The Standard Model: Example Sheet 1

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1. Show that $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ satisfies the Lorentz algebra,

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i\,(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho} + \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\mu\rho}\sigma^{\nu\sigma})$$

Hint: you may find it useful to first rewrite: $\sigma^{\mu\nu} = \frac{i}{2}(\eta^{\mu\nu} - \sigma^\nu\bar{\sigma}^\mu)$. At some stage of the calculation, you may also find it useful to prove the result $(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu) = 2\,\eta^{\mu\nu}\mathbb{1}_2$.

2. Under a Lorentz transformation, a left-handed Weyl spinor ψ_L and right-handed spinor ψ_R transform as

$$(\psi_L)_\alpha \to S_\alpha^{\ \beta}(\psi_L)_\beta \quad \text{and} \quad (\psi_R)_{\dot\alpha} \to (S^\star)_{\dot\alpha}^{\ \dot\beta}(\psi_R)_{\dot\beta} \tag{1}$$

with $S \in SL(2, \mathbb{C})$. (Here the dotted index $\dot\alpha = 1, 2$ is used to reflect the fact that these spinors transform in different representations. What we call $(\psi_R)_{\dot\alpha}$ here is called $\bar{\psi}_{\dot\alpha}$ in the Supersymmetry course.) Show that:

i) $(S^{-1})_\beta^{\ \alpha} = \epsilon^{\alpha\gamma}S_\gamma^{\ \lambda}\epsilon_{\lambda\beta}$

ii) $(\psi_L)^\alpha = \epsilon^{\alpha\beta}(\psi_L)_\beta$ transforms as $(\psi_L)^\alpha \to (\psi_L)^\beta(S^{-1})_\beta^{\ \alpha}$

iii) $\psi_L\,\chi_L = (\psi_L)^\alpha(\chi_L)_\alpha$ is a Lorentz scalar.

iv) $(\psi_R)^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}(\psi_R)_{\dot\beta}$ transforms as $(\psi_R)^{\dot\alpha} \to (S^{-1\,\dagger})^{\dot\alpha}_{\ \dot\beta}(\psi_R)^{\dot\beta}$.

v) $\bar{\psi}_R\,\psi_L = (\psi_R^\star)^\alpha(\psi_L)_\alpha$ is a Lorentz scalar.

3. Define $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ and $\bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$. Show that $\bar{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$. Let

$$S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$$

with $\omega_{\mu\nu}$ a collection of numbers that specify the Lorentz transformation. Show

$$S^{-1\,\dagger} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\ .$$

(It suffices to show this for infinitesimal $\omega_{\mu\nu}$ and then use general properties of Lie groups.) As an aside: combined with the result from Questions 2iv), this shows that a Dirac spinor should be viewed as having indices $\psi^T = ((\psi_L)_\alpha, (\psi_R)^{\dot\alpha})$.

4. A pair of Weyl spinors ψ_L and ψ_R have both a Dirac mass $M \in \mathbb{R}$ and Majorana masses $m_1, m_2 \in \mathbb{C}$. These appear in the Lagrangian as

$$\mathcal{L}_{\text{mass}} = -M(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) + \frac{m_1}{2}\psi_L\psi_L + \frac{m_1^\star}{2}\bar{\psi}_L\bar{\psi}_L + \frac{m_2}{2}\psi_R\psi_R + \frac{m_2^\star}{2}\bar{\psi}_R\bar{\psi}_R \; .$$

What are the physical masses of fermionic particles in this theory?

5*. The Lie algebra-valued gauge potential $A_\mu = A_\mu^A T^A$ transforms under a gauge symmetry G as

$$A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1}$$

where g is the coupling and $\Omega(x) = e^{ig\alpha(x)} \in G$ with $\alpha(x) = \alpha^A(x)T^A$. Show that:

i) the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ transforms as $F_{\mu\nu} \to \Omega F_{\mu\nu}\Omega^{-1}$.

ii) under an infinitesimal gauge transformation $\delta A_\mu = \mathcal{D}_\mu\alpha$ where the covariant derivative is defined by $\mathcal{D}_\mu\alpha = \partial_\mu\alpha - ig[A_\mu, \alpha]$.

iii) under an infinitesimal gauge transformation $\delta F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]\alpha = ig[\alpha, F_{\mu\nu}]$.

iv) the field strength obeys the Bianchi identity $\mathcal{D}_\mu {}^\star F^{\mu\nu} = 0$.

v) the action

$$S = -\frac{1}{2}\int d^4x \; \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

is gauge invariant.

vi) the equation of motion that follows by varying the action with respect to the fields A_μ^A is $\mathcal{D}_\mu F^{\mu\nu} = 0$.

A scalar ϕ in the fundamental $\mathbf{N}$ representation of $SU(N)$ transforms as $\phi \to \Omega\phi$. How does the covariant derivative $\mathcal{D}_\mu\phi = \partial_\mu\phi - igA_\mu\phi$ transform?

6*. The chiral basis of gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\sigma^\mu = (\mathbb{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$. For a Dirac spinor, $\bar{\psi} = \psi^\dagger\gamma^0$. The equation of motion for a $U(1)$ gauge field coupled to a Dirac fermion is

$$\partial_\nu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi \; .$$

Use the transformation properties of a Dirac spinor under C, P, and T, to derive the corresponding transformation properties of A_μ that ensure this equation remains invariant.

The *theta term* is an extra term that can be added to the Maxwell (or Yang-Mills) action. For Maxwell theory, it takes the form

$$S_\theta = \int d^4x \; F_{\mu\nu}{}^\star F^{\mu\nu} \; .$$

How does this transform under C, P, and T?

7. Show that $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$ are both real. Consider the mass term

$$\mathcal{L}_{\rm mass} = \; m_1\bar{\psi}\psi + im_2\bar{\psi}\gamma^5\psi \; .$$

Using the chiral basis of gamma matrices, write this mass term in terms of Weyl spinors, with $\psi^T = (\psi_L, \psi_R)$. Find a transformation of ψ_L and ψ_R such that this theory is invariant under parity. How does parity act on the Dirac spinor?

8. In $d = 2+1$ dimensions, with signature $(+,-,-)$, we can take the basis of purely imaginary 2×2 gamma matrices $\gamma^\mu = (\sigma^2, i\sigma^1, i\sigma^3)$. A massive Dirac fermion has action

$$S = -\int d^dx \, \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi\right) \; .$$

Parity is defined as $P : x^1 \mapsto -x^1$, with x^0 and x^2 untouched. Why is this the right definition, rather than the more usual $\mathbf{x} \to -\mathbf{x}$?

Find an action of parity, charge conjugation, and time reversal for a massless fermion that leaves the action invariant. Which of these symmetries are broken by the mass term?