Statistical Field Theory: Example Sheet 3

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1. The $O(N)$ model consists of N scalar fields, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$, with a free energy

$$F_1[\boldsymbol{\phi}] = \int d^dx \; \frac{1}{2}(\nabla\boldsymbol{\phi})^2 + \frac{\mu^2}{2}(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) + g(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$$

which is invariant under $O(N)$ rotations. In $d = 4 - \epsilon$ dimensions, the beta functions for the $O(N)$ model are

$$\begin{aligned}\frac{d\mu^2}{ds} &= 2\mu^2 + \frac{N+2}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2}\tilde{g} \\ \frac{d\tilde{g}}{ds} &= \epsilon\tilde{g} - \frac{N+8}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\tilde{g}^2\end{aligned}$$

where $\tilde{g} = \Lambda^{-\epsilon}g$ is the dimensionless coupling. What is the critical exponent ν at the Wilson-Fisher fixed point? Assuming that $\eta \sim \mathcal{O}(\epsilon^2)$, determine the critical exponents α, β, γ and δ to leading order in ϵ.

2. Along the flow from the Gaussian to the Wilson-Fisher fixed point, we have $\mu^2 \sim \mathcal{O}(\epsilon)$. To leading order in ϵ, the second beta function in question 1 becomes

$$\frac{d\tilde{g}}{ds} \approx \epsilon\tilde{g} - \frac{N+8}{2\pi^2}\tilde{g}^2 = -A\tilde{g}(\tilde{g} - \tilde{g}_\star)$$

for a suitable A and $\tilde{g}_\star$ that you should determine. Show that if the coupling initially sits at g_0, it evolves as

$$g = \frac{g_\star}{1 - (1 - g_\star/g_0)e^{-\epsilon s}}$$

3*. The $O(N)$ model is deformed by adding a term which breaks the $O(N)$ symmetry,

$$F_2[\boldsymbol{\phi}] = F_1[\boldsymbol{\phi}] + \int d^dx\ \lambda \sum_{a=1}^{N} \phi_a^4$$

Show that the free energy is positive definite only if $g + \lambda > 0$.

To leading order in ϵ, the coupled beta functions for the dimensionless couplings $\tilde{g} = \Lambda^{-\epsilon} g$ and $\tilde{\lambda} = \Lambda^{-\epsilon}\lambda$ are

$$\begin{aligned}\frac{d\tilde{g}}{ds} &= \epsilon\tilde{g} - \frac{N+8}{2\pi^2}\tilde{g}^2 - \frac{3}{\pi^2}\tilde{g}\tilde{\lambda}\\ \frac{d\tilde{\lambda}}{ds} &= \epsilon\tilde{\lambda} - \frac{6}{\pi^2}\tilde{g}\tilde{\lambda} - \frac{9}{2\pi^2}\tilde{\lambda}^2\end{aligned}$$

Show that these equations have four fixed points: the Gaussian fixed point, the Heisenberg fixed point (with $\tilde{\lambda} = 0$), the Ising fixed point (with $\tilde{g} = 0$) and the cubic fixed point with $\tilde{g}, \tilde{\lambda} \neq 0$. Check that the cubic fixed point lies within the regime of parameters for which the free energy is positive definite.

Determine the stability of each fixed point in the (g, λ) plane. [*Hint*: to determine the signs of the eigenvalues of a 2×2 matrix, you need look only at the determinant and trace.] Plot that RG flows between the four fixed points. You should find that you have to distinguish between the cases $N > 4$ and $N < 4$.

4. The free energy for the Sine-Gordon model in $d = 2$ dimensions, with UV cut-off Λ, is given by

$$F[\phi] = \int d^2x\ \frac{1}{2}(\nabla\phi)^2 - \lambda_0 \cos(\beta_0\phi)$$

What is the naive (i.e. "engineering") dimension of β_0 and λ_0?

Decompose the Fourier modes of the field as $\phi_{\mathbf{k}} = \phi^-_{\mathbf{k}} + \phi^+_{\mathbf{k}}$ where $\phi^+_{\mathbf{k}}$ has support on $\Lambda/\zeta < k < \Lambda$. Let $\phi^-(\mathbf{x})$ and $\phi^+(\mathbf{x})$ be the inverse Fourier transform of $\phi^-_{\mathbf{k}}$ and $\phi^+_{\mathbf{k}}$ respectively. Show that, after integrating out $\phi^+_{\mathbf{k}}$, the free energy for ϕ^- becomes, to leading order in λ_0,

$$F'[\phi^-] = \int d^2x\ \frac{1}{2}(\nabla\phi^-)^2 - \lambda_0 \left\langle \cos\beta_0(\phi^- + \phi^+)\right\rangle_+$$

where you should define the meaning of $\langle\ \rangle_+$. Evaluate this expectation value to show that,

$$F'[\phi^-] = \int d^2x \, \frac{1}{2}(\nabla\phi^-)^2 - \lambda_0 \, \zeta^{-\beta_0^2/4\pi} \cos(\beta_0\phi^-)$$

Hence show that the $\cos(\beta\phi)$ potential is relevant when $\beta_0^2 < 8\pi$ and irrelevant when $\beta_0^2 > 8\pi$.

[*Hint*: Write $\cos(\phi^- + \phi^+) = \frac{1}{2}(e^{i\phi^-}e^{i\phi^+} + e^{-i\phi^-}e^{-i\phi^+})$ and use Wick's identity (from Q8 on Sheet 2). You will also need the position space correlation function

$$\langle \phi^+(\mathbf{x})\phi^+(\mathbf{y})\rangle_+ = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^2k}{(2\pi)^2} \, \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}$$

You should make use of this in the limit $\mathbf{x} \to \mathbf{y}$.]

5*. The free energy for a $d = 2$ membrane fluctuating in three dimensional space is

$$\begin{aligned} F[h] &= \int d^2x \, \frac{r_0}{2}(\nabla h)^2 + \frac{\lambda_0}{2}\left(1 + (\nabla h)^2\right)^{-5/2}(\nabla^2 h)^2 \\ &\approx \int d^2x \, \frac{r_0}{2}(\nabla h)^2 + \frac{\lambda_0}{2}(\nabla^2 h)^2 - \frac{5\lambda_0}{4}(\nabla h)^2(\nabla^2 h)^2 + \ldots \end{aligned}$$

What is the engineering dimension of the height h, the tension r_0 and the bending modulus λ_0.

Write $F[h] = F_0[h] + F_I[h]$, where F_0 consists of the two quadratic terms, and $F_I[h]$ is the quartic term. Decompose the Fourier modes as $h_{\mathbf{k}} = h^-_{\mathbf{k}} + h^+_{\mathbf{k}}$ where $h^+_{\mathbf{k}}$ has support on $\Lambda/\zeta < k < \Lambda$. Using $e^{-F_0[h^+_{\mathbf{k}}]}$ as a probability distribution, show that

$$\langle h^+_{\mathbf{k}_1}\, h^+_{\mathbf{k}_2}\rangle_+ = \frac{1}{r_0k_1^2 + \lambda_0 k_1^4}\,(2\pi)^2\delta(\mathbf{k}_1 + \mathbf{k}_2)$$

Write $F_I[h_{\mathbf{k}}]$ in Fourier space. Expand $h_{\mathbf{k}} = h^-_{\mathbf{k}} + h^+_{\mathbf{k}}$ and identify the term in $F_I[h_{\mathbf{k}}]$ that will renormalise the interaction $(\nabla^2 h)^2$. Focus only on this term (which means you should ignore the effect of field renormalisation) and derive the beta function

$$\frac{d\lambda}{ds} \approx -\frac{5}{4\pi}$$

where, as usual, the scale is parameterised by $\zeta = e^s$. Do the fluctuations of the membrane on short distance scales render it more or less flexible on long distance scales?

6. This question is harder but is a good way to test your skills at integrating out fields. A system in d spatial dimensions is described by two local order parameters, $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$. The free energy is invariant under a $\mathbf{Z}_2^2$ symmetry, in which $\phi_1 \mapsto -\phi_1$ and, independently, $\phi_2 \mapsto -\phi_2$. The leading terms are

$$F[\phi_1, \phi_2] = \int d^dx \, \sum_{i=1}^{2} \left[\frac{1}{2} \nabla\phi_i \cdot \nabla\phi_i + \frac{\mu_i^2}{2} \phi_i^2 + g_i \phi_i^4 \right] + \lambda \phi_1^2 \phi_2^2$$

Set $\mu_i^2 = 0$ for $i = 1, 2$. Show that, in $d = 4 - \epsilon$ dimensions, the beta functions are given by

$$\begin{aligned} \frac{dg_i}{ds} &= \epsilon g_i - (36 g_i^2 + \lambda^2) I \qquad\qquad \text{for } i = 1, 2 \\ \frac{d\lambda}{ds} &= \epsilon \lambda - (8\lambda^2 + 12\lambda g_1 + 12 \lambda g_2) I \end{aligned}$$

for some constant I. What are the fixed points? Show that the stable fixed point has enhanced $O(2)$ symmetry.

7. This question leads you through the derivation of the renormalisation group flow of the coupling in the non-linear sigma model using the background field method. Recall that,

$$F[\mathbf{n}] = \int d^dx \, \frac{1}{2e^2} (\nabla n^a) \cdot (\nabla n^a) \, , \qquad Z = \int \mathcal{D}\mathbf{n} \, \delta(n^2 - 1) \, e^{-F[\mathbf{n}]} \, ,$$

where $a = 1, \ldots, N$. Consider a slowly-varying profile $n^a(\mathbf{x}) = \tilde{n}^a(\mathbf{x})$ with $\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1$. Define slowly-varying frame fields, $e^a_\alpha(\mathbf{x})$, with $\alpha = 1, \ldots, N-1$ which satisfy $\tilde{n}^a(\mathbf{x}) e^a_\alpha(\mathbf{x}) = 0 \; \forall \, \alpha$ and $e^a_\alpha(\mathbf{x}) e^a_\beta(\mathbf{x}) = \delta_{\alpha\beta}$. Then introduce fast-moving modes $\chi_\alpha(\mathbf{x})$. The full configuration is

$$n^a(\mathbf{x}) = \tilde{n}^a(\mathbf{x}) \sqrt{1 - \chi(\mathbf{x})^2} + \chi_\alpha(\mathbf{x}) e^a_\alpha(\mathbf{x}) \, .$$

Show that

$$\nabla n^a = (\nabla \tilde{n}^a)(1 - \tfrac{1}{2}\chi^2) - \tilde{n}^a \chi_\alpha \nabla \chi_\alpha + \nabla(\chi_\alpha e^a_\alpha) + O(\chi^3) \, ,$$

and hence that

$$\begin{aligned} (\nabla n^a) \cdot (\nabla n^a) = {}& (\nabla \tilde{n}^a) \cdot (\nabla \tilde{n}^a) \, (1 - \chi^2) + (\nabla \chi_\alpha) \cdot (\nabla \chi_\alpha) + \chi_\alpha \chi_\beta \, (\nabla e^a_\alpha) \cdot (\nabla e^a_\beta) \\ & + 2 \chi_\alpha \, (\nabla \chi_\beta) \cdot (\nabla e^a_\alpha) \, e^a_\beta + 2 (\nabla \tilde{n}^a) \cdot \nabla(\chi_\alpha e^a_\alpha) + O(\chi^3) \, . \end{aligned}$$

Now show that

$$Z = \int \mathcal{D}\tilde{\mathbf{n}} \, \delta(\tilde{n}^2 - 1) \, \exp\left[-\frac{1}{2e^2} \int d^dx \, (\nabla \tilde{n})^2 \right] \left\langle e^{-F_I[\tilde{\mathbf{n}}, \boldsymbol{\chi}]} \right\rangle_\chi \, ,$$

and that

$$\langle F_I \rangle = \frac{1}{2e^2} \int d^d x \, \left[-\delta_{\alpha\beta} \, (\nabla \tilde{n}^a) \cdot (\nabla \tilde{n}^a) + (\nabla e^a_\alpha) \cdot (\nabla e^a_\beta) \right] \langle \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{x}) \rangle \ .$$

Show that

$$(\nabla e^a_\alpha) \cdot (\nabla e^a_\alpha) = (\nabla e^a_\alpha) \cdot (\nabla e^b_\alpha) \, (\tilde{n}^a \, \tilde{n}^b + e^a_\beta \, e^b_\beta) = (\nabla \tilde{n}^a) \cdot (\nabla \tilde{n}^a) + (e^a_\beta \, \nabla e^a_\alpha) \cdot (e^b_\beta \, \nabla e^b_\alpha) \, .$$

[Hint: Use $\tilde{n}^a \, \tilde{n}^b + e^a_\beta \, e^b_\beta = \delta^{ab}$ and $\tilde{n}^a \, e^a_\alpha = 0$.]

Finally, show that the relevant part of $\langle F_I \rangle$ is $(2 - N) \, I_d \int d^d x \, \frac{1}{2} \, (\nabla \tilde{n}^a) \cdot (\nabla \tilde{n}^a)$, where $I_d = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2}$, and so, to leading order,

$$\frac{1}{e^2(\zeta)} = \zeta^{d-2} \left(\frac{1}{e_0^2} + (2 - N) I_d \right) \, .$$