Part III Maths **Lent Term 2018**
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Advanced Quantum Field Theory
Example Sheet 2

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk $(*)$ may be more difficult.

1. Let H be the Hamiltonian of a quantum simple harmonic oscillator of unit mass in one dimension, with frequency ω. Let $\mathcal{H}$ be the corresponding Hilbert space. Using standard canonical quantum mechanics, compute the partition function $\mathcal{Z}(\omega,\beta) = \mathrm{tr}_{\mathcal{H}}(\mathrm{e}^{-\beta H})$ in units with $\hbar = 1$.

   Write this partition function in terms of a formal Euclidean worldline path integral over the space of maps $x : S^1 \to \mathbb{R}$, making clear the role of β and the choice of action $S[x]$.

   Regularize this formal path integral by imposing a hard cut-off on the eigenvalues of the quadratic operator in the wordline action. What is the regularized path integral measure? Show that the regularized partition function is given by

$$\mathcal{Z}_N(\omega,\beta) = \frac{A_N}{\omega}\prod_{n=1}^{N}\left[\omega^2 + \left(\frac{2\pi n}{\beta}\right)^2\right]^{-1} .$$

   where N labels the cut-off and A_N is a constant that is independent of ω. [You need not determine the value of A_N.]

   Consider the ratio

$$\frac{\mathcal{Z}_N(\omega_1,\beta)}{\mathcal{Z}_N(\omega_2,\beta)}$$

   of regularized partition functions for harmonic oscillators of different frequencies. By examining the zeros and poles of this ratio as a function of (ω_1,ω_2), show that the limit

$$\lim_{N\to\infty}\mathcal{Z}_N(\omega,\beta)$$

   agrees with your answer obtained earlier, up to an overall constant.

2. Consider the theory given by the action

$$S[\phi] = \int \mathrm{d}^d x\, \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 .$$

   where ϕ is a real scalar field.

(a) Determine all connected one loop graphs, complete with their appropriate symmetry factors, which contribute to

$$\langle\phi(x)\phi(y)\rangle\,, \qquad \langle\phi(x)\phi(y)\phi(z)\rangle \qquad \text{and} \qquad \langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle\,,$$

expressing your answer in terms of integrals over d-dimensional loop momenta. [You are not required to evaluate the integrals.]

(b) Now set $\lambda = 0$ so that just the cubic interaction remains. Determine the momentum space correlation function $\int \prod_{i=1}^{3} \mathrm{d}^d x_i\, \mathrm{e}^{\mathrm{i}p_i\cdot x_i}\,\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle$ to one loop accuracy.

3. Let ψ^i denote a (fermionic) Dirac spinor field transforming in the fundamental representation of an SU(N) gauge group, and let $\bar{\psi}_j$ denote the Dirac conjugate spinor transforming in the antifundamental. Let $(A_\mu)^i{}_j$ denote the gauge field for this interaction. Write down all possible SU(N) gauge invariant local operators involving these fields that are relevant or marginal near the Gaussian critical point, in the cases that the space–time has dimension $d = 4$, $d = 3$ and $d = 2$.

4. Show that under the redefinition $g_i \to g'_i(g_j)$ of the couplings of a theory at scale Λ, the β-functions transform as

$$\beta_i \to \beta'_i = \frac{\partial g'_i}{\partial g_j}\beta_j\,.$$

Show that in a theory with a single coupling g, the first two terms in the β–function $\beta(g) = ag^3 + cg^5 + \mathcal{O}(g^7)$ are invariant under any coupling constant redefinition of the form $g \to g' = g + \mathcal{O}(g^3)$. Show that, in a neighbourhood of $g = 0$, it is possible to choose this redefinition so as to remove all terms *except* these first two. [*Hint: consider setting* $g'(g) = g + g^3 f(g)$ *where* $f(0) = 1$.] What is the significance of this calculation?

5. Consider a four dimensional theory whose only couplings are a mass parameter m^2 and a marginally relevant coupling g.

(a) Write down generic expressions for the β-functions in such a theory to lowest non–trivial order. (You should be able to identify the *values* of the classical contributions to the β-functions, and the *sign* of the leading–order quantum correction to $\beta(g)$.)

(b) Sketch the RG flows for this theory.

(c) Suppose that $g(\Lambda') = 0.1$ when the cut–off Λ' is fixed at 10^5 GeV. If $m^2(\Lambda')$ is measured to be 100 GeV, what value of $m^2(\Lambda)$ would be needed at the higher scale $\Lambda = 10^{19}$ GeV?

(d) Suppose you changed your value of $m^2(\Lambda)$ by one part in 10^{20}. What would be the change in $m^2(\Lambda')$?