Advanced Quantum Field Theory
Example Sheet 4

Please email me with any comments about these problems, particularly if you spot an error.

1. Let $t_A$ be the generators of a Lie algebra $g$, $[t_A, t_B] = i f^C_{AB} t_C$, and let $c^A$ be anticommuting variables. Show that

$$Q := c^A t_A - \frac{1}{2} f^A_{BC} c^B c^C \frac{\partial}{\partial c^A}$$

satisfies $Q^2 = 0$. Suppose $t_A = 0$ and also that $f_{ABC} = k_{CD} f^D_{AB}$ is completely antisymmetric, where $k_{CD}$ is the Killing form on $g$. If $X = f_{ABC} c^A c^B c^C$ show that $QX = 0$ but that $X \neq QY$.

2. Consider a gauge-fixed action for a free (Abelian) gauge field $A_\mu$ of the form

$$S = \int d^D x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + h \partial^\mu A_\mu + \frac{\xi}{2} h^2 + \bar{c} \partial^2 c \right)$$

where $h$ is an auxiliary bosonic field and $(c, \bar{c})$ are anticommuting ghost and antighost fields.

(a) Verify that this action is invariant under the BRST transformations $\delta A_\mu = \epsilon \partial_\mu c$, $\delta c = 0$, $\delta \bar{c} = -\epsilon h$, $\delta h = 0$ and that $\delta$ is nilpotent.

(b) Show that the action can be written in the form

$$S = -\int d^D x \left( \frac{1}{2} \Phi^T \Delta \Phi + \bar{c} (-\partial^2)c \right)$$

where $\Phi = \begin{pmatrix} A_\mu \\ h \end{pmatrix}$, $\Phi^T$ is its transpose and where

$$\Delta = \begin{pmatrix} -\partial^2 \delta^\nu + \partial^\mu \partial_\nu & \partial_\nu \\ -\partial^\mu & -\xi \end{pmatrix}.$$
4. Consider a gauge invariant Lagrangian density of the form

\[ \mathcal{L}(A) = -\frac{1}{4} \text{tr} \left( F_{\mu\nu} X(D^2) F_{\mu\nu} \right), \]

where \( D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}] \) and where \( X(D^2) = 1 + (-D^2)^r / \Lambda^{2r} \) for some scale \( \Lambda \). The full quantum Lagrangian with gauge fixing and ghost fields is

\[ \mathcal{L}_q(A, c, \bar{c}) = \mathcal{L}(A) - \frac{1}{2\xi} \text{tr} \left( \partial^\mu A_\mu X(\partial^2) \partial^\nu A_\nu \right) + \text{tr} \left( \bar{c} X(\partial^2) \partial^\mu D_\mu c \right). \]
(a) Show that the Feynman rules require that the gauge and ghost propagators are each proportional to $p^{-2-2r}$.

(b) Show that there must be vertices with $n$ gauge field legs $n = 3, \ldots, 2r + 4$ and with $2r + 4 - n$ powers of momentum, but that there is just a single vertex involving both ghost and gauge fields with $2r + 1$ momentum factors.

(c) Hence show that, in four dimensions, the superficial degree of divergence of an $\ell$-loop Feynman graph with $E_A$ external gauge field lines and $E_{gh}$ ghost field lines is

$$\delta = 4 - E_A - E_{gh} - 2r(\ell - 1).$$

5. Consider pure (= no charged matter) electrodynamics with Lagrangian $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Let $W_\gamma[A]$ be a Wilson loop around a closed curve $\gamma$.

(a) Show that

$$\langle W_\gamma[A] \rangle = \exp \left[ -\frac{e^2}{8\pi^2} \int_\gamma dx^\mu \int_\gamma dy^\mu \frac{1}{(x-y)^2} \right].$$

(b) Now suppose $\gamma$ is a large rectangle with space-like width $L$ and time-like length $T$. Compute $\langle W_\gamma[A] \rangle$ in the limit $T \gg L$. By comparing your result to the usual expression for time evolution, show that the potential between two point-like charges at fixed separation $L$ in electrodynamics is $V(L) = -e^2/4\pi L$.

(c) In Feynman gauge, the propagator for a non-Abelian gauge field is

$$\langle A_{\mu}^R(x) A_{\nu}^C(y) \rangle = i\eta_{\mu\nu} \delta^{BC} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2}.$$

Compute the expectation value of a Wilson loop in pure $\text{SU}(N)$ Yang-Mills theory to lowest non-trivial order in the coupling $g^2$. [Your result should depend on a choice of the representation $R$ of $\text{su}(N)$.]

(d) Show that, to this order, the Coulomb potential of non-Abelian gauge theory is $V(L) = -g^2 C_2(R)/4\pi L^2$, where $C_2(R)$ is the quadratic Casimir of the $R$ representation.

6. For a beta function $\beta(g) = -b_1g^3 - b_2g^5 + O(g^7)$ show that the solution for the running coupling can be expressed in the form

$$\frac{1}{g^2} = b_1 \ln \frac{\mu^2}{\Lambda^2} + b_2 b_1 \ln \left( \ln \frac{\mu^2}{\Lambda^2} \right) + O \left( \frac{1}{\ln \frac{\mu^2}{\Lambda^2}} \right)$$

or equivalently

$$\Lambda^2 = \mu^2 e^{-\frac{1}{b_1} \ln \left( b_1 g^2 \right)} \left( \frac{b_2}{b_1} \right) \left( 1 + O(g^2) \right).$$

Suppose $\bar{g} = g + \nu g^3 + O(g^5)$. Show that $\bar{g}^2$ can be expressed in terms of $\bar{\Lambda}^2$ as above where $\ln \bar{\Lambda}^2/\Lambda^2 = \nu/b_1$. 
