Part III Maths **Lent Term 2018**
David Skinner d.b.skinner@damtp.cam.ac.uk

Advanced Quantum Field Theory
Example Sheet 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Consider the partition function

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4} \tag{†}$$

for a zero–dimensional QFT with a quartic interaction with $\lambda > 0$.

(a) By expanding the integral in λ obtain the n^{th} order perturbative expansion

$$\mathcal{Z}_n(\lambda) = \sum_{\ell=0}^{n} \left(-\frac{\lambda}{4!}\right)^{\ell} \frac{(4\ell)!}{4^{\ell}\,(2\ell)!\,\ell!}$$

and show for $\ell \leq 3$ that the coefficients a_ℓ of λ^ℓ in this expression are the sums of automorphism factors of the relevant loop Feynman graphs. (At two loops there is only one graph, at three loops there are two graphs and at four loops there are four.)

(b) *Optional but instructive:* Using any computer package, plot $\mathcal{Z}_n(\lambda = \frac{1}{10})$ against n to see that there is a region in n where $\mathcal{Z}_n$ appears to converge, before blowing up as n is increased.

(c) Show that the minimum value of $a_\ell \lambda^\ell$ occurs when $\ell \approx \frac{3}{2\lambda}$. Hence show that the Borel transform $\mathcal{B}\mathcal{Z}(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\, a_\ell\, \lambda^\ell$ converges provided $|\lambda| < \frac{3}{2}$ and that in this case

$$\mathcal{Z}(\lambda) = \int_0^{\infty} \mathrm{d}z\, \mathrm{e}^{-z}\, \mathcal{B}Z(z\lambda)$$

so that $\mathcal{Z}(\lambda)$ may be recovered from its Borel transform.

(d) By expanding $\mathrm{e}^{-\frac{1}{2}x^2}$ in the integral (†) obtain the strong coupling expansion

$$\mathcal{Z}(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!}\, \Gamma\!\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}}$$

for $\mathcal{Z}(\lambda)$ as a series in $1/\sqrt{\lambda}$. For $\lambda = \frac{1}{10}$ how many terms does one need to obtain the value at which the weak coupling expansion appeared to converge?

2. Let $\mathrm{e}^{-\mathcal{W}(J_c)/\hbar} = \int \mathrm{d}^n\phi\, \mathrm{e}^{-(S(\phi)+J_c\phi^c)/\hbar}$ and let $\Gamma(\Phi^a)$ be the Legendre transform of $\mathcal{W}(J_c)$. Show directly that

$$\hbar^2 \frac{\partial^3 \mathcal{W}}{\partial J_a \partial J_b \partial J_c}\bigg|_{J=0} = \langle \phi^a \phi^b \phi^c \rangle^{\text{conn}}$$

and that

$$-\frac{1}{\hbar}\frac{\partial^3 \Gamma}{\partial \Phi^a \partial \Phi^b \partial \Phi^c}\bigg|_{\Phi=0} = \langle \phi^a \phi^b \phi^c \rangle^{\text{conn}}_{\text{1PI}}$$

3. In lectures we showed that

$$\int \mathrm{d}^{2k}\theta\, \mathrm{e}^{\frac{1}{2}A_{ab}\theta^a\theta^b} = \text{Pfaff}(A)$$

where A is an real, invertible antisymmetric matrix and θ^a are $2k$ Grassmann variables. By writing $\theta^a = N^a_{\ b}\theta'^b$ for some $N \in \mathrm{GL}(2k, \mathbb{R})$, show that $\text{Pfaff}(N^{\mathrm{T}}AN) = \det(N)\,\text{Pfaff}(A)$. Show that N may be chosen so as to put A into the form

$$N^{\mathrm{T}}AN = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix},$$

and hence show that $\text{Pfaff}(A) = \pm\sqrt{\det A}$.

4. Consider a theory of four Grassmann variables θ^a $(a = 1, \ldots, 4)$, governed by the action

$$S(\theta) = \frac{1}{2}A_{ab}\theta^a\theta^b + \frac{1}{4!}\lambda_{abcd}\theta^a\theta^b\theta^c\theta^d\,.$$

Compute the partition function of this theory *i)* by directly expanding $\mathrm{e}^{-S/\hbar}$ in the path integral, and *ii)* by writing down the Feynman rules and drawing all possible vacuum diagrams.

5. Let M be an $N \times N$ Hermitian matrix (with bosonic variables) and consider the integral

$$\mathcal{Z}(a; N) = \int \mathrm{d}^{2N}\mathrm{M}\, \exp\left(-\frac{1}{2}\mathrm{tr}(\mathrm{M}^2) - \frac{a}{N}\mathrm{tr}(\mathrm{M}^4)\right)$$

where a is a coupling constant. The measure $\mathrm{d}^{2N}\mathrm{M}$ represents an integral over the real and imaginary parts of each entry of M.

   (a) Represent the propagator as a "double line" where one line edge represents the rows and the other edge represents the columns of M. What are the Feynman rules for this action?

(b) Show that $\mathcal{Z}(a;N)/\mathcal{Z}(0;N)$ can be reduced to an integral over the eigenvalues $\{\lambda_i\}$ of M. [*You may use without proof that the measure* $\mathrm{d}^{2N}\mathrm{M}$ *is invariant under* $\mathrm{M} \to U^{-1}\mathrm{M}U$ *for any unitary matrix* U.]

(c) Show that $\mathcal{Z}(a;N)/\mathcal{Z}(0;N)$ admits a perturbative expansion of the form

$$\ln \frac{\mathcal{Z}(a;N)}{\mathcal{Z}(0;N)} = \sum_{g=0}^{\infty} N^{2-2g} \left(\sum_{n=0}^{\infty} (-a)^n F_{g,n} \right)$$

where $F_{g,n}$ is a combinatoric number, independent of N and a. (You are not required to find an explicit expression for $F_{g,n}$.)

(d) (*) Show that $F_{g,n}$ may be interpreted as the number of ways to cover a genus g Riemann surface with n squares. [*For help with this part of the question, you may wish to consult the first few sections of D. Bessis, C. Itzykson & B. Zuber,* Quantum Field Theory Techniques in Graphical Enumeration, *Adv. Applied Maths* **1**, *109-157, (1980).*]

6. Consider the Quantum Mechanics of a particle moving on $\mathbb{R}^n$ with Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, \mathrm{d}^n x)$. Obtain (Euclidean time) path integral expressions for the following Heisenberg picture transition functions:

   (a) $\mathrm{Tr}_{\mathcal{H}}\left(P\,\mathrm{e}^{-TH}\right)$, where P is the parity operator $P : \hat{x}^a \to -\hat{x}^a$.

   (b) $\langle\psi_f|\mathrm{e}^{-TH}|\psi_i\rangle$, where $\psi_{i,f}(x) = \langle x|\psi_{i,f}\rangle$ are arbitrary states in the Hilbert space.

   where T is the proper time on the wordline.

   Suppose $n = 1$ and the worldline action includes the potential term $\frac{1}{2}m\omega^2 x^2$. Given that the heat kernel for the (Euclidean) harmonic oscillator is

$$\langle x|\mathrm{e}^{-TH}|y\rangle = \sqrt{\frac{m\omega}{2\pi\,\sinh\omega t}}\,\exp\left(-m\omega\frac{(x^2+y^2)\cosh\omega t - 2xy}{2\sinh\omega t}\right),$$

   evaluate your expressions for (a) and (b) explicitly in the case that $|\psi_{i,f}\rangle$ are the ground state of the harmonic oscillator. Check that they agree with what you expect from QM, working directly in the energy basis.