Part III Maths **Lent Term 2018**
David Skinner d.b.skinner@damtp.cam.ac.uk

Advanced Quantum Field Theory
Example Sheet 3

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk $(*)$ may be more difficult.

1. Consider a scalar field ϕ with potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{6}\mu^{\epsilon/2}g(\mu)\phi^3$ in dimension $d = 6 - \epsilon$. Here μ is an arbitrary mass scale introduced so that the coupling $g(\mu)$ is dimensionless.

   (a) Draw the one-loop 1PI graph which contributes to the propagator at order g^2.

   (b) Using dimensional regularisation, show that the divergent part of the corresponding integral for the six dimensional theory is
   $$-\frac{1}{\epsilon}\frac{g^2}{(4\pi)^3}\left(m^2 + \frac{1}{6}p^2\right) ,$$
   where p is the external momentum. Also compute the divergence corresponding to the one particle irreducible one-loop graph that gives a g^3 correction to three point function, and find the one loop divergence for the one point function.

   (c) Show that in six dimensions all these divergences may be cancelled by introducing the counterterm action
   $$S_{\rm ct}[\phi] = \int \mathcal{L}_{\rm ct}\,\mathrm{d}^d x = \frac{1}{\epsilon}\frac{\hbar}{6(4\pi)^3}\int\left[\frac{1}{2}g^2(\partial\phi)^2 + \mu^{-\epsilon}V''(\phi)^3\right]\mathrm{d}^d x.$$
   Check that $\mathcal{L}_{\rm ct}$ has dimension d.

   (d) Determine the β-function for the coupling g and show that $\beta(g) < 0$ at small g. Does the theory have a continuum limit in perturbation theory? Do you expect this to survive non–perturbatively?

2. Scalar QED describes the interactions of a photon with a complex scalar field. In d dimensions it is defined by the action
$$S[A,\phi] = \int\left[\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D^\mu\phi)^* D_\mu\phi + \frac{m^2}{2}\phi^*\phi\right]\mathrm{d}^d x$$
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + \mathrm{i}eA_\mu\phi$.

(a) Show that, not including counterterms, there are two distinct 1-loop Feynman graphs that contribute to vacuum polarization in scalar QED. One of these diagrams leads to an integral that is independent of the external momentum. What is its role?

(b) By considering vacuum polarization, show that when $d = 4$, the 1-loop β-function for the dimensionless coupling g corresponding to the charge e is

$$\beta(g) = \frac{g^3}{48\pi^2}$$

in the $\overline{\mathrm{MS}}$ scheme. How does the theory behave at scales far below the mass of the scalar?

3. Consider the theory of a real scalar field ϕ and massless fermionic Dirac spinor ψ, with action

$$S[\phi, \psi] = \int \left[\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \bar{\psi} \not{\partial} \psi + \mathrm{i} g\, \phi \bar{\psi} \gamma_5 \psi + \frac{\lambda}{4!}\, \phi^4\right] \mathrm{d}^4 x$$

in four dimensional Euclidean space, where $\bar{\psi} = (\psi)^\dagger$, the Dirac matrices obey $\{\gamma^\mu, \gamma^\nu\} = 2\,\delta^{\mu\nu}$, $(\gamma^\mu)^\dagger = -\gamma^\mu$ and likewise $\{\gamma^\mu, \gamma_5\} = 0$, $(\gamma_5)^2 = +\mathrm{id}$ and $(\gamma_5)^\dagger = -\gamma_5$.

(a) Show that the action is real and invariant under the global transformation

$$\phi \to -\phi \qquad\qquad \psi \to \mathrm{e}^{\mathrm{i}\pi\gamma_5/2}\psi\,.$$

Assuming that the path integral measure is also invariant under this transformation, show that renormalization cannot generate any vertices involving odd powers of the scalar field unless they are accompanied by an odd power of $\mathrm{i}\bar{\psi}\gamma_5\psi$, as in the original action.

(b) What counterterms are necessary when studying the continuum limit of this theory? Using dimensional regularization and the on-shell renormalization scheme, evaluate these counterterms to 1-loop accuracy, and show that the physical amplitudes are finite.

4. Furry's theorem states that $\langle \tilde{A}_{\mu_1}(k_1) \cdots \tilde{A}_{\mu_n}(k_n)\rangle = 0$ when n is odd, where $\tilde{A}_\mu(k)$ is the photon field in momentum space. It is a consequence of charge conjugation invariance.

(a) In scalar QED, charge conjugation swaps ϕ and $\bar{\phi}$. How must the photon field A_μ transform if the action is to be invariant?

(b) Prove Furry's theorem in scalar QED using the path integral.

(c) Does Furry's theorem hold for off–shell photons with $k_\mu k^\mu \neq 0$?