Symmetries, Fields and Particles. Examples 4

1. Starting from the corresponding Cartan matrix construct the root and weight lattices of the Lie algebra $B_2 = \mathrm{L}_{\mathbb{C}}(SO(5))$. Using the algorithm introduced in the lectures, find the weights of the fundamental representation and those of the adjoint. Determine the degeneracy of the weight zero in the adjoint.

2. Show that the isospin I and hypercharge Y of the lightest mesons are correctly determined by the relations $I = H^1/2$ and $Y = (H^1 + 2H^2)/3$ where H^1 and H^2 are the standard basis for the Cartan subalgebra of A_2 and these generators act on states in the adjoint representation.

3. Show that the ten-dimensional representation $R_{3,0}$ of A_2 corresponds to a reducible representation of the $\mathrm{L}_{\mathbb{C}}(SU(2))$ subalgebra corresponding to any root. Find the irreducible components of this representation. Does your answer depend on the particular root chosen?

4. Decompose the following tensor products of $A_2 = \mathrm{L}_{\mathbb{C}}(SU(3))$ representations into irreducible components; i) $\mathbf{3} \otimes \bar{\mathbf{3}}$ and ii) $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$.

5. Find the non-trivial B_2 representation of smallest dimension and decompose the tensor product of two copies into irreducibles, giving the dimension of each component.

6. Consider a gauge theory whose gauge group, G is a matrix Lie group. The corresponding gauge field,

$$A_\mu : \ \mathbb{R}^{3,1} \to \mathrm{L}(G)$$

transforms as

$$A_\mu \to A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}$$

under a gauge transformation,

$$g : \ \mathbb{R}^{3,1} \to G \qquad (*)$$

For the case $G = SU(N)$, check that $A'_\mu(x)$ takes values in the Lie algebra $\mathrm{L}(G)$. Explain why this is true for any matrix Lie group G. Writing $g = \exp(\epsilon X)$ with $\epsilon << 1$, show that the corresponding infinitessimal gauge transformation coincides with the one defined in the lectures.

7. Let $G \subset Mat_N(\mathbb{C})$ be a compact matrix Lie group. A scalar field in the fundamental representation of G, corresponds to an N-component vector $\phi_{\mathrm{F}}(x) \in \mathbb{C}^N$ defined at each spacetime point $x \in \mathbb{R}^{3,1}$ which transforms as,

$$\phi_{\mathrm{F}} \to \phi'_{\mathrm{F}} = g\phi_{\mathrm{F}}$$

under the gauge transformation (*) defined above.
A scalar field in the adjoint representation of G, corresponds to an $N \times N$ matrix $\phi_{\mathrm{A}}(x) \in \mathrm{Mat}_N(\mathbb{C})$ defined at each spacetime point $x \in \mathbb{R}^{3,1}$ which transforms as,

$$\phi_{\mathrm{A}} \to \phi'_{\mathrm{A}} = g\phi_{\mathrm{A}}g^{-1}$$

Find explicit formulae for covariant derivatives $D^{(\mathrm{F})}_\mu$, $D^{(\mathrm{A})}_\mu$, such that $D^{(\mathrm{F})}_\mu \phi_{\mathrm{F}}$ and $D^{(\mathrm{A})}_\mu \phi_{\mathrm{A}}$ transform in the fundamental and adjoint of G respectively. Hence write down gauge-invariant Lagrangians describing the coupling of these scalar fields to the the gauge field A_μ. You may assume that the finite-dimensional representations of a compact Lie group are unitary.

8. Show that the field-strength tensor $F_{\mu\nu}$ for a matrix Lie group G can be written as,

$$F_{\mu\nu} = [D^{(\mathrm{A})}_\mu, D^{(\mathrm{A})}_\nu]$$

Hence show that $F_{\mu\nu}$ transforms in the adjoint representation of G ie,

$$F_{\mu\nu} \to F'_{\mu\nu} = gF_{\mu\nu}g^{-1}$$

under the gauge transformation (*). Thus show that the Lagrangian,

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{Tr}_N\left[F_{\mu\nu}F^{\mu\nu}\right]$$

is gauge invariant.

9. Using the fact that the Killing form,

$$\kappa\,(X, Y) = \mathrm{Tr}\left[\,\mathrm{ad}_X \circ \mathrm{ad}_Y\right] \qquad \forall X, Y \in \mathrm{L}(G)$$

is the unique invariant inner product any simple Lie algebra $\mathrm{L}(G)$ up to scalar multiplication, deduce that, when $\mathrm{L}(G)$ is simple, the gauge-field Lagrangian defined in the previous question is proportional to the one given in the lectures. Determine the constant of proportionality in the case $G = SU(N)$.