Part III Maths **Lent Term 2018**
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Advanced Quantum Field Theory
Example Sheet 4

Please email me with any comments about these problems, particularly if you spot an error. Questions marked with an asterisk may be more challenging.

1. Show that, when written in momentum space, the QED correlation function $\langle j^\mu(x)\, j^\nu(y)\rangle$ is proportional to
$$\Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k)\Delta_{\rho\sigma}(k)\Pi^{\sigma\nu}(k) + \cdots$$
where $j^\mu = \bar\psi\gamma^\mu\psi$, $\Pi^{\mu\nu}$ is the photon self–energy and $\Delta_{\rho\sigma}$ is the exact photon propagator. Hence show that $\Pi^{\mu\nu}$ is transverse.

2. Let $t_{\rm a}$ be the generators of a Lie algebra $\mathfrak{g}$ acting in a representation R, with $[t_{\rm a}, t_{\rm b}] = {\rm i} f_{\rm ab}^{\rm c} t_{\rm c}$, and let $c^{\rm a}$ be Grassmann variables. Show that
$$Q = c^{\rm a} t_{\rm a} - \frac{1}{2} f^{\rm a}_{\rm bc} c^{\rm b} c^{\rm c} \frac{\partial}{\partial c^{\rm a}}$$
satisfies $Q^2 = 0$. Suppose we are in the trivial representation and also that $f_{\rm abc} = k_{\rm cd} f^{\rm d}_{\rm ab}$ is completely antisymmetric, where $k_{\rm cd}$ is the Killing form on $\mathfrak{g}$. If $X = f_{\rm abc}\, c^{\rm a} c^{\rm b} c^{\rm c}$ show that $QX = 0$ but that $X \neq QY$.

3. Consider a gauge-fixed action for a free (Abelian) gauge field A_μ of the form
$$S = \int {\rm d}^D x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + h\, \partial^\mu A_\mu + \frac{\xi}{2} h^2 + \bar c\, \partial^2 c \right)$$
where h is an auxiliary bosonic field and $(c, \bar c)$ are anticommuting ghost and antighost fields.

   (a) Verify that this action is invariant under the BRST transformations $\delta A_\mu = \epsilon \partial_\mu c,\ \delta c = 0,\ \delta \bar c = -\epsilon h,\ \delta h = 0$ and that δ is nilpotent.

   (b) Show that the action can be written in the form
$$S = -\int {\rm d}^D x \left(\frac{1}{2} \Phi^{\rm T} \Delta \Phi + \bar c(-\partial^2) c \right)$$
where $\Phi = \begin{pmatrix} A_\mu \\ h \end{pmatrix}$, $\Phi^{\rm T}$ is its transpose and where
$$\Delta = \begin{pmatrix} -\partial^2 \delta^\mu{}_\nu + \partial^\mu \partial_\nu & \partial_\nu \\ -\partial^\mu & -\xi \end{pmatrix} .$$

(c) Obtain equations for the normalized correlation functions $\langle \Phi(x)\,\Phi(0)^{\rm T}\rangle$ and $\langle c(x)\,\bar c(0)\rangle$ and show that

$$\int {\rm d}^D x\, {\rm e}^{-{\rm i}p\cdot x}\langle \Phi(x)\,\Phi(0)^{\rm T}\rangle = -\frac{\rm i}{p^2}\begin{pmatrix} \delta_\mu{}^\nu - (1-\xi)\frac{p_\mu p^\nu}{p^2} & {\rm i}p_\mu \\ -{\rm i}p^\nu & 0 \end{pmatrix}$$

$$\int {\rm d}^D x\, {\rm e}^{-{\rm i}p\cdot x}\langle c(x)\,\bar c(0)\rangle = -\frac{\rm i}{p^2}\,.$$

(d) Assuming that $\langle \delta Y\rangle = 0$ for any Y, consider $\langle \delta\left(\Phi(x)\,\bar c(0)\right)\rangle$ and show that we must have $\langle h(x)\,h(0)\rangle = 0$. Obtain also a relation between $\langle c(x)\,\bar c(0)\rangle$ and $\langle A_\mu(x)\,h(0)\rangle$ which should be verified.

4. For a gauge theory coupled to scalars the single particle states are

$$|A^{\rm a}_\mu(k)\rangle\,, \qquad |\phi_i(k)\rangle\,, \qquad |c^{\rm a}(k)\rangle\,, \qquad |\bar c^{\rm a}(k)\rangle\,,$$

where a runs over a basis of the adjoint representation and i similarly indexes the R representation. These states have non-zero scalar products

$$\langle A^{\rm a}_\mu(k)|A^{\rm b}_\nu(k')\rangle = \eta_{\mu\nu}\delta^{\rm ab}\delta_{kk'} \qquad \langle \phi_i(k)|\phi_j(k')\rangle = \delta_{ij}\delta_{kk'}$$
$$\langle c^{\rm a}(k)|\bar c^{\rm b}(k')\rangle = \langle \bar c^{\rm a}(k)|c^{\rm b}(k')\rangle = \delta^{\rm ab}\delta_{kk'}$$

where $\delta_{kk'} = (2\pi)^{d-1}\,2k^0\,\delta^{(d-1)}(\mathbf{k}-\mathbf{k}')$. The non-zero action of the BRST charge Q is given by

$$Q|A^{\rm a}_\mu(k)\rangle = \alpha k_\mu|c^{\rm a}(k)\rangle\,, \quad Q|\phi_i(k)\rangle = \sum_{\rm a} v_{i{\rm a}}|c^{\rm a}(k)\rangle$$
$$Q|\bar c^{\rm a}(k)\rangle = \beta k^\mu|A^{\rm a}_\mu(k)\rangle + \sum_i \bar v_{{\rm a}i}|\phi_i(k)\rangle$$

while the ghost charge $Q_{\rm gh}$ acts non-trivially as

$$Q_{\rm gh}|c^{\rm a}(k)\rangle = {\rm i}|c^{\rm a}(k)\rangle\,, \qquad Q_{\rm gh}|\bar c^{\rm a}(k)\rangle = -{\rm i}|\bar c^{\rm a}(k)\rangle\,.$$

Verify that this is compatible with Q and $Q_{\rm gh}$ being Hermitian if α, β, $v_{i{\rm a}}$ and $\bar v_{{\rm a}i}$ are related appropriately. Assume a basis has been chosen so that $\sum_i \bar v_{{\rm a}i}v_{i{\rm b}} = \delta_{\rm ab}\rho_{\rm a}$ and so is diagonal. Find the conditions under which the BRST charge $Q^2=0$. Use this to determine the possible physical single particle states.

5. Consider a gauge invariant Lagrangian density of the form

$$\mathcal{L}(A) = -\frac{1}{4}{\rm tr}\left(F^{\mu\nu}X(D^2)F_{\mu\nu}\right)\,,$$

where $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$ and where $X(D^2) = 1 + (-D^2)^r/\Lambda^{2r}$ for some scale Λ. The full quantum Lagrangian with gauge fixing and ghost fields is

$$\mathcal{L}_q(A,c,\bar c) = \mathcal{L}(A) - \frac{1}{2\xi}{\rm tr}\left(\partial^\mu A_\mu\, X(\partial^2)\,\partial^\nu A_\nu\right) + {\rm tr}\left(\bar c\, X(\partial^2)\,\partial^\mu D_\mu c\right)\,.$$

(a) Show that the Feynman rules require that the gauge and ghost propagators are each proportional to p^{-2-2r}.

(b) Show that there must be vertices with n gauge field legs $n = 3, \ldots, 2r+4$ and with $2r + 4 - n$ powers of momentum, but that there is just a single vertex involving both ghost and gauge fields with $2r + 1$ momentum factors.

(c) Hence show that in four dimensions, an ℓ-loop Feynman graph with E_A external gauge field lines and E_{gh} ghost field lines behaves as $\int^{\infty} \mathrm{d}\kappa\, \kappa^{\delta-1}$ in the region where all loop momentum become large simultaneously, where $\delta = 4 - E_A - E_{\mathrm{gh}} - 2r(\ell - 1)$.