Statistical Field Theory: Example Sheet 2

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1. The purpose of this question is to evaluate the asymptotic behaviour of the integral

$$G(\mathbf{x}) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\gamma k^2 + \mu^2} = \frac{1}{\gamma} \int \frac{d^dk}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 + 1/\xi^2}$$

where the correlation length is defined as $\xi^2 = \gamma/\mu^2$. First, why does $G(\mathbf{x})$ depend only on $r = |\mathbf{x}|$? Show that

$$\frac{1}{k^2 + 1/\xi^2} = \int_0^\infty dt \; e^{-t(k^2 + 1/\xi^2)}$$

Use this to massage the original $\int d^dk$ integrations into Gaussian form and hence show that, ignoring an overall prefactor,

$$G(r) \sim \int_0^\infty dt \; e^{-S(t)} \quad \text{with} \quad S(t) = \frac{r^2}{4t} + \frac{t}{\xi^2} + \frac{d}{2}\log t$$

Evaluate this using the saddle point expression

$$\int_0^\infty dt \; e^{-S(t)} \approx \int_0^\infty dt \; e^{-S(t_\star) + S''(t_\star)t^2/2} = \sqrt{\frac{\pi}{2S''(t_\star)}} \, e^{-S(t_\star)}$$

where $S'(t_\star) = 0$ is the minimum of $S(t)$. Find the saddle point in the two regimes $r \ll \xi$ and $r \gg \xi$ to derive the Ornstein-Zernicke correlation

$$G(r) \sim \begin{cases} 1/r^{d-2} & r \ll \xi \\ \\ e^{-r/\xi}/r^{(d-1)/2} & r \gg \xi \end{cases}$$

2. Show that the correlation function is a Green's function, obeying

$$(-\gamma \nabla^2 + \mu^2) \langle \phi(\mathbf{x}) \phi(0) \rangle \sim \delta(\mathbf{x})$$

What is the physical interpretation of this?

3. Download a simulator for the 2d Ising model; you can find examples, in several different formats, at http://physics.weber.edu/thermal/computer.html. (Writing your own is also allowed.) Play with different temperatures and different initial conditions.

4. Scaling at a fixed point gives equalities between critical exponents. The purpose of this question is to show that less stringent inequalities follow from thermodynamics alone. Like many problems in thermodynamics, this involves taking lots of partial derivatives.

For a magnetic system, the first law of thermodynamics is

$$dE = TdS - MdB$$

with E the internal energy, M the magnetisation and B the applied magnetic field. By considering other functions of state, such as the free energy $F = E - TS$, derive the Maxwell relations

$$\left.\frac{\partial T}{\partial B}\right|_S = -\left.\frac{\partial M}{\partial S}\right|_B \quad , \quad \left.\frac{\partial S}{\partial B}\right|_T = \left.\frac{\partial M}{\partial T}\right|_B \quad , \quad \left.\frac{\partial T}{\partial M}\right|_S = \left.\frac{\partial B}{\partial S}\right|_M \quad , \quad \left.\frac{\partial S}{\partial M}\right|_T = -\left.\frac{\partial B}{\partial T}\right|_M$$

Define the heat capacities at constant magnetic field C_B and constant magnetisation C_M, and the susceptibility χ

$$C_B = T\left.\frac{\partial S}{\partial T}\right|_B \quad , \quad C_M = T\left.\frac{\partial S}{\partial T}\right|_M \quad , \quad \chi = \left.\frac{\partial M}{\partial B}\right|_T$$

Show that

$$\chi(C_B - C_M) = T\left(\left.\frac{\partial M}{\partial T}\right|_B\right)^2$$

Hint: To show this you will need the following identities involving partial derivatives

$$\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial S}{\partial T}\right|_B + \left.\frac{\partial S}{\partial B}\right|_T \left.\frac{\partial B}{\partial T}\right|_M \quad \text{and} \quad \left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$$

For $B = 0$, as we approach the critical point from below, $T \to T_c$, various thermodynamic quantities scale as

$$C_B \sim (T_c - T)^{-\alpha} \quad , \quad M \sim (T_c - T)^{\beta} \quad , \quad \chi \sim (T_c - T)^{-\gamma}$$

Show that if $C_M \geq 0$ then the exponents must obey the so-called *Rushbrooke inequality*,

$$\alpha + 2\beta + \gamma \geq 2$$

What stronger statement can we make using scaling arguments?

5. An anisotropic material is described by the *Lifshitz* theory. This has a preferred direction, $\mathbf{x} = (x, \vec{y})$, where $\vec{y}$ is a $(d-1)$ dimensional vector. Ignoring interactions, the free energy is

$$F[\phi] = \int d^d x \; \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\vec{\nabla}^2\phi)^2 + \frac{1}{2}\mu_0^2\,\phi^2$$

Note that the gradient term is quadratic in the x-direction, and quartic in the $\vec{y}$-directions.

Write the free energy in Fourier space, with momentum $\mathbf{k} = (k, \vec{q})$. Set up a renormalisation group transformation in which the momenta and field scale as

$$k' = \zeta k \quad , \quad \vec{q}\,' = \zeta^a \vec{q} \quad , \quad \phi'_{\mathbf{k}'} = \zeta^{-b}\phi_{\mathbf{k}}$$

Determine a and b by requiring that both gradient terms remain canonically normalised. Determine $\mu^2(\zeta)$.

Returning to real space, what is the scaling dimension Δ_ϕ of the field $\phi(\mathbf{x})$ about the Gaussian fixed point? Consider the interaction

$$\int d^d x \; g_n \phi^{2n}$$

What is the scaling dimension of g_n? Show that the ϕ^4 interaction is relevant for $d < 7$ and irrelevant for $d > 7$.

6. Consider the free energy for a complex scalar field ψ coupled to a gauge field A_i,

$$F[\psi, A_i] = \int d^d x \; \frac{1}{4}F_{ij}F^{ij} + |\partial_i \psi - ieA_i\psi|^2 + \mu^2|\psi|^2$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. (As an aside: if we add a quartic term $g|\psi|^4$ then, in $d = 3$, this is the original Ginzburg-Landau free energy for a superconductor.)

What is the critical dimension d_c, such that the coupling between the scalar and gauge field is relevant for $d < d_c$ and irrelevant for $d > d_c$? Speculate on what might happen at $d = d_c$. (Computing the RG flow for the coupling e in dimension $d = d_c$ will be part of next term's Advanced Quantum Field Theory course.)

7. A microscopic system sits on a cubic lattice in d dimensions and, at large distances, is described by a local order parameter $\phi(\mathbf{x})$, with $\phi \to -\phi$ symmetry. What is the simplest interaction of ϕ that is compatible with the underlying discrete rotational symmetry, but not $SO(d)$ rotational symmetry? Explain using scaling arguments why the long distance physics exhibits the full $SO(d)$ invariance.

8. Consider n variables ϕ_a drawn from a Gaussian ensemble such that, for any function $f(\phi)$, the expectation value is

$$\langle f(\phi)\rangle = \frac{1}{\mathcal{N}}\int_{-\infty}^{\infty} d^n\phi\ f(\phi)\, e^{-\frac{1}{2}\phi\cdot G^{-1}\phi}$$

where G is an invertible $n \times n$ matrix and $\mathcal{N} = \det^{1/2}(2\pi G)$. Show that

$$\langle \phi_a \phi_b \rangle = G_{ab}$$

Hence prove "Wick's identity",

$$\left\langle e^{B_a\phi_a}\right\rangle = e^{\frac{1}{2}B_a\langle\phi_a\phi_b\rangle B_b}$$

for any constant B^a. By Taylor expanding both sides, show that

$$\langle\phi_a\phi_b\phi_c\phi_d\rangle = \langle\phi_a\phi_b\rangle\langle\phi_c\phi_d\rangle + \langle\phi_a\phi_c\rangle\langle\phi_b\phi_d\rangle + \langle\phi_a\phi_d\rangle\langle\phi_b\phi_c\rangle$$

Show further that $\langle\phi_{a_1}\dots\phi_{a_l}\rangle = 0$ for l odd. Derive an expression for $\langle\phi_{a_1}\dots\phi_{a_l}\rangle$ when l is even.