J_3} = \alpha_1(\theta)E_1 + \alpha_2(\theta)E_2\,,$$
   where $\theta, a_1, a_2 \in \mathbb{R}$ and
   $$\begin{pmatrix}\alpha_1(\theta)\\ \alpha_2(\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}.$$

   (c) Defining a unitary operator $O[\theta, a_1, a_2] = e^{a_1 E_1 + a_2 E_2}\, e^{\theta J_3}$, show that
   $$O[\theta', a_1', a_2']\, O[\theta, a_1, a_2] = O[\theta' + \theta,\ \alpha_1(\theta) + a_1',\ \alpha_2(\theta') + a_2']\,. \tag{1}$$
   Deduce from this that $ISO(2)$ is isomorphic to $SO(2) \ltimes T^2$, where T^2 is the 2-dimensional translation group.

   (d) Take $|k, e_1, e_2\rangle$ to be an eigenvector of E_1 and E_2 with eigenvalues e_1 and e_2, respectively. Show that (1) implies there are a continuum of eigenvalues for E_1 and E_2. Given that massless states like neutrinos do not have a continuous internal degree-of-freedom, what does that imply about physically allowed values e_1 and e_2?

5. Show that there is a choice of basis for $L(SO(4))$ consisting of 4×4 antisymmetric matrices that contain precisely two non-zero entries: 1 and -1. Evaluate the commutation relations of these generators. By choosing a new basis consisting of sums and differences of pairs of $L(SO(3))$ generators, show that $L(SO(4)) \cong L(SO(3)) \oplus L(SO(3))$.

6. Let $\{T^i{}_j\}$ be $n \times n$ matrices such that $T^i{}_j$ has a 1 in the i'th row and j'th column and is zero otherwise.

   (a) Show that they satisfy the Lie algebra
   $$[T^i{}_j, T^k{}_l] = \delta^k{}_j T^i{}_l - \delta^i{}_l T^k{}_j.$$

   (b) Define $X = T^i{}_j X^j{}_i$ with arbitrary components $X^j{}_i$. Determine the adjoint matrix $(X^{\mathrm{ad}})^n{}_m{}^k{}_l$ by
   $$[X, T^k{}_l] = T^m{}_n (X^{\mathrm{ad}})^n{}_m{}^k{}_l,$$
   and show that
   $$\kappa(X,Y) = \mathrm{Tr}\big(X^{\mathrm{ad}} Y^{\mathrm{ad}}\big) = 2\big(n\textstyle\sum_{i,j} X^j{}_i Y^i{}_j - \sum_i X^i{}_i \sum_j Y^j{}_j\big).$$

(c) Show that $1 + \epsilon X \in U(n)$ for infinitesimal ϵ if $(X^j{}_i)^* = -X^i{}_j$.

(d) Hence show that in this case

$$\kappa(X,X) = -2n\textstyle\sum_{i,j}|\hat{X}^j{}_i|^2, \qquad \hat{X}^j{}_i = X^j{}_i - \frac{1}{n}\delta^j{}_i\textstyle\sum_k X^k{}_k,$$

and therefore $\kappa(X,X) = 0 \;\Leftrightarrow\; X^{\mathrm{ad}} = 0$.

(e) What restrictions must be made for $SU(n)$ and verify that in this case the generators satisfy $\kappa(X,X) < 0$ so the group is semi-simple?

7. For a simple Lie algebra $\mathfrak{g}$, with elements X_a such that $[X_a, X_b] = f_{abc}X_c$ where f_{abc} is totally antisymmetric, let $\tilde{T}_a$ be matrices forming a basis for representation R of $\mathfrak{g}$, and assume $\tilde{T}_a\tilde{T}_a = C_R I$. Define

$$\langle X_a, X_b\rangle = \mathrm{Tr}(\tilde{T}_a\tilde{T}_b)\frac{\dim\mathfrak{g}}{C_R\dim R}\,.$$

(a) Let $\mathfrak{g} = \mathfrak{su}(2)$. Evaluate $\langle J_3, J_3\rangle$ in the j-th irreducible representation of $\mathfrak{su}(2)$ and show that the result is independent of j.

(b) For $\mathfrak{su}(3)$ show that the Gell-Mann representation, $\tilde{T}_a = \frac{i}{2}\lambda_a$, where the Gell-Mann matrices λ_a are given below, gives the same value for $\langle X_a, X_b\rangle$ as does the adjoint representation $(T^{\mathrm{ad}}_a)_{bc} = f_{abc}$.
[The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\quad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad \lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\quad \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.]$$

(c) It can be shown that the Killing form on a simple Lie algebra is the unique symmetric bilinear form, up to an overall scalar multiple. How do you interpret your calculations above in relation to this fact?

8. The Lie algebra of $U(n)$ may be represented by a basis consisting first of the $n^2 - n$ off-diagonal matrices $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i \neq j$ and also the n diagonal matrices $(h_i)_{kl} = \delta_{ik}\delta_{kl}$, (no sum on k), where $i, j, k, l = 1, \ldots n$. For $SU(n)$ it is necessary to restrict to traceless matrices given by $h_i - h_j$ for some i, j. The $n-1$ independent $h_i - h_j$ correspond to the Cartan subalgebra.

(a) Show that

$$[h_i, E_{jk}] = (\delta_{ij} - \delta_{ik})E_{jk}, \qquad [E_{ij}, E_{ji}] = h_i - h_j \;\; \text{(no summation convention).}$$

(b) Let $\mathbf{e}_i$ be orthogonal n-dimensional unit vectors, $(\mathbf{e}_i)_j = \delta_{ij}$. Show that E_{ij} is associated with the root vector $\mathbf{e}_i - \mathbf{e}_j$ while E_{ji} corresponds to the root vector $\mathbf{e}_j - \mathbf{e}_i$.

(c) Hence show that there are $n(n-1)$ root vectors belonging to the $n-1$ dimensional hyperplane orthogonal to $\sum_i \mathbf{e}_i$.

(d) Verify that we may take as simple roots

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \ \ldots, \quad \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \ \ldots, \quad \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,$$

by showing that all roots may be expressed in terms of the α_i with either positive or negative integer coefficients.

(e) Determine the Cartan matrix and write down the corresponding Dynkin diagram. [You may assume the Killing form is diagonal.]

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