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STRING THEORY

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1. What is String Theory?

String theory is a theory of strings of a special kind. Consider a string of uniform tension T and uniform mass density ρ, stretched between two points. Small amplitude oscillations of this string propagate along the string with speed

$$v = \sqrt{\frac{T}{\rho}}\,. \tag{1.1}$$

If the string is tightened by pulling on the two ends, the tension goes up and the mass-density decreases slightly, causing v to increase. However, it cannot increase indefinitely because Special Relativity requires $v \leq c$, which implies

$$T \leq \rho c^2 \,. \tag{1.2}$$

For everyday strings, such as violin strings, $T \ll \rho c^2$; these are non-relativistic. String theory strings saturate the inequality, i.e. $T = \rho c^2$; they are therefore ultra-relativistic. Cosmic strings, which arise as topological defects in scalar fields of relevance to cosmology, are also ultra-relativistic, but they have a finite core: a cross section of non-zero area. When probed at wavelengths less than the core size they betray some internal structure. In contrast, string theory strings are assumed to have a zero core size (classically, at least) and no internal structure; they are "elementary" strings, and the oscillation modes are identified with elementary particles.

For an elementary ultra-relativistic string the only dimensionful constant of relevance is the string tension T. In the quantum theory this serves to define a "string length"

$$\ell_s \sim \sqrt{\frac{\hbar c}{T}} \,. \tag{1.3}$$

Usually we will choose natural units for which $\hbar = c = 1$ and sometimes we will write

$$T = \frac{1}{2\pi\alpha'} \qquad (\Rightarrow \;\; \ell_s \sim \sqrt{\alpha'}) \,, \tag{1.4}$$

where α' is called the "Regge slope parameter". This terminology comes from the late 1960's when it was found that hadron resonances could be predicted by extrapolating straight lines of slope α' in a plot of angular momentum J against m^2. At that time it was supposed that $\ell_s \sim 1\,\mathrm{fm}$ (sub-nuclear distances). The "dual model" theory that was developed to explain such properties was interpreted as a string theory around 1970.

The application of string theory to hadron physics soon ran into difficulties. There was a persistent prediction of a zero-mass spin-2 particle, and it was found that the theory was intractable, or even inconsistent, unless Minkowski spacetime was assumed to have dimension $D = 26$. This is now called the "critical dimension". It was also found that the high-energy scattering of string theory particles was particularly "soft", in disagreement with experiments in which electrons were scattered off nucleons. String theory also had to compete with the emerging theory of QCD, which was ultimately much more successful.

By 1974 string theory was virtually abandoned, but that year it was resurrected with a new interpretation. Now it was argued that string theory was a theory of quantum gravity and that one should take (G is Newton's gravitational constant)

$$\ell_s \sim \ell_P = \sqrt{\frac{\hbar G}{c^3}} \quad \text{(Planck length)}. \tag{1.5}$$

The massless spin-2 particle was now welcome; linearisation of Einstein's gravitational field equations yields a free field theory whose quanta are gravitons, and a graviton is a massless particle of spin 2. The soft high-energy behaviour was also an indication that string theory might resolve the problem of non-renormalizable UV divergences that arise in attempts to quantize GR as an interacting spin-2 QFT. The critical dimension of $D = 26$, later to be reduced to $D = 10$ in the context of superstring theory, was now an opportunity to unify gravity with other forces that could emerge on compactification to $D = 4$, along the lines suggested by Kaluza and Klein in the 1920s.

In reality, the self-consistency of string theory as formulated in 1974 (and until 1995) requires $\ell_s \ll \ell_P$ because otherwise the assumption of a Minkowski background makes no sense; if $\ell_s \sim \ell_P$ then the gravitational back-reaction of the string is not negligible. This restriction translates to a weak coupling requirement on string theory. In fact, string theory as understood then (and in this course) is only defined as an asymptotic perturbation series, with the first few terms providing a good approximation at weak coupling. In other words, "string theory" is really a weak-coupling expansion of some other theory in which strings might not be as fundamental as they appear to be in perturbation theory.. More on this below.

Particles propagate along worldlines. To introduce interactions, one has to posit point-like interaction vertices at which particles meet. One then gets a scattering amplitude by summing over all possibilities consistent with given initial and final states. This is how Feynman diagrams originated; fields were not necessary. This particle viewpoint ended up in appendices to Feynman's QED papers, because by the time he wrote them his diagrams had been re-interpreted by Dyson as visual aids to the organisation of QFT calculations. However, string theory did *not* emerge as a generalisation of QFT. Instead, it is a generalisation of Feynman's ideas about particles. There *is* a string-theory analog of QFT called String Field Theory, but it is much more complicated and still under construction; it is not a good entry point to the subject.

Interactions of particles have to be pointlike because any "smearing" of the interaction in a relativistic theory will violate causality (this is the locality requirement on relativistic QFT). But point-like interactions lead to UV divergences; these can be removed by renormalisation for some theories, at the cost of introducing a few free parameters, but this doesn't work for gravity. Strings can potentially avoid this problem because strings propagate along worldsheets, and it is a simple matter to draw smooth worldsheets that represent the scattering of strings. When the worldsheet is viewed as a sequence of spacelike curves, which involves a choice of Lorentz frame, the interaction looks like either (i) a splitting/joining of two ends, or (ii) the merging/separation of two strings, but in either case the position of the "interaction point" on the worldsheet depends on the choice of frame; it is not a Lorentz invariant concept. There can be no "interaction point" on a smooth worldsheet.

A string either has two ends ("open") or it has no ends ("closed"). Given a theory of closed strings with interactions only of type (ii), open strings will never appear. In contrast, given a theory of open strings, either type of interaction will allow the appearance of a closed string. So all string theories must contain closed strings. The graviton appears in the spectrum of every closed string, so all string theories are (perturbative) quantum gravity theories.

How many string theories are there? Initially there was only what we now call the "bosonic string" because there are no fermions in its spectrum; The critical spacetime dimension is 26. Its ground state is tachyonic (particle of negative m^2) which implies an instability of the Minkowski vacuum. It is now believed that there is no stable vacuum state (it is similar in this respect to ϕ^3 scalar QFT). These problems can be resolved by introducing worldsheet fermions, in two ways: this gives the Ramond (R) and the Neveu-Schwarz (NS) strings, and the critical dimension is now 10. Ultimately, consistency requires a mixture of R and NS, resulting in the RNS superstring, so called because the spectrum turns out to have ($D = 10$) spacetime supersymmetry. An alternative formulation (the Green-Schwarz superstring) makes this feature manifest.

In the 1980s it was found that there are five consistent superstring theories

- Type 1. Open and closed strings with $\mathcal{N} = 1$ spacetime supersymmetry.
- Types IIA and IIB. Closed strings with $\mathcal{N} = 2$ spacetime supersymmetry. We shall be having a look at these two.
- Heterotic E and O. Closed strings with $\mathcal{N} = 1$ spacetime supersymmetry.

All five have a low-energy effective description in terms of a $D = 10$ supergravity theory.

String theory is not the only candidate for a theory of quantum gravity (two other competing proposals go by the names "Asymptotic Safety" and "Loop Quantum Gravity"). However, string theory has the property that consistency allows the string of a string theory to couple *only* to itself[1]. So if one proposes a string theory as *the* theory of quantum gravity then one is necessarily proposing that this theory describes everything else, e.g. standard model of particle physics, dark matter, dark energy, and whatever else is discovered. This state of affairs led string theory to be called "A Theory of Everything" (TOE). This was an unfortunate choice of terminology but there is a serious point to it: ambition is not optional in string theory. On the other hand, there are 5 distinct consistent superstring theories whereas one might expect a TOE to be unique.

[1]This is a statement about string perturbation theory, but what is currently understood about M-theory (see below) indicates that some modified statement of this kind will still be true.

In fact, the five superstring theories look different because they were all found as weakly coupled theories. Around 1995 it was found that all five are actually weak-coupling limits of a single theory for which the spacetime dimension is 11 rather than 10, and in the 11-dimensional vacuum there are membranes rather than strings, and the effective description is provided by $D = 11$ supergravity. Remarkably, $D = 11$ is the highest dimension allowing a supergravity theory, and this theory is unique at low energy. This unified theory acquired the name "M-theory". It is more like a set of interlocking ideas than a "theory" so the name is provisional. However, one thing is clear: M-theory is not just a theory of strings but rather a "brane democracy" because various types of "brane" (extended object, a string is a 1-brane) appear in a symmetrical way.

Enough motivation. Now we start on particles.

2. The relativistic point particle

What is an elementary particle?

- (Maths) "A unitary irrep of the Poincaré group". These are classified by mass and spin.
- (Physics) "A particle without structure". The classical action for such a particle should depend only on the geometry of its worldline (plus possible variables describing its spin).

Let's pursue the physicist's answer, in the context of a D-dimensional Minkowski space-time. For zero spin the simplest geometrical action for a particle of mass m is

$$I = -mc^2 \int_A^B d\tau = -mc \int_A^B \sqrt{-ds^2} = -mc \int_{t_A}^{t_B} \sqrt{-\dot{x}^2}\, dt\,, \qquad \dot{x}^m = \frac{dx^m}{dt}\,, \tag{2.1}$$

where t is an arbitrary worldline parameter. In words, the action is the elapsed proper time between an initial point A and a final point B on the particle's worldline.

- The Minkowski coordinates are x^m ($m = 0, 1, \ldots, D-1$) and the Minkowski metric is $\eta = \text{diag.}(-1, 1, \ldots, 1)$. That is, we use a "mostly-plus" signature convention and the 'time' coordinate x^0 actually has dimensions of length (it is really c, the speed of light, times some time coordinate).
- Latin indices ($m, n, \ldots$) will be used for spacetime coordinates (but x will become X for the string). When we later need to introduce coordinates for the string worldsheet we will use greek indices ($\mu, \nu, \ldots$).
- In many string theory texts, τ is used as a time coordinate for the string worldsheet, and for the parameter of a particle worldline, but the use of τ will be reserved here for proper time (and, much later on, for "Euclidean time").

We could include terms involving the extrinsic curvature K of the worldline (inverse of the radius of curvature), which is essentially the D-acceleration, or yet higher derivative terms, i.e.

$$I = -mc \int dt \sqrt{-\dot{x}^2} \left[1 + (\ell K)^2 + \dots\right] , \tag{2.2}$$

where ℓ is a new length scale, which must be characteristic of some internal structure. As long as $K^{-1} \gg \ell$ this structure is invisible and we can neglect any extrinsic curvature corrections. Or perhaps the particle is truly elementary, and $\ell = 0$. In either case, quantization should yield a Hilbert space carrying a unitary irrep of the Poincaré group. For zero spin this means that the particle's wavefunction Ψ should satisfy the Klein-Gordon equation $(\Box - m^2)\,\Psi = 0$. There are many ways to see that this is true.

2.1 Gauge invariance

Because the worldline time t was an arbitrary parameter, we can change it to any other parameter $t^*(t)$ (which should also be monotonic, so $\dot{t}^* \neq 0$). Consider the transformation

$$t \to t^*(t) \qquad x \to x^*(t^*) = x(t)\,. \tag{2.3}$$

This definition of x^* implies that it is a scalar function on the worldline. For an infinitesimal transformation with

$$t^*(t) = t - \xi(t)\,, \tag{2.4}$$

where $\xi(t)$ is an arbitrary (but infinitesimal) function, then[2]

$$x^*(t-\xi) = x(t) \qquad \Rightarrow \qquad \delta_\xi x(t) \equiv x^*(t) - x(t) = \xi\dot{x}\,. \tag{2.5}$$

Let's now compute the variation of the integrand induced by the variation $\delta_\xi x$ of the function x:

$$\begin{aligned} \delta_\xi\sqrt{-\dot{x}^2} &= -\frac{1}{\sqrt{-\dot{x}^2}}\,\dot{x}\cdot\frac{d(\delta_\xi x)}{dt} = -\frac{1}{\sqrt{-\dot{x}^2}}\left(\dot{\xi}\dot{x}^2 + \xi\dot{x}\cdot\ddot{x}\right) \\ &= \dot{\xi}\sqrt{-\dot{x}^2} + \xi\frac{d\sqrt{-\dot{x}^2}}{dt} = \frac{d}{dt}\left(\xi\sqrt{-\dot{x}^2}\right)\,. \end{aligned} \tag{2.6}$$

This is the infinitesimal form of the transformation of a worldline scalar density; its variation contributes only a time-boundary term to the variation of I. This time-boundary term is cancelled by a shift of the integration limits. Let's verify this.

$$I[x] \to -mc\int_{t_A^*}^{t_B^*} dt^*\sqrt{-\left(\frac{dx^*}{dt^*}\right)^2} = -mc\int_{t_A-\xi(t_A)}^{t_b-\xi(t_B)} dt^*\sqrt{-\left(\frac{dx^*}{dt^*}\right)^2}\,. \tag{2.7}$$

[2]Use $\xi\dot{x}^* = \xi\dot{x} + \mathcal{O}(\xi^2)$.

Now we *relabel* the integration variable $t^* \to t$ to get

$$\begin{aligned} I[x] \to -mc \int_{t_A - \xi(t_A)}^{t_b - \xi(t_B)} dt \sqrt{-\left(\dot{x}^*\right)^2} &= -mc \int_{t_A - \xi(t_A)}^{t_b - \xi(t_B)} dt \left\{ \sqrt{-\dot{x}^2} + \delta_\xi \sqrt{-\dot{x}^2} \right\} \\ &= -mc \int_{t_A - \xi(t_A)}^{t_b - \xi(t_B)} dt \sqrt{-\dot{x}^2} - mc \int_{t_A}^{t_B} \delta_\xi \sqrt{-\dot{x}^2} \, . \end{aligned} \tag{2.8}$$

We are keeping only the terms linear in ξ and discarding anything of higher order (that's what we mean by "infinitesimal"). Now we expand the first term to first order in ξ, which gives us I plus a time-boundary term, and we use (2.6) to rewrite the last integral as another time-boundary term:

$$I[x] \to \left[I[x] + mc \left[\xi \sqrt{-\dot{x}^2} \right]_A^B \right] - mc \left[\xi \sqrt{-\dot{x}^2} \right]_A^B = I[x] \, . \tag{2.9}$$

The time-boundary terms have cancelled.

Gauge invariance is **not** a symmetry. Instead it implies a redundancy in the description. We can remove the redundancy by imposing a gauge-fixing condition. For example, in standard Minkowski spacetime coordinates. $(x^0, \vec{x})$ we may choose the "temporal gauge"

$$x^0(t) = c\, t \, . \tag{2.10}$$

Since $\delta_\xi x^0 = c\,\xi$ when $x^0 = c\,t$, insisting on this gauge choice implies $\xi = 0$; i.e. no gauge transformation is compatible with the gauge choice, so the gauge is fixed. In this gauge

$$I = -mc^2 \int dt \sqrt{1 - v^2/c^2} = \int dt \left\{ -mc^2 + \frac{1}{2} m v^2 \left[1 + \mathcal{O}\left(v^2/c^2 \right) \right] \right\} , \tag{2.11}$$

where $v = |\dot{\vec{x}}|$. The potential energy is therefore the rest mass energy mc^2, which we can subtract because it is constant. We can then take the $c \to \infty$ limit to get the non-relativistic particle action

$$I_{NR} = \frac{1}{2} m \int dt \, |d\vec{x}/dt|^2 \, . \tag{2.12}$$

From now on we set $c = 1$.

2.2 Hamiltonian formulation

If we start from the gauge-invariant action with $L = -m\sqrt{-\dot{x}^2}$, then

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{-\dot{x}^2}} \qquad \Rightarrow \qquad p^2 + m^2 \equiv 0 \, . \tag{2.13}$$

So not all components of p are independent, which means that *we cannot solve for $\dot{x}$ in terms of p.* Another problem is that

$$H = \dot{x} \cdot p - L = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} \equiv 0 \, , \tag{2.14}$$

so *the canonical Hamiltonian is zero.*

What do we do? Around 1950 Dirac developed methods to deal with such cases. We call the mass-shell condition $p^2 + m^2 = 0$ a "primary" constraint because it is a direct consequence of the definition of conjugate momenta. Sometimes there are "secondary" constraints but we will never encounter them. According to Dirac we should relax the constraint to allow all components of p to be independent but then impose it with a Lagrange multiplier. Since the canonical Hamiltonian H is zero we end up with the phase-space action

$$I[x,p;e] = \int dt \left\{ \dot{x} \cdot p - \frac{1}{2} e \left(p^2 + m^2 \right) \right\} , \tag{2.15}$$

where $e(t)$ is the Lagrange multiplier.

We can easily check this result by eliminating the variables p and e:

- Use the p equation of motion $p = e^{-1}\dot{x}$ to get the new action

  $$I[x;e] = \frac{1}{2} \int dt \left\{ e^{-1}\dot{x}^2 - em^2 \right\} . \tag{2.16}$$

  At this point it looks as though we have '1-dim. scalar fields" coupled to 1-dim. "gravity", with "cosmological constant" m^2; in this interpretation e is the square root of the 1-dim. metric, i.e. the "einbein".

- Now eliminate e from (2.16) using the e equation of motion[3] $me = \sqrt{-\dot{x}^2}$, to get the standard point particle action $I = -m \int dt \sqrt{-\dot{x}^2}$.

Elimination lemma. When is it legitimate to solve an equation of motion and substitute the result back into the action to get a new action? Let the action $I[\psi, \phi]$ depend on two sets of variables ψ and ϕ, such that the equation $\delta I / \delta \phi = 0$ can be solved algebraically for the variables ϕ as functions of the variables ψ, i.e. $\phi = \phi(\psi)$. In this case

$$\left. \frac{\delta I}{\delta \phi} \right|_{\phi = \phi(\psi)} \equiv 0 . \tag{2.17}$$

The remaining equations of motion for ψ are then equivalent to those obtained by variation of the new action $\hat{I}[\psi] = I[\psi, \phi(\psi)]$, i.e. that obtained by back-substitution. This follows from the chain rule and (2.17):

$$\frac{\delta \hat{I}}{\delta \psi} = \left. \frac{\delta I}{\delta \psi} \right|_{\phi = \phi(\psi)} + \frac{\delta \phi(\psi)}{\delta \psi} \left. \frac{\delta I}{\delta \phi} \right|_{\phi = \phi(\psi)} = \left. \frac{\delta I}{\delta \psi} \right|_{\phi = \phi(\psi)} . \tag{2.18}$$

Moral: If you use some equations of motion to eliminate a set of variables $\{A\}$ then you can substitute the result into the action, to get a new action for the remaining

[3]The equation is actually $me = \pm\sqrt{-\dot{x}^2}$, where the sign corresponds to the sign of the energy (relativity always allows both signs). Here we are choosing the energy to be positive.

variables, provided that the equations used are those found by varying the original action with respect to the set of variables $\{A\}$. However, if you solve for $\{A\}$ by using the equations of motion of $\{B\}$ then back-substitution into the *action* is not legitimate unless $\{B\} = \{A\}$, even though it *is* legitimate to make this substitution into the equations of motion.

The action (2.15) is still gauge invariant. The gauge transformations are now[4]

$$\delta_\xi x = \xi \dot{x}\,, \qquad \delta_\xi p = \xi \dot{p}\,, \qquad \delta_\xi e = \frac{d}{dt}\left(e\xi\right)\,, \tag{2.19}$$

which shows that x and p are worldline scalars but e is is scalar density ($e \equiv \det e$ in 1D). We shall call the above variations the Diff_1 variations because they are induced by a diffeomorphism (1D general coordinate transformation, or reparametrization) of the worldline.

The action (2.15) is *also* invariant under the much simpler gauge transformations

$$\delta_\alpha x = \alpha(t) p\,, \qquad \delta_\alpha p = 0\,, \qquad \delta_\alpha e = \dot{\alpha}\,. \tag{2.20}$$

Let's call this the "canonical" gauge transformation (for reasons that will become clear). In fact,

$$\delta_\alpha I = \frac{1}{2}\left[\alpha\left(p^2 - m^2\right)\right]_{t_A}^{t_B}\,, \tag{2.21}$$

which is zero if $\alpha(t_A) = \alpha(t_B) = 0$.

The Diff_1 and canonical gauge transformations are equivalent (at least if we ignore time-boundary terms) because they differ by a "trivial" gauge transformation (see Q.I.1):

- Trivial gauge invariances. Consider $I[\psi, \phi]$ again and transformations

$$\delta_f \psi = f \frac{\delta I}{\delta \phi}\,, \qquad \delta_f \phi = -f \frac{\delta I}{\delta \psi}\,, \tag{2.22}$$

  for *arbitrary* function f (we assume that ϕ and ψ satisfy b.c.s for which the functional derivatives are defined). This gives $\delta_f I = 0$, so the action is gauge invariant. As the gauge transformations are zero "on-shell" (i.e. using equations of motion) they have no physical effect. *Any two sets of gauge transformations that differ by a trivial gauge transformation have equivalent physical implications.*

[4]Recall that the variation $\delta t = -\xi(t)$ produces a time-boundary term; we'll ignore these terms now, or assume that they are zero because of endpoint conditions on the gauge transformation parameters.

If we fix the gauge invariance by choosing the temporal gauge $x^0(t) = t$ then we have

$$\dot{x}^m p_m = \dot{\vec{x}} \cdot \vec{p} - p^0 \,, \tag{2.23}$$

so in this gauge the canonical Hamiltonian is

$$H = p^0 = \pm\sqrt{|\vec{p}|^2 + m^2} \,, \tag{2.24}$$

where we have used the constraint to solve for p^0. The sign ambiguity is typical for a relativistic particle.

The canonical Hamiltonian *depends on the choice of gauge*. Another possible gauge choice is *light-cone gauge*. Choose phase-space coordinates

$$\begin{aligned} x^\pm &= \frac{1}{\sqrt{2}}\left(x^1 \pm x^0\right) \,, \qquad \mathbf{x} = \left(x^2, \ldots, x^{D-1}\right) \\ p_\pm &= \frac{1}{\sqrt{2}}\left(p_1 \pm p_0\right) \,, \qquad \mathbf{p} = \left(p_2, \ldots, p_{D-1}\right) \,. \end{aligned} \tag{2.25}$$

Then

$$\dot{x}^m p_m = \dot{x}^+ p_+ + \dot{x}^- p_- + \dot{\mathbf{x}} \cdot \mathbf{p} \,, \qquad p^2 \equiv \eta^{mn} p_m p_n = 2p_+ p_- + |\mathbf{p}|^2 \,. \tag{2.26}$$

The latter equation follows from the fact that the non-zero components of the Minkowski metric in light-cone coordinates are

$$\eta^{+-} = \eta^{-+} = 1 = \eta_{-+} = \eta_{+-} \,, \qquad \eta^{IJ} = \eta_{IJ} = \delta_{IJ} \,, \quad I, J = 1, \ldots, D-2. \tag{2.27}$$

It also follows from this that

$$p^+ = p_- \,, \qquad p^- = p_+ \,. \tag{2.28}$$

The light-cone gauge (which can be viewed as an infinite Lorentz boost of the temporal gauge) is

$$x^+(t) = t \,. \tag{2.29}$$

Since $\delta_\alpha x^+ = \alpha p^+ = \alpha p_-$ the gauge is fixed provided that $p_- \neq 0$. In this gauge

$$\dot{x}^m p_m = \dot{\mathbf{x}} \cdot \mathbf{p} + \dot{x}^- p_- + p_+ \,, \tag{2.30}$$

so the canonical Hamiltonian is now

$$H = -p_+ = \frac{|\mathbf{p}|^2 + m^2}{2p_-} \,, \tag{2.31}$$

where we have used the mass-shell constraint to solve for p_+.

- *Poisson brackets.* For mechanical model with action[5]

$$I[q,p] = \int dt \left[\dot{q}^I p_I - H(q,p)\right] \tag{2.32}$$

the Poisson bracket of any two functions (f,g) on phase space is

$$\{f,g\}_{PB} = f\left[\frac{\overleftarrow{\partial}}{\partial q^I}\frac{\partial}{\partial p_I} - \frac{\overleftarrow{\partial}}{\partial p_I}\frac{\partial}{\partial q^I}\right] g \tag{2.33}$$

where the backwards arrow over a derivative indicates that it acts to the left[6]. In particular,

$$\left\{q^I, p_J\right\}_{PB} = \delta^I_J \,. \tag{2.34}$$

- *More generally*, we start from a symplectic manifold, a phase-space with coordinates z^A and a symplectic (closed, invertible) 2-form $\Omega = \frac{1}{2}\Omega_{AB}\, dz^A \wedge dz^B$. Locally, since $d\Omega = 0$,

$$\Omega = d\omega \,, \qquad \omega = dz^A f_A(z) \,, \tag{2.35}$$

and the action in local coordinates is

$$I = \int dt \left[\dot{z}^A f_A(z) - H(z)\right] \,. \tag{2.36}$$

The PB of functions (f,g) is defined as

$$\{f,g\}_{PB} = \Omega^{AB}\frac{\partial f}{\partial z^A}\frac{\partial g}{\partial z^B} \,, \tag{2.37}$$

where Ω^{AB} is the inverse of Ω_{AB}. The PB is an antisymmetric bilinear product, from its definition. Also, for any three functions (f,g,h),

$$d\Omega = 0 \quad \Leftrightarrow \quad \left\{\{f,g\}_{PB}\,, h\right\}_{PB} + \quad \text{cyclic permutations} \equiv 0 \,. \tag{2.38}$$

In other words, the PB satisfies the Jacobi identity, and is therefore a Lie bracket, as a consequence of the closure of the symplectic 2-form.

- *Darboux theorem.* This states that there exist local coordinates[7] such that

$$\Omega = dp_I \wedge dq^I \quad \Rightarrow \quad \omega = p_I dq^I + d() \,. \tag{2.39}$$

This leads to the action (2.32) and the definition (2.33) of the PB.

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[5] This action is not invariant under time reparametrizations unless $H \equiv 0$ but it can be converted to one that *is*; see Q.I.2 for an illustration of this point.

[6] This way of writing the PB has advantages when it is generalised to phase spaces with anti-commuting coordinates.

[7] "Local" means at a point **and** in some neighbourhood of that point.

- *Canonical transformations.* Any function Q on phase-space is the generator of an infinitesimal change of phase-space coordinates, which implies an infinitesimal variation of any function f of these coordinates; for parameter ϵ, this variation is
$$\delta_\epsilon f = \{f, Q\}_{PB}\, \epsilon\,. \tag{2.40}$$
Suppose that we have Darboux coordinates (q^I, p_I). Then
$$\delta_\epsilon q^I = \epsilon \frac{\partial Q}{\partial p_I} \qquad \delta_\epsilon p_I = -\epsilon \frac{\partial Q}{\partial q^I}\,. \tag{2.41}$$
Notice that
$$\delta_\epsilon \left(dp_I \wedge dq^I\right) = \epsilon \left[dp_I \wedge d\left(\frac{\partial Q}{\partial p_I}\right) - d\left(\frac{\partial Q}{\partial q^I}\right) \wedge dq^I\right] = 0\,. \tag{2.42}$$
The last equality follows from the symmetry of mixed partial derivatives.

  In other words, the transformation generated by Q preserves the form of the symplectic 2-form, equivalently the Poisson bracket. Such transformations are called *symplectic diffeomorphisms* (Maths) or *canonical transformations* (Phys.)

  In differential geometry a symplectic diffeomorphism is equivalent to a choice of vector field ξ such that $\mathcal{L}_\xi \Omega = 0$, where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. This condition (combined with $d\Omega = 0$) implies that ξ is a Hamiltonian vector field, which means that it takes the form $\xi = \Omega^{AB}\partial_A Q \partial_B$ for some function Q, which is the function generating the corresponding canonical transformation.

2.2.1 Gauge invariance and first-class constraints

Consider the action
$$I = \int dt \left\{\dot{q}^I p_I - \lambda^i \varphi_i(q,p)\right\}\,, \qquad I = 1,\dots,N \quad ;\; i = 1,\dots,n < N\,. \tag{2.43}$$
The Lagrange multipliers λ^i impose the phase-space constraints $\varphi_i = 0$. Let us suppose that
$$\{\varphi_i, \varphi_j\}_{PB} = f_{ij}{}^k \varphi_k \tag{2.44}$$
for some phase-space *structure functions* $f_{ij}{}^k = -f_{ji}{}^k$. In this case we say that the constraints are “first-class”. Otherwise the constraints are either “second-class” or a mixture of first and second class, but here we consider only the simple case in which all constraints are first-class[8]. The special feature of first-class constraints is that *they generate gauge invariances.*

[8]Strictly speaking, the adjective “first class” applies to sets of constraints, not individual constraints, unless there is only one.

Lemma. The infinitesimal transformations of (2.41), but with constant parameter ϵ promoted to a function $\epsilon(t)$, are such that[9]

$$\delta_\epsilon \left(\dot{q}^I p_I\right) = \dot{\epsilon} Q + \frac{d}{dt}() \; . \tag{2.45}$$

Here's the proof:

$$\begin{aligned} LHS &= \frac{d}{dt}\left(\epsilon \frac{\partial Q}{\partial p_I}\right) p_I - \epsilon \dot{q}^I \frac{\partial Q}{\partial q^I} = \frac{d}{dt}\left(\epsilon\, p_I \frac{\partial Q}{\partial p_I}\right) - \epsilon \left(\dot{p}_I \frac{\partial Q}{\partial p_I} + \dot{q}^I \frac{\partial Q}{\partial q^I}\right) \\ &= \frac{d}{dt}\left(\epsilon p_I \frac{\partial Q}{\partial p_I}\right) - \epsilon \dot{Q} = \frac{d}{dt}\left[\epsilon \left(p_I \frac{\partial Q}{\partial p_I} - Q\right)\right] + \dot{\epsilon} Q \, . \end{aligned} \tag{2.46}$$

This becomes a total time derivative for $\dot{\epsilon} = 0$ because the transformation is then canonical.

Applying this for $\epsilon Q = \epsilon^i \varphi_i$ we get

$$\delta_\epsilon \left(\dot{q}^I p_I\right) = \dot{\epsilon}^i \varphi_i + \frac{d}{dt}() \; , \tag{2.47}$$

and we also have

$$\delta_\epsilon \left(\lambda^i \varphi_i\right) = \delta_\epsilon \lambda^i \varphi_i + \lambda^i \epsilon^j \left\{\varphi_i, \varphi_j\right\}_{PB} = \left(\delta_\epsilon \lambda^k + \lambda^i \epsilon^j f_{ij}{}^k\right) \varphi_k \, , \tag{2.48}$$

where we use (2.44) in the second equality. Putting these result together, we have

$$\delta_\epsilon I = \int dt \left\{ \left(\dot{\epsilon}^k - \delta_\epsilon \lambda^k - \lambda^i \epsilon^j f_{ij}{}^k\right) \varphi_k + \frac{d}{dt}() \right\} \; . \tag{2.49}$$

As the Lagrange multipliers are not functions of canonical variables, their transformations can be chosen independently. If we choose

$$\delta_\epsilon \lambda^k = \dot{\epsilon}^k + \epsilon^i \lambda^j f_{ij}{}^k \, , \tag{2.50}$$

then $\delta_\epsilon I$ is a surface term, which is zero if we impose the b.c.s $\epsilon^i(t_A) = \epsilon^i(t_B) = 0$.

The point particle is a very simple (abelian) example. The one constraint is

$$\varphi = \frac{1}{2}\left(p^2 + m^2\right) \; , \tag{2.51}$$

and it is trivially first-class. It generates the canonical gauge transformations:

$$\begin{aligned} \delta_\alpha x &= \frac{1}{2}\alpha \left\{x, p^2 + m^2\right\}_{PB} = \alpha p \, , \\ \delta_\alpha p &= \frac{1}{2}\alpha \left\{p, p^2 + m^2\right\}_{PB} = 0 \, , \end{aligned} \tag{2.52}$$

[9]There is also a $\epsilon \partial Q/\partial t$ term if Q has an explicit t-dependence, but this possibility is not relevant when t is just the integration parameter of a time-reparametrization invariant action.

and if we apply the formula (2.50) to get the gauge transformation of the einbein, we find that $\delta_\alpha e = \dot{\alpha}$.

The general model (2.43) also includes the string, as we shall see later. This is still a rather simple case because the structure functions are constants, which means that the constraint functions φ_i span a (non-abelian) Lie algebra. In such cases the transformation (2.50) is a Yang-Mills gauge transformation for a 1D YM gauge potential.

2.2.2 Gauge fixing

We can fix the gauge generated by a set of n first-class constraints by imposing n gauge-fixing conditions

$$\chi^i(q,p) = 0 \qquad i = 1, \dots, n\,. \tag{2.53}$$

The gauge transformation of these gauge-fixing functions is

$$\delta_\epsilon \chi^i = \left\{\chi^i, \varphi_j\right\}_{PB} \epsilon^j\,, \tag{2.54}$$

so if we want $\delta_\epsilon \chi^i = 0$ to imply $\epsilon^j = 0$ for all j (which is exactly what we do want in order to fix the gauge completely) then we must choose the functions χ^i such that

$$\det \left\{\chi^i, \varphi_j\right\}_{PB} \neq 0\,. \tag{2.55}$$

This is a useful test for any proposed gauge fixing condition.

In addition to requiring that the gauge-fixing conditions $\chi^i = 0$ actually do fix the gauge, it should also be possible to make a gauge transformation to ensure that $\chi^i = 0$ if this is not already the case. In particular, if $\chi^i = f^i$ for infinitesimal functions f^i , and $\hat{\chi}^i = \chi^i + \delta_\epsilon \chi^i$, then $\hat{\chi}^i = f^i + \delta_\epsilon \chi^i$ and we should be able to find parameters ϵ^i such that $\hat{\chi}^i = 0$. This requires us to solve the equation

$$\left\{\chi^i, \varphi_j\right\}_{PB} \epsilon^j = -f^i \tag{2.56}$$

for ϵ^i, but a solution exists for arbitrary f^i iff the matrix $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant.

Corollary. Whenever $\{\chi^i, \varphi_j\}_{PB}$ has zero determinant, two problems arise. One is that the gauge fixing conditions don't completely fix the gauge, and the other is that you can't always arrange for the gauge fixing conditions to be satisfied by making a gauge transformation. This is a *very general point*. Consider the Lorenz gauge $\partial \cdot A = 0$ in electrodynamics (yes, that's Ludwig Lorenz, not Henrik Lorentz of the Lorentz transformation). A gauge transformation $A \to A + d\alpha$ of the gauge condition gives $\Box\alpha = 0$, which does *not* imply that $\alpha = 0$; the gauge has not been fixed completely. It is also true, and for the same reason, that you can't always make a gauge transformation to get to the Lorenz gauge if $\partial \cdot A$ is not zero, even if it is arbitrarily close to zero: the reason is that the operator $\Box$ is not invertible because there are non-zero solutions of the wave equation that cannot be eliminated by imposing the b.c.s permissible for hyperbolic partial differential operators. The Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ does not have this problem because ∇^2 is invertible for appropriate b.c.s (but it breaks manifest Lorentz invariance).

A similar problem arises if we try to fix the gauge invariance of the action (2.43) by imposing conditions on the Lagrange multipliers. More on this later.

2.2.3 Continuous symmetries and Noether's theorem

In addition to its gauge invariance, the point particle action is invariant (in a Minkowski background) under the Poincaré transformations, i.e, spacetime translations $x^m \to x^m + a^m$ and Lorentz "rotations"

$$\begin{aligned} x^m &\to \Lambda^m{}_n x^n \,, \qquad \Lambda^T \eta \Lambda = \eta \,, \\ p_m &\to p_n \left(\Lambda^{-1}\right)^n{}_m = \Lambda_m{}^n p_n \,. \end{aligned} \tag{2.57}$$

Because these are continuous transformations, we may consider transformations close to the identity. For Lorentz transformations we write

$$\Lambda^m{}_n = \delta^m_n + \omega^m{}_n + \mathcal{O}(\omega^2) \,, \qquad \eta_{mp}\omega^p{}_n \equiv \omega_{mn} = -\omega_{nm} \,, \tag{2.58}$$

where $\omega^m{}_n$ is assumed "small". Similarly, we may suppose a^m to be "small". The resulting "infinitesimal" transformations are

$$\begin{aligned} x^m &\to x^m + \delta x^m \,, \qquad \delta x^m = a^m + \omega^m{}_n x^n \,, \\ p_m &\to p_m + \delta p_m \,, \qquad \delta p_m = \omega_m{}^n p_n \,. \end{aligned} \tag{2.59}$$

For constant parameters, this infinitesimal transformation leaves the point particle action unchanged.

Noether's theorem says that any continuous symmetry is associated with a corresponding constant of the motion, i.e. a "conserved charge" There is an easy way to find these Noether charges. Let $I[\phi]$ be an action functional invariant under an infinitesimal transformation $\delta_\epsilon \phi$ for constant parameter ϵ. Then, when ϵ is promoted to an *arbitrary* function of t, it must be possible to write the variation of I in the form

$$\delta_\epsilon I = \int dt\, \dot\epsilon\, Q \,. \tag{2.60}$$

The phase space function Q is just what the computation of $\delta_\epsilon I$ produces[10].

[10] If any terms with higher derivatives of ϵ appear at an intermediate point in this computation then we just integrate by parts; any resulting boundary terms must cancel because $\delta_\epsilon I = 0$ when $\dot\epsilon = 0$, by hypothesis.

- **Noether's Theorem**. Q is a constant of motion. Proof: choose $\epsilon(t)$ to be zero at the endpoints of integration. In this case, integration by parts gives us

$$\delta_\epsilon I = -\int dt\, \epsilon\, \dot{Q}\,. \tag{2.61}$$

  The LHS must vanish if the equations of motion hold because these equations are found by extremizing the action for arbitrary variations of the functions on which I depends. But the RHS is zero for all possible $\epsilon(t)$ iff $\dot{Q}(t) = 0$ for all t within the integration limits (which we can choose arbitrarily). It follows that the equations of motion imply $\dot{Q} = 0$, i.e. that Q is a constant of motion.

This proves Noether's theorem[11]: a continuous symmetry implies a conserved charge (constant of motion). The symmetry has to be continuous for us to be able to consider its infinitesimal form.

Also, given Q we can recover the symmetry transformation from the formula

$$\delta_\epsilon f = \{f, \epsilon Q\}_{PB} \tag{2.62}$$

for any phase-space function f. There may be conserved charges for which the RHS of this formula is zero. These are "topological charges", which do not generate symmetries; they are not Noether charges.

To apply this proof of Noether's theorem to Poincaré invariance of the point particle action, we allow the parameters a^m and $\omega^m{}_n$ of (2.59) to be time-dependent. A calculation then shows that

$$\delta I = \int dt \left\{ \dot{a}^m \mathcal{P}_m + \frac{1}{2}\dot{\omega}^n{}_m \mathcal{J}^m{}_n \right\}, \tag{2.63}$$

where

$$\mathcal{P}_m = p_m\,, \qquad \mathcal{J}^m{}_n = x^m p_n - x_n p^m\,, \tag{2.64}$$

which are therefore the Poincaré charges. Notice that they are gauge-invariant; this is obvious for $\mathcal{P}_m$, and for $\mathcal{J}^m{}_n$ we have

$$\delta_\alpha \mathcal{J}^m{}_n = \alpha\,(p^m p_n - p_n p^m) = 0\,. \tag{2.65}$$

The gauge-invariance of Noether charges could have been anticipated from the fact that gauge-fixing cannot break symmetries., which we will now prove for gauge fixing conditions of the type just considered.

Gauge-fixing and symmetries. If we have fixed a gauge invariance by imposing gauge-fixing conditions $\chi^i = 0$, then what happens if our gauge choice does not respect a symmetry with Noether charge Q, i.e. what happens if $\{Q, \chi^i\}_{PB}$ is non-zero.

[11] See arXiv:1605.07128 for details, and a history of this proof

The answer is that the symmetry is *not* broken. The reason is that there is an intrinsic ambiguity in the symmetry transformation generated by Q whenever there are gauge invariances. We may take the symmetry transformation to be

$$\delta_\epsilon f = \{f, Q\}_{PB}\, \epsilon + \{f, \varphi_j\}_{PB}\, \alpha^j(\epsilon)\,. \tag{2.66}$$

That is, a symmetry transformation with parameter ϵ combined with a gauge transformtion for which the parameters α^i are fixed, in a way to be determined, in terms of ϵ. Because gauge transformations have no physical effect, such a transformation is as good as the one generated by Q alone. The parameters $\alpha^i(\epsilon)$ are determined by requiring that the modified symmetry transformation respect the gauge conditions $\chi^i = 0$, i.e.

$$0 = \left\{\chi^i, Q\right\}_{PB} \epsilon + \left\{\chi^i, \varphi_j\right\}_{PB} \alpha^j(\epsilon)\,. \tag{2.67}$$

As long as $\{\chi^i, \varphi_j\}_{PB}$ has non-zero determinant, we can solve this equation for all α^i in terms of ϵ.

Moral: gauge-fixing never breaks symmetries, because it just removes redundancies (see Q.I.3 for an example). If a symmetry *is* broken by some gauge choice then there is something wrong with the gauge choice!

2.2.4 Canonical Quantization

We will use Dirac's quantization prescription

$$\left\{q^I, p_J\right\}_{PB} \to -\frac{i}{\hbar}\left[\hat{q}^I, \hat{p}_J\right]\,, \tag{2.68}$$

which gives us the canonical commutation relations for the operators $\hat{q}^I$ and $\hat{p}_I$ that replace the classical phase-space coordinates:

$$\left[\hat{q}^I, \hat{p}_J\right] = i\hbar\, \delta^I_J\,. \tag{2.69}$$

Let's apply this to the point particle in temporal gauge. In this case the canonical commutation relations are $[\hat{x}^i, \hat{p}_j] = i\hbar\, \delta^i_j$ for $i, j, = 1, \dots, D-1$. We can realise these relations on eigenfunctions $\Psi(t, \vec{x})$ of $\hat{\vec{x}}$ by setting $\hat{\vec{p}} = -i\hbar\vec{\nabla}$. The Schroedinger equation is

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}\,, \qquad \hat{H} = \pm\sqrt{-\hbar^2\nabla^2 + m^2}\,. \tag{2.70}$$

Acting on this with $\hat{H}$ we get $\hat{H}^2 = i\hbar\partial_t\Psi = -\partial_t^2\Psi$, and hence

$$\left[-\partial_t^2 + \nabla^2 - (m/\hbar)^2\right]\Psi(t, \vec{x}) = 0\,. \tag{2.71}$$

This is the Klein-Gordon equation for a scalar field Ψ with mass parameter $m/\hbar$ (the mass parameter of the field equation is the particle mass divided by $\hbar$). The final result is Lorentz invariant even though this was not evident at each step.

2.2.5 Quantization for systems with first-class constraints

An alternative procedure that maintains Lorentz covariance at each step is provided by Dirac's method for quantization of systems with first-class constraints. We'll use the point particle to illustrate the idea.

- Step 1. We start from the manifestly Lorentz invariant phase space action and quantise canonically as if there were no constraint. This gives us the canonical commutation relations
$$[\hat{x}^m, \hat{p}_n] = i\hbar\delta_n^m \,. \tag{2.72}$$
We can realise this on eigenfunctions $\Psi(x)$ of $\hat{x}^m$ by setting $\hat{p}_m = -i\hbar\partial_m$.

- Step 2. Because of the gauge invariance there are unphysical states in the Hilbert space. We need to remove these with a constraint. The mass-shell constraint encodes the full dynamics of the particle, so we now impose this in the quantum theory as the *physical state condition*
$$\left(\hat{p}^2 + m^2\right)|\Psi\rangle = 0 \,. \tag{2.73}$$
This is equivalent to the Klein-Gordon equation
$$\left[\Box_D - (m/\hbar)^2\right]\Psi(x) = 0 \,, \qquad \Psi(x) = \langle x|\Psi\rangle \,. \tag{2.74}$$
where $\Box_D = \eta^{mn}\partial_m\partial_n$ is the wave operator in D-dimensions.

More generally, for the general model with first-class constraints, we impose the physical state conditions

$$\hat{\varphi}_i|\Psi\rangle = 0 \,, \quad i = 1, \ldots, n \,. \tag{2.75}$$

The consistency of these conditions requires that

$$[\hat{\varphi}_i, \hat{\varphi}_j]\,|\Psi\rangle = 0 \quad \forall\, i, j \,. \tag{2.76}$$

This would be guaranteed if we could apply the PB-to-commutator prescription of (2.68) to arbitrary phase-space functions, because this would give

$$[\hat{\varphi}_i, \hat{\varphi}_j] = i\hbar f_{ij}{}^k \hat{\varphi}_k \,, \qquad (?) \tag{2.77}$$

and the RHS annihilates physical states. However, because of operator ordering ambiguities there is no guarantee that (2.77) will be true when the functions φ_i are non-linear. We can use some of the ambiguity to redefine what we mean by $\hat{\varphi}_i$, but this may not be sufficient. *There could be a quantum anomaly.* The string will provide an example of this.
From now on we set $\hbar = 1$.

3. The Nambu-Goto string

The string analog of a particle's worldline is its "worldsheet": the 2-dimensional surface in spacetime that the string sweeps out in the course of its time evolution. Strings can be *open*, with two ends, or *closed*, with no ends. We shall start by considering a closed string. This means that the parameter σ specifying position on the string is subject to a periodic identification. The choice of period has no physical significance[12] we will choose it to be 2π; i.e. ($\sim$ means "is identified with")

$$\sigma \sim \sigma + 2\pi\,. \tag{3.1}$$

The worldsheet of a closed string is topologically a cylinder, parametrised by σ and some arbitrary time parameter t. We can consider these together as σ^μ ($\mu = 0,1$), i.e.

$$\sigma^\mu = (t, \sigma). \tag{3.2}$$

The map from the worldsheet to Minkowski space-time is specified by worldsheet fields $X^m(t,\sigma)$. Using this map we can pull back the Minkowski metric on space-time to the worldsheet to get the induced worldsheet metric

$$g_{\mu\nu} = \partial_\mu X^m \partial_\nu X^n \eta_{mn}\,. \tag{3.3}$$

The natural string analog of the point particle action proportional to the proper length of the worldline (i.e. the elapsed proper time) is the Nambu-Goto action, which is proportional to the area of the worldsheet in the induced metric, i.e.

$$I_{NG}[X] = -T \int dt \oint d\sigma \sqrt{-\det g}\,, \tag{3.4}$$

where the constant T is the string tension, and

$$X(t, \sigma + 2\pi) = X(t,\sigma)\,. \tag{3.5}$$

We are still free to change the coordinates locally, and the action doesn't change if we suppose that the X^m are worldsheet scalars. In this case, a change of coordinates $\sigma^\mu \to \sigma^\mu - \zeta^\mu$ for infinitesimal worldsheet vector field ξ induces the Diff_2 gauge transformation

$$\delta_\xi X^m = \xi^\mu \partial_\mu X^m\,. \tag{3.6}$$

This yields

$$\delta_\xi \left(\sqrt{-\det g}\right) = \partial_\mu \left(\xi^\mu \sqrt{-\det g}\right)\,, \tag{3.7}$$

[12] It would fix units for length if σ were supposed to have dimensions of length. For this reason we assume here, as we may, that the parameters σ and t are both dimensionless; in this case, and for $c = 1$, we have $[X] = L$ and $[P] = M$.

confirming that the variation of the action is at most a boundary term (with potential contributions only at initial and final times since there is no space boundary).

Varying I_{NG} with respect to X we get the NG equation of motion[13]

$$\partial_\mu\left(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu X\right) = 0\,. \tag{3.8}$$

This is just the massless wave equation for a set of 2D "scalar fields" X^m but the metric on the "2D spacetime" (the worldsheet) depends on the scalar fields, so it looks hopelessly non-linear!

3.1 Hamiltonian formulation

Denoting derivatives with respect to t by an overdot and derivatives with respect to σ by a prime, we have

$$g_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}\,. \tag{3.9}$$

This shows that the NG action can be written as

$$I_{NG} = \int dt\, L_{NG}\,, \quad L_{NG} = -T\oint d\sigma\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}\,. \tag{3.10}$$

The worldsheet momentum density $P_m(t,\sigma)$ canonically conjugate to the wordsheet fields X^m is therefore

$$P_m = \frac{\delta L}{\delta \dot{X}^m} = \frac{T}{\sqrt{-\det g}}\left[\dot{X}_m X'^2 - X'_m\left(\dot{X}\cdot X'\right)\right]\,. \tag{3.11}$$

This implies the following identities [<u>Exercise</u>: check this]

$$P^2 + (TX')^2 \equiv 0\,, \qquad X'^m P_m \equiv 0\,. \tag{3.12}$$

In addition, the canonical Hamiltonian is

$$H = \oint d\sigma \dot{X}^m P_m - L \equiv 0\,. \tag{3.13}$$

As for the particle, we should take the Hamiltonian to be a sum of Lagrange multipliers times the constraints, so we should expect the phase-space form of the action to be

$$I[X,P;e,u] = \int dt \oint d\sigma\left\{\dot{X}^m P_m - \frac{1}{2}e\left[P^2 + (TX')^2\right] - u\, X'^m P_m\right\}\,, \tag{3.14}$$

where $e(t,\sigma)$ and $u(t,\sigma)$ are Lagrange multipliers[14]. To check this, we eliminate P by using its equation of motion:

$$P = e^{-1} D_t X\,, \qquad D_t X \equiv \dot{X} - uX'\,. \tag{3.15}$$

[13]Check, as exercise, or wait until we derive it later in a simpler way.

[14]Analogous to the "lapse" and "shift" functions in the Hamiltonian formulation of GR.

We are assuming here that e is nowhere zero (as for the particle). Back substitution takes us to the action

$$I[X;e,u] = \frac{1}{2}\int dt \oint d\sigma \left\{e^{-1}\left(D_t X\right)^2 - e\left(TX'\right)^2\right\} . \tag{3.16}$$

Varying u in this new action we find that

$$u = \frac{\dot{X}\cdot X'}{X'^2} \quad \Rightarrow \quad (D_t X)^2 = \frac{\det g}{(X')^2} . \tag{3.17}$$

Here we assume that X'^2 is non-zero (but we pass over this point). Eliminating u we arrive at the action

$$I[X,e] = \frac{1}{2}\int dt \oint d\sigma \left\{e^{-1}\frac{\det g}{X'^2} - e\left(TX'\right)^2\right\} . \tag{3.18}$$

Varying this action with respect to e we find that

$$e = \frac{\sqrt{-\det g}}{T(X')^2} , \tag{3.19}$$

and back-substitution returns us to the Nambu-Goto action in its original form.

3.1.1 Alternative form of phase-space action

Notice that the phase-space constraints are equivalent to

$$\mathcal{H}_\pm = 0 , \qquad \mathcal{H}_\pm \equiv \frac{1}{4T}\left(P \pm TX'\right)^2 , \tag{3.20}$$

so we may rewrite the action as

$$I = \int dt \oint d\sigma \left\{\dot{X}^m P_m - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+\right\} , \qquad \lambda^\pm = Te \pm u . \tag{3.21}$$

Notice that the Lagrange multipliers $\lambda^\pm$ are dimensionless if the worldsheet coordinates are taken to be dimensionless.

3.1.2 Gauge invariances

From the Hamiltonian form of the NG string action (3.14) we read off the canonical Poisson bracket relations[15]

$$\{X^m(\sigma), P_n(\sigma')\}_{PB} = \delta^m_n \delta(\sigma - \sigma') , \tag{3.22}$$

where, for any two functionals $F[X,P]$ and $G[X,P]$,

$$\{F,G\}_{PB} = \oint d\sigma\, F \left[\frac{\overleftarrow{\delta}}{\delta X^m(\sigma)}\frac{\delta}{\delta P_m(\sigma)} - \frac{\overleftarrow{\delta}}{\delta P_m(\sigma)}\frac{\delta}{\delta X^m(\sigma)}\right] G .$$

[15] By means of the Fourier series representations for X and P to be given later, one can verify that this is equivalent to the definition (2.33).

This implies, in particular, that $\{(X^m)'(\sigma), P_n(\sigma')\}_{PB} = \delta_n^m \delta'(\sigma - \sigma')$.

Using (3.22), one may now compute the PBs of the constraint functions. One finds that

$$\begin{aligned}
\{\mathcal{H}_+(\sigma), \mathcal{H}_+(\sigma')\}_{PB} &= [\mathcal{H}_+(\sigma) + \mathcal{H}_+(\sigma')]\, \delta'(\sigma - \sigma')\,, \\
\{\mathcal{H}_-(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= -\,[\mathcal{H}_-(\sigma) + \mathcal{H}_-(\sigma')]\, \delta'(\sigma - \sigma')\,, \\
\{\mathcal{H}_+(\sigma), \mathcal{H}_-(\sigma')\}_{PB} &= 0\,.
\end{aligned} \tag{3.23}$$

This shows that

- The constraints are "first-class", with constant structure functions, which are therefore the structure constants of a Lie algebra

- This Lie algebra is a direct sum of two isomorphic algebras ($-\mathcal{H}_-$ obeys the same algebra as $\mathcal{H}_+$). In fact, it is the algebra

$$\text{Diff}_1 \oplus \text{Diff}_1\,. \tag{3.24}$$

  We will verify this later. Notice that this is a proper subalgebra of Diff_2. Only the $\text{Diff}_1 \oplus \text{Diff}_1$ subalgebra has physical significance because all other gauge transformations of Diff_2 are "trivial" in the sense explained earlier for the particle.

The gauge transformation of any function F on phase space is

$$\delta_\xi F = \left\{ F, \oint d\sigma \left(\xi^- \mathcal{H}_- + \xi^+ \mathcal{H}_+\right) \right\}_{PB}\,, \tag{3.25}$$

where the parameters $\xi^\pm(t,\sigma)$ are arbitrary functions. This gives

$$\begin{aligned}
\delta X &= \frac{1}{2T}\xi^- \left(P - TX'\right) + \frac{1}{2T}\xi^+ \left(P + TX'\right)\,, \\
\delta P &= -\frac{1}{2}\Big[\xi^- \left(P - TX'\right)\Big]' + \frac{1}{2}\left[\xi^+ \left(P + TX'\right)\right]'\,.
\end{aligned} \tag{3.26}$$

Notice that

$$\delta_{\xi^-}\left(P + TX'\right) = 0\,, \qquad \delta_{\xi^+}\left(P - TX'\right) = 0\,, \tag{3.27}$$

and hence $\delta_{\xi^\mp}\mathcal{H}_\pm = 0$, as expected from the fact that the algebra is a direct sum ($\mathcal{H}_+$ has zero PB with $\mathcal{H}_-$).

To verify invariance of the action one needs

$$\delta_\xi \left(\dot{X}^m P_m\right) = \dot{\xi}^+ \mathcal{H}_+ + \dot{\xi}^- \mathcal{H}_- + \frac{d}{dt}()\,, \tag{3.28}$$

and, using (3.23),

$$\begin{aligned}\delta_{\xi^\pm}\mathcal{H}_\pm(\sigma) &= \oint d\sigma'\xi^\pm(\sigma')\left\{\mathcal{H}_\pm(\sigma),\mathcal{H}_\pm(\sigma')\right\}_{PB} \\ &= \pm\oint d\sigma'\xi^\pm(\sigma')\left[\mathcal{H}_\pm(\sigma)+\mathcal{H}_\pm(\sigma')\right]\delta'(\sigma-\sigma') \\ &= \pm\left[\xi^\pm(\sigma)\right]'\mathcal{H}_\pm(\sigma)\pm\left[\xi^\pm(\sigma)\mathcal{H}_\pm(\sigma)\right]' . \end{aligned} \tag{3.29}$$

Using these results, one finds that the action is invariant provided that

$$\delta\lambda^- = \dot{\xi}^- + \lambda^-\left(\xi^-\right)' - \xi^-\left(\lambda^-\right)' , \qquad \delta\lambda^+ = \dot{\xi}^+ - \lambda^+\left(\xi^+\right)' + \xi^+\left(\lambda^+\right)' . \tag{3.30}$$

We see that $\lambda^\pm$ is a gauge potential for the $\xi^\pm$-transformation, with each being inert under the gauge transformation associated with the other, as expected from the direct sum structure of the gauge algebra. The sign differences in these two transformations are a consequence of the fact that $\mathcal{H}_+$ and $-\mathcal{H}_-$ have the same PB algebra.

3.1.3 Symmetries of NG action

The closed NG action has manifest Poincaré invariance, with Noether charges

$$\mathcal{P}_m = \oint d\sigma P_m , \qquad \mathcal{J}_{mn} = 2\oint d\sigma X_{[m}P_{n]} . \tag{3.31}$$

These are constants of the motion. [Exercise: verify that the NG equations of motion imply that $\dot{\mathcal{P}}_m = 0$ and $\dot{\mathcal{J}}_{mn} = 0$.]

N.B. We use the following notation

$$T_{[mn]} = \frac{1}{2}\left(T_{mn} - T_{nm}\right) , \qquad T_{(mn)} = \frac{1}{2}\left(T_{mn} + T_{nm}\right) . \tag{3.32}$$

In other words, we use square brackets for antisymmetrisation and round brackets for symmetrisation, in both cases with "unit strength" (which means, for tensors of any rank, that $A_{[m_1\dots m_n]} = A_{m_1\dots m_n}$ if A is totally antisymmetric, and $S_{(m_1\dots m_n)} = S_{m_1\dots m_n}$ if S is totally symmetric).

The closed NG string is also invariant under **worldsheet parity**: $\sigma \to -\sigma$ (mod 2π). The worldsheet fields (X,P) are parity even, which means that X' is parity odd and hence $(P+TX')$ and $(P-TX')$ are exchanged by parity. This implies that $\mathcal{H}_\pm$ are exchanged by parity.

3.2 Open string boundary conditions

An open string has two ends. We shall choose the parameter length to be π, so the action in Hamiltonian form is

$$I = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2}e\left[P^2 + (TX')^2\right] - uX'\cdot P \right\} . \tag{3.33}$$

What are the possible boundary conditions at the ends of the string?

Principle: the action should be stationary when the equations of motion are satisfied. In other words, when we vary the action to get the equations of motion, the boundary terms arising from integration by parts must be zero; otherwise the functional derivative of the action is not defined.

Applying this principle to the above action, we see that boundary terms can arise only when we vary X' and integrate by parts to get the derivative (with respect to σ) off the δX variation (we can ignore any boundary terms in time). These boundary terms are [Exercise: check this].

$$\delta I\big|_{\text{on-shell}} = -\int dt\, \left[\left(T^2 e X' + uP\right)\cdot \delta X\right]_{\sigma=0}^{\sigma=\pi}, \tag{3.34}$$

where, "on-shell" is shorthand for "using the equations of motion".

It would make no physical sense to fix X^0 at the endpoints, and if X^0 is free then so is $\dot{X}^0$ and hence P^0 when we use the equations of motion, so the boundary term with the factor of δX^0 will be zero only if we impose the conditions

$$u\big|_{ends} = 0\,, \qquad \left(X^0\right)'\Big|_{ends} = 0\,. \tag{3.35}$$

Given that $e \neq 0$ we conclude that

$$\left(\vec{X}'\cdot\delta\vec{X}\right)\Big|_{ends} = 0\,. \tag{3.36}$$

What this means is that *at each end* we may *choose cartesian space coordinates* $\vec{X}$ such that *for any given component*, call it X_*, we have

$$\begin{aligned} \textbf{either}\quad & X_*' = 0 \qquad \text{(Neumann b.c.s)}\,, \\ \textbf{or}\quad & \delta X_* = 0 \quad \Rightarrow \quad X_* = \bar{X}_*\ \text{(constant)} \qquad \text{(Dirichlet b.c.s)} \end{aligned} \tag{3.37}$$

There are many possibilities. The only one that does not break Lorentz invariance is *free-end* boundary conditions

$$X'\big|_{ends} = 0\,. \tag{3.38}$$

This implies that $(X')^2$ is zero at the ends of the string. The open string mass-shell constraint then implies that P^2 is zero at the endpoints, and since

$$P\big|_{ends} = e^{-1}\left(\dot{X} - uX'\right)\Big|_{ends} = e^{-1}\dot{X}\Big|_{ends}\,, \tag{3.39}$$

we deduce that $\dot{X}^2$ is zero at the ends of the string; i.e. *the string endpoints move at the speed of light.*

3.3 Monge gauge

A natural analogue of the temporal gauge for the particle is a gauge in which we set not only $X^0 = t$, to fix the time-reparametrization invariance, but also $X^1 = \sigma$, to fix the reparametrization invariance of the string[16]. This is often called the "static gauge" but this is not a good name because there is no restriction to static configurations. A better name (standard in Mathematical Biology) is "Monge gauge", after the French geometer who used it in the study of surfaces. So, the Monge gauge for the NG string is[17]

$$X^0(t,\sigma) = t \qquad X^1(t,\sigma) = \sigma \,. \tag{3.40}$$

In this gauge the action (3.14) becomes

$$\begin{aligned} I = \int dt \oint d\sigma \Big\{ \dot X^I P_I + P_0 - u\left(P_1 + X'^I P_I\right) \\ -\frac{1}{2} e\left[-P_0^2 + P_1^2 + |\mathbf{P}|^2 + T^2\left(1 + |\mathbf{X}'|^2\right)\right]\Big\}\,, \end{aligned} \tag{3.41}$$

where $I = 2, \dots, D-2$, and $\mathbf{X}$ is the $(D-2)$-vector with components X^I (and similarly for $\mathbf{P}$). We may solve the constraints for P_1, and P_0^2. Choosing the sign of P_0 corresponding to positive energy, we arrive at the action

$$I = \int dt \oint d\sigma \left\{ \dot X^I P_I - T\sqrt{1 + |\mathbf{X}'|^2 + T^{-2}\left[|\mathbf{P}|^2 + (\mathbf{X}'\cdot\mathbf{P})^2\right]}\right\}\,. \tag{3.42}$$

The expression for the Hamiltonian in Monge gauge simplifies if $\mathbf{P}$ is momentarily zero[18]; we then have

$$H = T\oint d\sigma \sqrt{1 + |\mathbf{X}'|^2} \qquad (\mathbf{P} = \mathbf{0})\,. \tag{3.43}$$

The integral equals the proper length L of the string. To see this, we observe that the induced worldsheet metric in Monge gauge is

$$\begin{aligned} ds^2\big|_{\text{ind}} &= -dt^2 + d\sigma^2 + \left|\dot{\mathbf{X}} dt + \mathbf{X}' d\sigma\right|^2 \\ &= -\left(1 - |\dot{\mathbf{X}}|^2\right) dt^2 + 2\dot{\mathbf{X}}\cdot\mathbf{X}' d\sigma dt + \left(1 + |\mathbf{X}'|^2\right) d\sigma^2\,, \end{aligned} \tag{3.44}$$

and hence

$$L = \oint \sqrt{ds^2|_{\text{ind}}(t = \text{const.})} = \oint d\sigma \sqrt{1 + |\mathbf{X}'|^2}\,. \tag{3.45}$$

[16] We could choose any linear combination of the space components of X to equal σ but locally we can always orient the axes such that this combination equals X^1.

[17] This fixes the units of length if (t,σ) are dimensionless; for this reason, it is convenient to assume here that they have dimensions of length and to leave unspecified the periodicity in σ.

[18] We assume here a moment at which the entire string is simultaneously at rest; such a moment exists only for very special string configurations, but these suffice for the argument.

Also, when $\mathbf{P} = \mathbf{0}$ the equations of motion in Monge gauge imply that $\dot{\mathbf{X}} = \mathbf{0}$, so the string is momentarily at rest. The energy of such a string is $H = TL$, and hence the (potential) energy per unit length, or energy density, of the string is

$$\mathcal{E} = T\,, \tag{3.46}$$

as expected for an ultra-relativistic string.

An ultra-relativistic string cannot support tangential momentum. Given $X^0 = t$, the constraint $X' \cdot P = 0$ becomes

$$\vec{X}' \cdot \vec{P} = 0\,, \tag{3.47}$$

which tells us that the (space) momentum density at any point on the string is orthogonal to the tangent to the string at that point; the momentum density has no tangential component. This has various consequences. One is that there can be no longitudinal waves on the string (i.e. sound waves). Only transverse fluctuations are physical.

Another consequence is that a plane circular loop of NG string cannot be supported against collapse by rotation in the plane (which can be done if $T < \mathcal{E}$). This does not mean that a plane circular loop of string cannot be supported against collapse by rotation in other planes; we'll see an example later.

3.3.1 Polyakov action

Recall that elimination of P, using $P = e^{-1} D_t X$, gives the action

$$I = \frac{1}{2}\int d^2\sigma \left\{ e^{-1}\left(D_t X\right)^2 - e\left(TX'\right)^2 \right\} \qquad \left(D_t = \dot{X} - uX'\right) \tag{3.48}$$

This is quadratic in derivatives of X. In fact,

$$I = -\frac{T}{2}\int d^2\sigma\, \tilde{\gamma}^{\mu\nu}\partial_\mu X \cdot \partial_\nu X\,, \qquad \tilde{\gamma}^{\mu\nu} = \frac{1}{Te}\begin{pmatrix} -1 & u \\ u & T^2e^2 - u^2 \end{pmatrix}\,. \tag{3.49}$$

Notice that $\det \tilde{\gamma} \equiv -1$, which allows us to write

$$\tilde{\gamma}^{\mu\nu} = \sqrt{-\det\gamma}\,\gamma^{\mu\nu}\,, \tag{3.50}$$

where $\gamma^{\mu\nu}$ is the inverse of an *independent and unconstrained* worldsheet metric. This gives us the "Polyakov" action for the NG string[19]

$$I_{poly}[X;\gamma] = -\frac{T}{2}\int d^2\sigma \sqrt{-\det\gamma}\; \gamma^{\mu\nu}\partial_\mu X \cdot \partial_\nu X\,, \tag{3.51}$$

[19] The Polyakov action was actually introduced by Brink, DiVecchia and Howe, and by Deser and Zumino. Polyakov used it in the context of a path-integral quantization of the NG string, which we will consider later.

The construction is such that only the conformal class of $\gamma_{\mu\nu}$ has been defined. In fact,

$$\gamma_{\mu\nu} = \Omega^2 \begin{pmatrix} u^2 - T^2 e^2 & u \\ u & 1 \end{pmatrix} = \Omega^2 \begin{pmatrix} -\lambda^+\lambda^- & \frac{1}{2}\left[\lambda^+ - \lambda^-\right] \\ \frac{1}{2}\left[\lambda^+ - \lambda^-\right] & 1 \end{pmatrix} \tag{3.52}$$

for some arbitrary conformal factor Ω. Equivalently,

$$ds^2(\gamma) = \Omega^2 \left(d\sigma + \lambda^+ dt\right)\left(d\sigma - \lambda^- dt\right) . \tag{3.53}$$

The conformal factor Ω cancels from the Polyakov action because this action is "Weyl invariant"; i.e. invariant under an arbitrary *local* rescaling of the metric. This is easily seen since

$$\gamma^{\mu\nu} \to \Omega^{-2}\gamma^{\mu\nu}, \qquad \sqrt{-\det\gamma} \to \Omega^2\sqrt{-\det\gamma}\,, \tag{3.54}$$

so the factors of Ω cancel from $\tilde{\gamma}^{\mu\nu}$.

We have found the Polyakov action by elimination of P from the phase-space action. Recall that we established equivalence with the NG action by subsequent elimination of the Lagrange multipliers. This step can now be done more easily by eliminating the Polyakov metric directly. Variation with respect to $\gamma^{\mu\nu}$ gives us the equation

$$g_{\mu\nu} = \Omega^{-2}\gamma_{\mu\nu} \qquad \left(\Omega^{-2} = \frac{1}{2}\gamma^{\mu\nu}g_{\mu\nu}\right), \tag{3.55}$$

and hence

$$\gamma_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \Rightarrow \quad \tilde{\gamma}^{\mu\nu} = \sqrt{-\det g}\, g^{\mu\nu} . \tag{3.56}$$

In words, the equation of motion for $\gamma_{\mu\nu}$ sets it equal to the induced metric $g_{\mu\nu}$ up to a conformal factor, which cancels from $\tilde{\gamma}^{\mu\nu}$. Back substitution now gives us

$$I_{poly} \to -\frac{T}{2}\int d^2\sigma\sqrt{-\det g}\, g^{\mu\nu}g_{\mu\nu} = -T\int d^2\sigma\sqrt{-\det g}\,, \tag{3.57}$$

which is the NG action.

The Polyakov action also simplifies the derivation of the NG equations of motion. Varying it with respect to X, we find the equation of motion

$$\partial_\mu\left(\tilde{\gamma}^{\mu\nu}\partial_\nu X^m\right) = 0\,. \tag{3.58}$$

On using (3.56), we recover the NG equation of motion (3.8).

3.4 Conformal gauge

On a 2D manifold any metric is conformally flat, which means that we may choose local coordinates[20] for which the metric is a conformal factor times a standard flat metric. For a Lorentzian metric on the worldsheet of a string, this means that local

[20]Recall that "local" means at a point *and* in some neighbourhood of that point

coordinates exist for which the Polyakov metric $\gamma_{\mu\nu}$ is some conformal factor Ω^2 times the standard 2D Minkowski metric $\eta_{\mu\nu} = \text{diag.}(-1, 1)$, i.e.

$$\gamma_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \quad \Leftrightarrow \quad ds^2(\gamma) = \Omega^2 (d\sigma + dt)(d\sigma - dt)\,. \tag{3.59}$$

From (3.53) we see that this is equivalent to imposing the following gauge-fixing conditions on the Lagrange multipliers:

$$\lambda^+ = \lambda^- = 1 \qquad \Leftrightarrow \qquad e = \frac{1}{T} \quad \& \quad u = 0\,. \tag{3.60}$$

This is the *conformal gauge*.

A feature of the conformal gauge choice is that it does not completely fix the gauge; there is a *residual* gauge invariance. One way to see this is to use the conformal gauge in the expressions of (3.30) for the gauge transformations of $\lambda^\pm$. This gives us

$$\begin{aligned} \delta\lambda^- &= \dot{\xi}^- + \left(\xi^-\right)' = 2\,\partial_+ \xi^- \\ \delta\lambda^+ &= \dot{\xi}^+ - \left(\xi^+\right)' = 2\partial_- \xi^+\,, \end{aligned} \tag{3.61}$$

where $\partial_\pm$ are partial derivatives with respect to the light-cone worldsheet coordinates[21]

$$\sigma^\pm = t \pm \sigma \qquad \Leftrightarrow \qquad \partial_\pm = \frac{1}{2}\left(\partial_t \pm \partial_\sigma\right)\,. \tag{3.62}$$

We see that $\delta\lambda^\pm = 0$ if

$$\partial_\pm \xi^\mp = 0 \qquad \left[\Rightarrow \quad \xi^\pm = \xi^\pm(\sigma^\pm)\right]\,. \tag{3.63}$$

In other words the conformal gauge conditions are preserved by those gauge transformations for which ξ^+ is a function only of σ^+ and ξ^- is a function only of σ^-

Using $P = T\dot{X}$, the residual gauge transformation of X can be read off from (3.26):

$$\delta_\xi X^m = \xi^- \partial_- X^m + \xi^+ \partial_+ X^m\,. \tag{3.64}$$

By interpreting $\xi^\pm$ as the lightcone components of a wordsheet vector field $\xi = \xi^\mu \partial_\mu$, we can rewrite this residual transformation as

$$\delta_\xi X^m = \xi^\mu \partial_\mu X^m\,. \tag{3.65}$$

Using this, we can compute the residual transformation of the induced metric:

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= 2\partial_{(\mu} X \cdot \partial_{\nu)} \left(\xi^\lambda \partial_\lambda X\right) \\ &= \xi^\lambda \partial_\lambda g_{\mu\nu} + 2\partial_{(\mu} \xi^\lambda g_{\nu)\lambda} \equiv \left[\mathcal{L}_\xi g\right]_{\mu\nu}\,. \end{aligned} \tag{3.66}$$

[21] Notice that we are defining worldsheet lightcone coordinates with different conventions to those used for spacetime lightcone coordinates.

This is the Lie derivative of the induced metric with respect to the vector field ξ, otherwise known as the infinitesimal Diff_2 transformation generated by ξ. The transformation of X is also its Lie derivative with respect to ξ (in its action on a worldsheet scalar).

However, the induced metric in conformal gauge is conformal to the 2D Minkowski metric. This is because it is conformal to the Polyakov metric (by the Polyakov metric equations, which are also the Hamiltonian constraints) and the Polyakov metric is conformal to the 2D Minkowski metric (by definition of conformal gauge). In other words, $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$ in conformal gauge and we can choose to set $\Omega^2 = 1$ as long as we allow this to change under a residual gauge transformation. This means that the worldsheet vector field ξ must satisfy

$$[\mathcal{L}_\xi \eta]_{\mu\nu} = \chi\, \eta_{\mu\nu}\,, \tag{3.67}$$

where the function χ is a small change in the conformal factor caused by the residual gauge transformation. This is the conformal Killing equation, in this case for the standard 2D Minkowski metric. As this metric is independent of the worldsheet coordinates, the conformal Killing equations simplifies to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \chi\, \eta_{\mu\nu}\,. \tag{3.68}$$

By choosing worldsheet lightcone coordinates, and using the fact that $\eta_{++} = \eta_{--} = 0$, one finds that the lightcone components $\xi^\pm$ of the worksheet vector field $\xi = \xi^\mu \partial_\mu$ are restricted precisely by (3.63).

This provides us with an interpretation for the parameters $\xi^\pm(\sigma^\pm)$ of a residual gauge transformation. They are solutions of the conformal Killing equation for 2D Minkowski spacetime, and hence associated with infinitesimal conformal transformations of coordinates for this spacetime. It is a special feature of 2D Minkowski spacetime that there are an infinite number of conformal Killing vector fields. For example, in 4D, there are precisely 15 independent ones, and their commutator algebra is the algebra of $SO(4,2)$. More generally one finds that they span the algebra $SO(D,2)$. For $D = 2$ this is

$$SO(2,2) = Sl(2;\mathbb{R}) \oplus Sl(2;\mathbb{R})\,, \tag{3.69}$$

but each $Sl(2;\mathbb{R})$ factor is actually a subalgebra of an infinite-dimensional Diff_1 algebra. So the algebra of residual gauge transformations of the NG string in conformal gauge is

$$\text{Diff}_1 \oplus \text{Diff}_1\,. \tag{3.70}$$

This is *exactly the same as the algebra of canonical gauge transformations prior to gauge fixing.* All that the gauge fixing has done is to restrict the parameters[22].

[22] In this respect, the conformal gauge is analogous to the Lorenz gauge $\partial \cdot A$ in electrodynamics; the gauge condition is Lorentz invariant but there is a residual $U(1)$ gauge invariance $A \to A + d\alpha$ with the parameter restricted by $\Box\alpha = 0$.

3.5 Solving the NG equations in conformal gauge

In conformal gauge, the Polyakov action simplifies to

$$I = -\frac{T}{2}\int d^2\sigma\, \eta^{\mu\nu}\partial_\mu X\cdot\partial_\nu X\,. \tag{3.71}$$

The field equation for X is now *linear*:

$$\Box_2 X = 0 \qquad\qquad \left(\Box_2 = -\partial_t^2 + \partial_\sigma^2\right)\,. \tag{3.72}$$

Equivalently,

$$\partial_+\partial_- X = 0 \quad\Rightarrow\quad X = X_L(\sigma^+) + X_R(\sigma^-)\,. \tag{3.73}$$

In other words, the general solution of the NG equation of motion in conformal gauge is a linear superposition of a left-moving wave profile (given by X_L) with a right-moving wave profile (given by X_R). These wave profiles move at the speed of light, which is characteristic of an ultra-relativistic string.

However, there is more than this to solving the NG equations. To linearize the equations we had to impose $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$ for some conformal factor Ω^2, and this tells us that

$$\dot X\cdot X' = 0\,, \qquad -\dot X^2 = (X')^2 = \Omega^2 \quad\Rightarrow\quad \dot X^2 + (X')^2 = 0\,. \tag{3.74}$$

These are just the Hamiltonian constraints $\mathcal{H}_\pm = 0$, simplified by using the conformal-gauge relation $P = T\dot X$. So, to solve the NG equations we must ensure that our solution $X = X_L + X_R$ also satisfies $(\dot X \pm X')^2 = 0$ (or verify that the induced metric is conformally flat). Using the lightcone worldsheet coordinates $\sigma^\pm$, we can rewrite these conditions as

$$(\partial_\pm X)^2 = 0\,. \tag{3.75}$$

Notice that $\Box_2 X^0 = 0$ is solved by $X^0 = t$. There are more general solutions but we can use the residual gauge invariance to arrange for $X^0 = t$ without loss of generality. Let's check that $X^0 = t$ fixes the residual gauge invariance. The residual gauge transformation of X^0 is

$$\delta_\xi X^0 = \xi^+(\sigma^+)\partial_+ X^0 + \xi^-(\sigma^-)\partial_- X^0\,. \tag{3.76}$$

Setting $X^0 = t$ and insisting that $\delta_\xi X^0 = 0$ (to maintain the gauge choice) we find that

$$0 = \xi^+(\sigma^+) + \xi^-(\sigma^-) \quad\Rightarrow\quad \pm\,\xi^\pm = \bar\xi \text{ (constant)}. \tag{3.77}$$

Thus, the choice $X^0 = t$ has now fixed all of the residual gauge invariance except for the transformation

$$\delta_{\bar\xi} X = \bar\xi\,(\partial_+ - \partial_-)\,X = \bar\xi X'\,, \tag{3.78}$$

which is the transformation induced by a constant shift of σ; this is an invariance of the closed string (it makes no difference which point on the string corresponds to $\sigma = 0$) but the b.c.s for an open string require $\bar{\xi} = 0$. So we have just enough residual gauge invariance (and slightly more for a closed string) to set $X^0 = t$. If we do so then the constraints (3.75) become

$$\left|2\,\partial_\pm \vec{X}\right|^2 = 1\,. \tag{3.79}$$

See Q.II.1 for a geometrical interpretation of this result.

Let's apply these ideas to the closed string configuration in a 5D Minkowski spacetime with $X^0 = t$ and

$$Z \equiv X^1 + iX^2 = \frac{1}{2n} e^{in(\sigma - t)}\,, \qquad W \equiv X^3 + iX^4 = \frac{1}{2m} e^{im(\sigma + t)}\,. \tag{3.80}$$

This configuration solves the 2D wave equation and is periodic in σ with period 2π. If the induced metric is conformal to 2D Minkowski then it will also solve the full NG equations. A calculation gives

$$\begin{aligned} ds^2\big|_{\text{ind}} &= -\left(dX^0\right)^2 + |dZ|^2 + |dW|^2 \\ &= -dt^2 + \frac{1}{4}\left(d\sigma - dt\right)^2 + \frac{1}{4}\left(d\sigma + dt\right)^2 \\ &= \frac{1}{2}\left(-dt^2 + d\sigma^2\right)\,. \end{aligned} \tag{3.81}$$

In other words,

$$g_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}\,. \tag{3.82}$$

This is a conformal factor (1/2) times the Minkowski metric, so the given configuration is a solution of the NG equations[23].

Let's compute the proper length of the string. Setting $t = t_0$ in the induced worldsheet metric (for some constant t_0) we see that $d\ell^2 = \frac{1}{2} d\sigma^2$, so

$$L = \frac{1}{\sqrt{2}} \oint d\sigma = \sqrt{2}\,\pi\,. \tag{3.83}$$

It is rather surprising that this should be constant, i.e. independent of t_0. This can only happen if the motion of the string supports it against collapse due to its tension. To check this, we may compute the total energy, which is

$$H = \oint d\sigma P^0 = T \oint d\sigma \dot{X}^0 = 2\pi T\,. \tag{3.84}$$

We see that

$$H = \sqrt{2}\,TL = TL + \left(\sqrt{2} - 1\right) TL\,. \tag{3.85}$$

The first term is the potential energy of the string. The second term is therefore kinetic energy. The string is supported against collapse by rotation in the Z and W planes. This solution has the special property of being stationary; the string is motionless in a particular rotating frame, supported against collapse by its rotation in the Z and W planes (neither of which coincide with the plane of the string).

[23] It is not necessary to verify (3.79), but this *is* a good check.

3.5.1 Conformal invariance of conformal gauge action

Let's now verify the residual gauge invariance of the conformal gauge action for the closed string. We can write this action as[24]

$$I[X] = 2T \int dt \oint d\sigma\, \partial_+ X \cdot \partial_- X \,. \tag{3.86}$$

The variation of this action that results from the transformation $\delta X = \xi^+ \partial_+ X$ is

$$\delta_{\xi^+} I = 2T \int dt \oint d\sigma \left\{ \partial_+ \left(\xi^+ \partial_+ X\right) \cdot \partial_- X + \partial_+ X \cdot \partial_- \left(\xi^+ \partial_+ X\right) \right\} \,. \tag{3.87}$$

Integrating by parts in the first term[25] we find that

$$\delta_{\xi^+} I = 2T \int dt \oint d\sigma\, \partial_- \xi^+ \left(\partial_+ X\right)^2 \,. \tag{3.88}$$

We thus confirm that the action is invariant if $\partial_- \xi^+ = 0$.

But is this a *gauge* invariance rather than a symmetry? For it to be a symmetry there would have to be corresponding Noether charges. Let's attempt to compute them. As $\sigma \sim \sigma + 2\pi$, we also have $\sigma^\pm \sim \sigma^\pm + 2\pi$, so $\xi^\pm(\sigma^\pm)$ must be periodic functions of their arguments, which we can express as a Fourier series:

$$\xi^\pm(\sigma^\pm) = \sum_{n\in\mathbb{Z}} e^{in\sigma^\pm} \xi^\pm_n \,. \tag{3.89}$$

We have verified invariance of the action for *constant* coefficients $\xi^\pm_n$. Now we allow these coefficients to depend on t:

$$\xi^\pm(\sigma^+, t) = \sum_{n\in\mathbb{Z}} e^{in\sigma^\pm} \xi^+_n(t) \qquad \Rightarrow \qquad \partial_\mp \xi^\pm = \frac{1}{2} \sum_{n\in\mathbb{Z}} e^{in\sigma^\pm} \dot{\xi}^\pm_n(t) \,. \tag{3.90}$$

Now we have

$$\delta_{\xi^\pm} I = \int dt \sum_{n\in\mathbb{Z}} \dot{\xi}^\pm_n Q^\pm_n \,, \qquad Q^\pm_n = T \oint d\sigma\, e^{in\sigma^\pm} (\partial_\pm X)^2 \,, \tag{3.91}$$

and it looks as though we have Noether charges $Q^\pm_n$ $(n \in \mathbb{Z})$. Let's verify that the $Q^\pm_n$ are constants of motion. We have

$$\begin{aligned}
\dot{Q}^\pm_n &= T \oint d\sigma\, \partial_t \left[e^{in\sigma^+} (\partial_\pm X)^2 \right] = 2T \oint d\sigma\, \partial_\mp \left[e^{in\sigma^\pm} (\partial_\pm X)^2 \right] \\
&= 2T \oint d\sigma\, e^{in\sigma^\pm} \partial_\mp (\partial_\pm X)^2 = -T \oint d\sigma\, e^{in\sigma^\pm} \partial_\pm X \cdot \Box_2 X \\
&= 0 \quad \text{given } \Box_2 X = 0 \,.
\end{aligned} \tag{3.92}$$

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[24] $(\partial X)^2 = -4\, \partial_+ X \cdot \partial_- X$ for our definition of worldsheet lightcone coordinates.

[25] As usual, we ignore time-boundary terms. There is no space boundary for a closed string.

The $Q_n^\pm$ *are* Noether charges associated to symmetries of the action (3.86) but in the process of arriving at this action we have lost the constraints $(\partial_\pm X)^2 = 0$. This cannot happen if the gauge-fixing procedure is legitimate, so there must be something wrong with the conformal gauge! What we have overlooked is that the conformal gauge condition can be made only *locally* on the worldsheet. On the cylindrical worldsheet of a closed string propagating from some initial time, we could arrange (for example) to set $\lambda^\pm = 1$ everywhere immediately after an initial time t_0, but then we could not set $\lambda^\pm = 1$ *at* the initial time. In this case, the action would still depend on $\lambda^\pm(t_0)$ and variation with respect to $\lambda^\pm(t_0)$ would lead to the initial conditions

$$(\partial_\pm X)^2|_{t=t_0} = 0\,. \tag{3.93}$$

However, the identity

$$\partial_t \left[(\partial_\pm X)^2\right] \equiv -\frac{1}{2}\,\partial_\pm X \cdot \Box_2 X \;\pm\; \left[(\partial_\pm X)^2\right]' \tag{3.94}$$

ensures that $(\partial_\pm X)^2$ remains zero if it is initially zero, as a consequence of the equation of motion $\Box_2 X = 0$. In other words, the equation of motion plus constraints as initial conditions is equivalent to the full NG equations (which must be the outcome of *any* legitimate resolution of the problem). And once we include the constraints, the Noether charges $Q_n^\pm$ are all zero, which means that the residual gauge invariance of the NG action in conformal gauge is indeed a gauge invariance, and not a symmetry.

Another statement of the NG constraints in the context of the conformal gauge action is that the stress tensor for X is zero. Let's check this. The stress tensor is

$$\Theta_{\mu\nu} = T\left[\partial_\mu X \cdot \partial_\nu X - \frac{1}{2}\eta_{\mu\nu}\,\left(\partial^\lambda X \cdot \partial_\lambda X\right)\right]\,. \tag{3.95}$$

This tensor is traceless because X is massless (as a set of 2D fields) and this translates to $\Theta_{+-} = 0$; to check this you need to use the fact that $\eta_{+-} = 1/2$ for our worldsheet lightcone conventions. The non-zero components are

$$\Theta_{\pm\pm} = T(\partial_\pm X)^2\,, \tag{3.96}$$

so the constraints are equivalent to the vanishing of $\Theta_{\mu\nu}$. This implies, in particular, that the total energy of the 2D fields X^m is zero (which is possible because X^0 enters the action with the "wrong" sign). However, this does **not** mean that the energy of the string is zero. The string energy is

$$E_{\text{string}} = \oint d\sigma\, P^0 = T\oint d\sigma\, \dot{X}^0\,, \tag{3.97}$$

where we have used the conformal gauge relation $P = T\dot{X}$. As we have shown, the residual gauge invariance in the conformal gauge can be fixed by setting $X^0 = t$, but this implicitly fixes a unit of length if (t,σ) are assumed dimensionless (such that $E_{\text{string}} = 2\pi T$ in these units). It is usually more convenient to suppose (t,σ) to have units of length and then allow the solutions to determine the periodicity in σ.

3.6 Fourier expansion: closed string

The worldsheet fields of the closed string are periodic functions of σ with (by convention) period 2π, so we can express them as Fourier series. It is convenient to express (X, P) as Fourier series by starting with the combinations $P \pm TX'$ (because the gauge transformations act separately on $P + TX'$ and $P - TX'$), so we write

$$P - TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{ik\sigma} \alpha_k(t) \qquad (\alpha_{-k} = \alpha_k^*)$$
$$P + TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{-ik\sigma} \tilde{\alpha}_k(t) \qquad (\tilde{\alpha}_{-k} = \tilde{\alpha}_k^*) . \tag{3.98}$$

Recall that worldsheet parity $\sigma \to -\sigma$ exchanges $P + TX'$ with $P - TX'$. Because of the relative minus sign in the exponents of the Fourier series, this means that worldsheet parity exchanges α_k with $\tilde{\alpha}_k$:

$$\alpha_k \quad \leftrightarrow \quad \tilde{\alpha}_k . \tag{3.99}$$

We can integrate either of equations (3.98) to determine the total D-momentum in terms of Fourier modes because X' integrates to zero for a closed string; this gives us

$$p = \oint P d\sigma = \begin{cases} \sqrt{4\pi T}\alpha_0 \\ \sqrt{4\pi T}\tilde{\alpha}_0 \end{cases} \quad \Rightarrow \quad \alpha_0 = \tilde{\alpha}_0 = \frac{p}{\sqrt{4\pi T}} . \tag{3.100}$$

By adding the Fourier series expressions for $P \pm TX'$ we now get

$$P(t, \sigma) = \frac{p(t)}{2\pi} + \sqrt{\frac{T}{4\pi}} \sum_{k \neq 0} e^{ik\sigma} \left[\alpha_k(t) + \tilde{\alpha}_{-k}(t)\right] . \tag{3.101}$$

By subtracting we get

$$X' = -\frac{1}{\sqrt{4\pi T}} \sum_{k \neq 0} e^{ik\sigma} \left[\alpha_k(t) - \tilde{\alpha}_{-k}(t)\right] , \tag{3.102}$$

which we may integrate to get the Fourier series expansion for X:

$$X(t, \sigma) = x(t) + \frac{1}{\sqrt{4\pi T}} \sum_{k \neq 0} \frac{i}{k} e^{ik\sigma} \left[\alpha_k(t) - \tilde{\alpha}_{-k}(t)\right] . \tag{3.103}$$

The integration constant (actually a function of t) can be interpreted as the position of the centre of mass of the string; we should expect it to behave like a free particle.

Using the Fourier series expansions of (X, P) we now find that

$$\oint d\sigma\, \dot{X}^m P_m = \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k}\right) + \frac{d}{dt}() \tag{3.104}$$

Exercise: check this [*Hint.* Cross terms that mix α with $\tilde{\alpha}$ are all in the total time derivative term, and the $k < 0$ terms in the resulting sum double the $k > 0$ terms].

Next we Fourier expand the constraint functions $\mathcal{H}_\pm$:

$$\mathcal{H}_- = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in\sigma} L_n\,, \qquad \mathcal{H}_+ = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in\sigma}\tilde{L}_n\,. \tag{3.105}$$

Inverting to get the Fourier coefficients in terms of the functions $\mathcal{H}_\pm$, we get

$$\begin{aligned} L_n &= \oint d\sigma\, e^{-in\sigma}\mathcal{H}_- = \frac{1}{4T}\oint d\sigma\, e^{-in\sigma}\left(P - TX'\right)^2 \\ \tilde{L}_n &= \oint d\sigma\, e^{in\sigma}\mathcal{H}_+ = \frac{1}{4T}\oint d\sigma\, e^{in\sigma}\left(P + TX'\right)^2 . \end{aligned} \tag{3.106}$$

Inserting the Fourier expansions of $(P \pm TX')$ given in (3.98), we find that (Exercise: check this)

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\cdot\alpha_{n-k}\,, \qquad \tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\tilde{\alpha}_k\cdot\tilde{\alpha}_{n-k}\,. \tag{3.107}$$

We may similarly expand the Lagrange multipliers as Fourier series but it should be clear in advance that there will be one Fourier mode of λ^- for each L_n (let's call this λ_{-n}) and one Fourier mode of λ^+ for each $\tilde{L}_n$ (let's call this $\tilde{\lambda}_{-n}$). We may now write down the closed string action in terms of Fourier modes. It is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty}\frac{i}{k}\left(\dot{\alpha}_k\cdot\alpha_{-k} + \dot{\tilde{\alpha}}_k\cdot\tilde{\alpha}_{-k}\right) - \sum_{n\in\mathbb{Z}}\left(\lambda_{-n}L_n + \tilde{\lambda}_{-n}\tilde{L}_n\right)\right\}. \tag{3.108}$$

This action is manifestly Poincaré invariant. The Noether charges are

$$\begin{aligned} \mathcal{P}_m &= \oint d\sigma P_m = p_m\,, \\ \mathcal{J}^{mn} &= 2\oint d\sigma X^{[m}P^{n]} = 2x^{[m}p^{n]} + S^{mn}\,, \end{aligned} \tag{3.109}$$

where the spin part of the Lorentz charge is (Exercise: check this)

$$S^{mn} = -2\sum_{k=1}^{\infty}\frac{i}{k}\left(\alpha_{-k}^{[m}\alpha_k^{n]} + \tilde{\alpha}_{-k}^{[m}\tilde{\alpha}_k^{n]}\right) . \tag{3.110}$$

- **Lemma**. For a Lagrangian of the form

$$L = \frac{i}{c}\,\dot{\alpha}\alpha^* - H(\alpha,\alpha^*) \tag{3.111}$$

for constant c, the PB of the canonical variables takes the form

$$\{\alpha,\alpha^*\}_{PB} = -ic\,. \tag{3.112}$$

To see this set $\alpha = \sqrt{|c|/2}\,[q + i\,\mathrm{sign}(c)p]$ to get $L = \dot{q}p - H$, for which we know that $\{q,p\}_{PB} = 1$. This implies the above PB for α and α^*.

Using this lemma we may read off from the action that the non-zero Poisson brackets of canonical variables are $\{x^m, p_n\}_{PB} = \delta^m_n$ and

$$\left\{\alpha^m_k, \alpha^n_{-k}\right\}_{PB} = -ik\eta^{mn}\,, \quad \left\{\tilde{\alpha}^m_k, \tilde{\alpha}^n_{-k}\right\}_{PB} = -ik\eta^{mn}\,. \tag{3.113}$$

Using these PBs, and the Fourier series expressions for (X, P), we may compute the PB of $X(\sigma)$ with $P(\sigma')$. [Exercise: check that the result agrees with (3.22).]

We may also use the PBs (3.113) to compute the PBs of the constraint functions $(L_n, \tilde{L}_n)$. The non-zero PBs are (Exercise: check this)

$$\{L_k, L_j\}_{PB} = -i\,(k-j)\,L_{k+j}\,, \qquad \left\{\tilde{L}_k, \tilde{L}_j\right\}_{PB} = -i\,(k-j)\,\tilde{L}_{k+j}\,. \tag{3.114}$$

We may draw a number of conclusions from this result:

- The constraints are first class, so the L_n and $\tilde{L}_n$ generate gauge transformations, for each $n \in \mathbb{Z}$.
- The structure functions of the algebra of first-class constraints are *constants*. This means that the $(L_n, \tilde{L}_n)$ span an infinite dimensional *Lie algebra*.
- The Lie algebra of the gauge group is a direct sum of two copies of the same algebra, sometimes called the Witt algebra.

The Witt algebra is also the algebra of diffeomorphisms of the circle. Suppose we have a circle parameterized by $\theta \sim \theta + 2\pi$ (we could take θ to be σ^+ or σ^-). The algebra Diff_1 of diffeomorphisms is spanned by the vector fields on the circle, and since these are periodic we may take as a basis set the vector fields $\{V_n; n \in \mathbb{Z}\}$, where

$$V_n = e^{in\theta}\frac{d}{d\theta}\,. \tag{3.115}$$

The commutator of two basis vector fields is

$$[V_k, V_j] = -i\,(k-j)\,V_{k+j}\,. \tag{3.116}$$

Corollary: the algebra of the gauge group is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$, as claimed earlier.

3.7 Fourier expansion: open string

The open string has two ends. We will choose the ends to be at $\sigma = 0$ and $\sigma = \pi$, so the parameter length of the string is π (this is just a convention). We shall consider the case of free-end (Neumann) boundary conditions, as these preserve Lorentz invariance.

If the ends of the string are free, we must require X' to be zero at the ends, i.e. at $\sigma = 0$ and $\sigma = \pi$. We shall proceed in a way that will allow us to take over results from the closed string; we shall use a "doubling trick":

- First, we extend the definition of (X, P) from the interval $[0, \pi]$ to the interval $[0, 2\pi]$ in such a way that (X, P) are periodic on this doubled interval. This will allow us to use the closed string Fourier series expressions.
- Next, we impose a condition that relates (X, P) in the interval $[\pi, 2\pi]$ to (X, P) in the interval $[0, \pi]$; this will ensure that any additional degrees of freedom that we have introduced by doubling the interval are removed. Notice that if $\sigma \in [0, \pi]$ then $-\sigma \sim -\sigma + 2\pi \in [\pi, 2\pi]$, so we need to relate the worldsheet fields at σ to their values at $-\sigma$. The condition that does this should be consistent with periodicity in the doubled interval, but it should also imply the free-end b.c.s at $\sigma = 0, \pi$.

The solution to these requirements is to impose the condition

$$(P + TX')(\sigma) = (P - TX')(-\sigma) . \tag{3.117}$$

This is consistent with periodicity, and setting $\sigma = 0$ it implies that $X'(0) = 0$. It also implies that $X'(\pi) = 0$ because $-\pi \sim \pi$ by periodicity. In terms of Fourier modes, the condition (3.117) becomes

$$\tilde{\alpha}_k = \alpha_k \qquad (k \in \mathbb{Z}). \tag{3.118}$$

Using this in (3.98) we have

$$P \pm TX' = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} e^{\mp ik\sigma} \alpha_k \qquad (\alpha_{-k} = \alpha_k^*) . \tag{3.119}$$

Equivalently,

$$P = \sqrt{\frac{T}{\pi}} \sum_{k \in \mathbb{Z}} \cos(k\sigma)\, \alpha_k \,, \qquad X' = -\frac{i}{\sqrt{\pi T}} \sum_{k \in \mathbb{Z}} \sin(k\sigma)\, \alpha_k \,. \tag{3.120}$$

Integrating to get X, and defining $p(t)$ by

$$\alpha_0 = \frac{p}{\sqrt{\pi T}} \,, \tag{3.121}$$

we have

$$\begin{aligned} X(t,\sigma) &= x(t) + \frac{1}{\sqrt{\pi T}} \sum_{k \neq 0} \frac{i}{k} \cos(k\sigma)\, \alpha_k \,, \\ P(t,\sigma) &= \frac{p(t)}{\pi} + \sqrt{\frac{T}{\pi}} \sum_{k \neq 0} \cos(k\sigma)\, \alpha_k \,. \end{aligned} \tag{3.122}$$

Notice that p is again the total momentum since

$$\int_0^\pi d\sigma\, P(t,\sigma) = p \,, \tag{3.123}$$

but its relation to α_0 differs from that of the closed string.

Using the Fourier series expansions for (X, P) we find that

$$\int_0^\pi \dot{X}^m P_m \, d\sigma = \dot{x}^m p_m + \sum_{k=1}^\infty \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} + \frac{d}{dt}() \ . \tag{3.124}$$

Because of (3.117) we also have $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$, so we should impose a similar relation on the Lagrange multipliers

$$\lambda^+(\sigma) = \lambda^-(-\sigma) \qquad (\Rightarrow \ u|_{\text{ends}} = 0 \quad \& \quad e'|_{\text{ends}} = 0) \ . \tag{3.125}$$

Then

$$\begin{aligned} \int_0^\pi d\sigma \left(\lambda^- \mathcal{H}_- + \lambda^+ \mathcal{H}_+\right) &= \int_0^\pi d\sigma \, \lambda^-(\sigma) \mathcal{H}_-(\sigma) + \int_0^\pi d\sigma \, \lambda^-(-\sigma) \mathcal{H}_-(-\sigma) \\ &= \int_0^\pi d\sigma \, \lambda^-(\sigma) \mathcal{H}_-(\sigma) + \int_{-\pi+2\pi}^{0+2\pi} d\sigma \, \lambda^-(\sigma) \mathcal{H}_-(\sigma) \\ &= \oint d\sigma \, \lambda^- \mathcal{H}_- \, , \end{aligned} \tag{3.126}$$

and we can now use the Fourier series expansions of the closed string.

The final result for the open string action in Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n \in \mathbb{Z}} \lambda_{-n} L_n \right\} \ . \tag{3.127}$$

The difference with the closed string is that we have one set of oscillator variables instead of two. We can now read off the non-zero PB relations

$$\{x^m, p_n\}_{PB} = \delta^m_n \, , \qquad \left\{\alpha^m_k, \alpha^n_{-k}\right\}_{PB} = -ik\eta^{mn} \, , \tag{3.128}$$

and we can use this to show that

$$\{L_j, \alpha^m_k\}_{PB} = ik\alpha_{k+j} \, . \tag{3.129}$$

This means that the gauge variation of α_k is

$$\delta_\xi \alpha^m_k = \sum_{n \in \mathbb{Z}} \xi_{-n} \{\alpha^m_k, L_n\}_{PB} = -ik \sum_{n \in \mathbb{Z}} \xi_{-n} \alpha^m_{n+k} \, , \tag{3.130}$$

where ξ_n are parameters. To compute the gauge transformation of (x, p) we use the relation $p = \sqrt{\pi T} \, \alpha_0$ and the fact that

$$L_0 = \frac{1}{2} \alpha_0^2 + \dots \, , \qquad L_n = \alpha_0 \cdot \alpha_n + \dots \, , \tag{3.131}$$

where the dots indicate terms that do not involve α_0, to compute

$$\delta_\xi x^m = \frac{1}{\sqrt{\pi T}} \sum_{k \in \mathbb{Z}} \xi_{-k} \alpha^m_k \, , \qquad \delta p_m = 0 \, . \tag{3.132}$$

Finally, one may verify that the action is invariant if

$$\delta_\xi \lambda_j = \dot{\xi}_j + i \sum_{k \in \mathbb{Z}} (2k - j) \, \xi_k \lambda_{j-k} \, . \tag{3.133}$$

3.8 The NG string in light-cone gauge

We shall start with the open string (with free-end b.c.s). We shall impose the gauge conditions

$$X^+(t,\sigma) = x^+(t)\,, \qquad P_-(t,\sigma) = p_-(t)/\pi\,. \tag{3.134}$$

where $x^(t)$ and $p_-(t)$ are freely variable functions of t. It customary to also set $x^+(t) = t$, as for the particle, but it is simpler not to do this. This means that we will not be fixing the gauge completely since we will still be free to make σ-independent reparametrizations of the worldsheet time t.

The above gauge-fixing conditions are equivalent to

$$(P \pm TX')^+ = p_-(t)/\pi \quad \Leftrightarrow \quad \alpha_k^+ = 0 \quad \forall k \neq 0\,. \tag{3.135}$$

In other words, we impose a light-cone gauge condition only on the oscillator variables of the string, not on the zero modes (centre of mass variables). Let's check that the gauge has been otherwise fixed. We can investigate this using the criterion summarised by the formula (2.55); we compute

$$\begin{aligned}\left\{L_j, \alpha_{-k}^+\right\}_{PB} &= -ik\alpha_{j-k}^+ \quad = \; -ik\alpha_0^+\,\delta_{jk} \quad \text{(using gauge condition)}\\ &= -ik\frac{p_-}{\sqrt{\pi T}}\delta_{jk}\,.\end{aligned} \tag{3.136}$$

This is invertible if we exclude $j = 0$ and $k = 0$, so we have fixed all but the gauge transformation generated by L_0. Now we have, since $\alpha_k^+ = 0$,

$$\sum_{k=1}^{\infty} \frac{i}{k}\dot{\alpha}_k \cdot \alpha_{-k} = \sum_{k=1}^{\infty} \frac{i}{k}\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k}\,, \tag{3.137}$$

where the $(D-2)$-vectors $\boldsymbol{\alpha}_k$ are the *transverse* oscillator variables.

We also have,

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{1}{2\pi T}\left(p^2 + 2\pi T N\right)\,, \tag{3.138}$$

where N is the *level number*:

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k\,. \tag{3.139}$$

For $n \neq 0$,

$$\begin{aligned}L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\alpha_k^+\alpha_{n-k}^- + \alpha_k^-\alpha_{n-k}^+\right) + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}\\ &= \alpha_0^+\alpha_n^- + \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad \text{(using gauge condition)}.\end{aligned} \tag{3.140}$$

We can solve this for α_n^-; using $p = \sqrt{\pi T}\,\alpha_0$, we get

$$\alpha_n^- = -\frac{\sqrt{\pi T}}{2p_-}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \qquad (n \neq 0). \tag{3.141}$$

As we have solved the constraints $L_n = 0$ for $n \neq 0$, only the $L_0 = 0$ constraint will be imposed by a Lagrange multiplier in the gauge-fixed action, which is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} - \frac{1}{2} e_0 \left(p^2 + M^2\right) \right\}, \tag{3.142}$$

where $e_0 = \lambda_0/(\pi T)$ and

$$M^2 = 2\pi T N = N/\alpha' \qquad \left(\alpha' \equiv \frac{1}{2\pi T}\right). \tag{3.143}$$

Notice that the action does not involve α_n^- (for $n \neq 0$) but the Lorentz charges do. Recall that the spin part of the Lorentz charge $\mathcal{J}^{mn}$ is $S^{mn} = -2\sum_{k=1}^{\infty} \frac{i}{k}\alpha_{-k}^{[m}\alpha_k^{n]}$. Its non-zero components of S^{mn} in light-cone gauge are $(I, J, = 1, \ldots D-2)$

$$S^{IJ} = -2\sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k}^{[I}\alpha_k^{J]}, \qquad S^{-I} = -\sum_{k=1}^{\infty} \frac{i}{k}\left(\alpha_{-k}^-\alpha_k^I - \alpha_{-k}^I \alpha_k^-\right). \tag{3.144}$$

The canonical PB relations that we read off from the action (3.142) are

$$\{x^m, p_n\}_{PB} = \delta_n^m, \qquad \{\alpha_k^I, \alpha_{-k}^J\}_{PB} = -ik\,\delta^{IJ}. \tag{3.145}$$

These may be used to compute the PBs of the Lorentz generators; since $\mathcal{J} = L + S$ where $\{L, S\}_{PB} = 0$, the PB relations among the components of S alone must be the same as those of $\mathcal{J}$. The PBs of S^{IJ} are those of the Lie algebra of the transverse rotation group $SO(D-2)$, and their PBs with S^{-K} are those expected from the fact that S^{-K} is a $(D-2)$ vector. Finally, Lorentz invariance requires that

$$\left\{S^{-I}, S^{-J}\right\}_{PB} = 0. \tag{3.146}$$

This can be confirmed by making use of the PB relations

$$\left\{\alpha_k^-, \alpha_\ell^-\right\}_{PB} = i\frac{\sqrt{\pi T}}{p_-}(k-\ell)\,\alpha_{k+\ell}^-, \qquad \left\{\alpha_k^-, \alpha_\ell^I\right\}_{PB} = -i\frac{\sqrt{\pi T}}{p_-}\ell\,\alpha_{k+\ell}^I. \tag{3.147}$$

This has to work because gauge fixing cannot break symmetries; it can only obscure them.

3.8.1 Closed string in light-cone gauge

Now we fix the gauge invariances associated with L_n and $\tilde{L}_n$ for $n \neq 0$ by setting

$$\alpha_k^+ = 0 \quad \& \quad \tilde{\alpha}_k^+ = 0 \qquad \forall k \neq 0\,. \tag{3.148}$$

This leaves unfixed the gauge invariances generated by L_0 and $\tilde{L}_0$, which are now

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k = \frac{p^2}{8\pi T} + N\,, \\ \tilde{L}_0 &= \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{k=1}^{\infty} \tilde{\boldsymbol{\alpha}}_{-k} \cdot \tilde{\boldsymbol{\alpha}}_k = \frac{p^2}{8\pi T} + \tilde{N}\,. \end{aligned} \tag{3.149}$$

Here we have used the closed string relation (3.100) between p and $\alpha_0 = \tilde{\alpha}_0$. By adding and subtracting the two constraints $L_0 = 0$ and $\tilde{L}_0 = 0$ we get the two equivalent constraints

$$p^2 + 4\pi T\left(N + \tilde{N}\right) = 0 \quad \& \quad \tilde{N} - N = 0\,, \tag{3.150}$$

which will be imposed by the Lagrange multipliers $e_0 = 2(\lambda_0 + \tilde{\lambda}_0)$ and $u_0 = 4\pi T(\lambda - \tilde{\lambda}_0)$ in the gauge-fixed action. The remaining constraints $L_n = 0$ and $\tilde{L}_n = 0$ for $n \neq 0$ we solve for α_k^- and $\tilde{\alpha}_k^-$, as for the open string. The closed string action in light-cone gauge is therefore

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k}\left(\dot{\boldsymbol{\alpha}}_k \cdot \boldsymbol{\alpha}_{-k} + \dot{\tilde{\boldsymbol{\alpha}}}_k \cdot \tilde{\boldsymbol{\alpha}}_{-k}\right) - \frac{1}{2}e_0\left(p^2 + M^2\right) - u_0\left(N - \tilde{N}\right)\right\}\,, \tag{3.151}$$

where

$$\begin{aligned} M^2 &= 4\pi T\left(N + \tilde{N}\right) \\ &= 8\pi T N \qquad \left(\text{using } \tilde{N} = N \text{ constraint}\right). \end{aligned} \tag{3.152}$$

We now have two first-class constraints, a mass-shell constraint and a *level-matching* constraint[26].

4. Interlude: Light-cone gauge in field theory

We shall consider Maxwell's equation, for the vector potential A_m, and the linearised Einstein equations for a symmetric tensor potential h_{mn}, which may be interpreted as the perturbation of the space-time metric about a Minkowski vacuum metric.

[26]The constraint function $(N - \tilde{N})$ generates the residual gauge transformation $\delta_\beta \alpha_k(t) = -ik\beta(t)\alpha_k(t)$ and $\delta_\beta \tilde{\alpha}_k(t) = ik\beta(t)\tilde{\alpha}_k(t)$ which originates from invariance under t-dependent shifts of σ.

We choose light-cone coordinates (x^+, x^-, x^I) $(I = 1, \ldots, D-2)$. Recall that for the particle we assumed that $p_- \neq 0$, which is equivalent to the assumption that the differential operator ∂_- is invertible. We shall make the same assumption in the application to field theory.

4.1 Maxwell in light-cone gauge

Maxwell's equations are

$$\Box_D A_m - \partial_m (\partial \cdot A) = 0\,, \qquad m = 0, 1, \ldots, D-1\,. \tag{4.1}$$

They are invariant under the gauge transformation $A_m \to A_m + \partial_m \alpha$. The light-cone gauge is

$$A_- = 0\,. \tag{4.2}$$

To see that the gauge is fixed we set to zero a gauge variation of the gauge-fixing condition: $0 = \delta_\alpha A_- = \partial_- \alpha$. This implies that $\alpha = 0$ if the differential operator ∂_- is invertible.

In this gauge the $m = -$ Maxwell equation is $\partial_- (\partial \cdot A) = 0$, which implies that $\partial \cdot A = 0$ since we assume invertibility of ∂_-. And since $0 = \partial \cdot A = \partial_- A_+ + \partial_I A_I$, we can solve for A_+:

$$A_+ = -\partial_-^{-1} (\partial_I A_I)\,. \tag{4.3}$$

This leaves A_I as the only independent variables. The $m = +$ equation is $\Box_D A_+ = 0$, but this is a consequence of the $m = I$ equation, which is

$$\Box_D A_I = 0\,, \qquad I = 1, \ldots, D-2\,. \tag{4.4}$$

So this is what Maxwell's equations look like in light-cone gauge: wave equations for $D-2$ independent polarisations.

4.2 Linearized Einstein in light-cone gauge

The Einstein field equations of GR are equations for a spacetime metric g_{mn}. Given that Minkowski spacetime is a solution, we can write $g_{mn} = \eta_{mn} + h_{mn}$ and view h_{mn} as a symmetric tensor field on a Minkowski background; this makes sense if h_{mn} is "small". To first order in this metric perturbation, the (source-free) Einstein field equations become

$$\Box_D h_{mn} - 2\partial_{(m} h_{n)} + \partial_m \partial_n h = 0\,, \qquad h_m \equiv \partial^n h_{nm}\,, \quad h = \eta^{mn} h_{mn}\,. \tag{4.5}$$

This equation is invariant under the gauge transformation (<u>Exercise</u>: verify this)

$$h_{mn} \to h_{mn} + 2\partial_{(m} \xi_{n)}\,. \tag{4.6}$$

The light-cone gauge choice is

$$h_{-n} = 0 \qquad (n = -, +, I) \qquad \Rightarrow \quad h_- = 0 \qquad \& \quad h = h_{JJ}\,. \tag{4.7}$$

In the light-cone gauge the "$m = -$" equation is $-\partial_- h_n + \partial_- \partial_n h_{JJ} = 0$, which can be solved for h_n:

$$h_n = \partial_n h_{JJ} \,. \tag{4.8}$$

But since we already know that $h_- = 0$, this tells us that $h_{JJ} = 0$, and hence that $h_n = 0$ and $h = 0$. At this point we see that the equations reduce to $\Box_D h_{mn} = 0$, but

$$\begin{aligned} h_+ = 0 \quad &\Rightarrow \quad h_{++} = -\partial_-^{-1} \left(\partial_I h_{I+}\right) \,, \\ h_I = 0 \quad &\Rightarrow \quad h_{+I} = -\partial_-^{-1} \left(\partial_J h_{JI}\right) \,, \end{aligned} \tag{4.9}$$

so the only independent components of h_{mn} are h_{IJ}, and this has zero trace. We conclude that the linearised Einstein equations in light-cone gauge are

$$\Box_D h_{IJ} = 0 \quad \& \quad h_{II} = 0 \,. \tag{4.10}$$

The number of polarisation states of the graviton in D dimensions is therefore

$$\frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2} D(D-3) \,. \tag{4.11}$$

For example, for $D = 4$ there are two polarisation states[27], and the graviton is a massless particle of spin 2.

It is a general feature of massless particles that their independent polarisation states form a representation of the transverse rotation group $SO(D-2)$.

4.3 Massive fields

Notice that for both Maxwell and linearised Einstein, the independent fields in light-cone gauge form an irreducible representation of the transverse rotation group $SO(D-2)$. In the former case we get a $(D-2)$-vector. In the latter case we get a traceless symmetric tensor whose components span a space of dimension $D(D-3)/2$.

Adding a mass usually breaks the gauge invariance that we used to fix the light-cone gauge, but the light-cone decomposition is still useful. Consider the Proca equations

$$(\Box_D - m^2) A_m = 0 \,, \qquad \partial^m A_m = 0 \,. \tag{4.12}$$

The second of these equations is called the subsidiary condition. In light-cone coordinates we have $A = dx^m A_m = dx^- A_- + dx^+ A_+ + dx^I A_I$, and the subsidiary condition is

$$\partial_- A_+ + \partial_+ A_- + \partial_I A_I = 0 \,. \tag{4.13}$$

We can't now set $A_- = 0$ because there isn't a gauge invariance to justify it, but we can still suppose that ∂_- is invertible, in which case the subsidiary condition can be solved for A_+. This leaves us with the $D-1$ equations

$$\left(\Box_D - m^2\right) A_I = 0 \quad \& \quad \left(\Box_D - m^2\right) A_- = 0 \,. \tag{4.14}$$

[27]Massless particles of any non-zero spin have two independent polarization states in $D = 4$.

There are now $(D-1)$ independent polarisation. This is the $(D-1)$-vector representation of the (full) rotation group $SO(D-1)$.. For $D=4$ we have a 3-vector and hence (since $3 = 2s+1$ for $s=1$) the Proca equations are field equations for a massive particle of spin 1.

It is *a general feature of massive particles that their polarisation states form a representation of the rotation group.*

5. Quantum NG string

Now we pass to the quantum theory. This is simplest in light-cone gauge because this gauge choice removes all unphysical components of the oscillator variables prior to quantization, but it obscures Lorentz invariance. Then we consider how the same results might be found in the conformal gauge, where Lorentz invariance is still manifest.

5.1 Light-cone gauge quantization: open string

The canonical PB relations of the open string in Fourier modes are (3.145). These become the canonical commutation relations

$$[\hat{x}^m, \hat{p}_n] = i\delta^m_n \,, \qquad \left[\hat{\alpha}^I_k, \hat{\alpha}^J_{-k}\right] = k\,\delta^{IJ} \,, \tag{5.1}$$

where the hats now indicate operators, and the hermiticity of the operators $(\hat{X}, \hat{P})$ requires that

$$\hat{\boldsymbol{\alpha}}_{-k} = \hat{\boldsymbol{\alpha}}^\dagger_k \,. \tag{5.2}$$

A state of the string of definite momentum is the tensor product of a momentum eigenstate $|p\rangle$ with a state in the oscillator Fock space, built upon the Fock vacuum state $|0\rangle$ annihilated by all annihilation operators:

$$\hat{\boldsymbol{\alpha}}_k|0\rangle = \mathbf{0} \quad \forall\, k \in \mathbb{Z}^+ \,. \tag{5.3}$$

We get other states in the Fock space by acting on the oscillator vacuum with the creation operators $\hat{\boldsymbol{\alpha}}_{-k}$ any number of times, and for any $k > 0$. This gives us a basis for the entire infinite-dimensional space.

Next, we need to replace the level number N by a level number operator $\hat{N}$, but there is an operator ordering ambiguity; different orderings lead to operators $\hat{N}$ that differ by a constant. We shall choose to call $\hat{N}$ the particular operator that annihilates the oscillator vacuum; i.e.

$$\hat{N} = \sum_{k=1}^{\infty} \hat{\boldsymbol{\alpha}}_{-k} \cdot \hat{\boldsymbol{\alpha}}_k \qquad \Rightarrow \qquad \hat{N}|0\rangle = 0 \,. \tag{5.4}$$

So the oscillator vacuum has level number zero. This removes the ambiguity in the definition of $\hat{N}$ but it does *not* remove the ambiguity in passing from the classical to the quantum theory; whenever we see N in the classical theory we may still replace it by $\hat{N}$ plus a constant in the quantum theory.

Notice now that

$$\left[\hat{N}, \hat{\boldsymbol{\alpha}}_{-k}\right] = k\, \hat{\boldsymbol{\alpha}}_{-k} \, . \tag{5.5}$$

This tell us that acting on a state with any component of $\hat{\boldsymbol{\alpha}}_{-k}$ raises the level number by k, and this tells that $\hat{N}$ is diagonal in the Fock state basis constructed in the way described above, and that the possible level numbers (eigenvalues of $\hat{N}$) are $N = 0, 1, 2, \ldots, \infty$. We can therefore organise the states according to their level number. There is only one state in the Fock space with $N = 0$, the oscillator vacuum. At $N = 1$ we have the $(D-2)$ states $\hat{\alpha}^I_{-1}|0\rangle$. At $N = 2$ we have the states

$$\hat{\alpha}^I_{-2}|0\rangle \, , \qquad \hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}|0\rangle \, . \tag{5.6}$$

At $N = 3$ we have the states

$$\hat{\alpha}^I_{-3}|0\rangle \, , \qquad \hat{\alpha}^I_{-2}\hat{\alpha}^J_{-1}|0\rangle \, , \qquad \hat{\alpha}^I_{-1}\hat{\alpha}^J_{-1}\hat{\alpha}^K_{-1}|0\rangle \, , \tag{5.7}$$

and so on.

A generic state of the string at level N in a momentum eigenstate $|p\rangle$ of D-momentum p takes the form

$$|p\rangle \otimes |\Psi_N\rangle \, , \tag{5.8}$$

where $|\Psi_N\rangle$ is some state in the oscillator Fock space with level number N. The mass-shell constraint for such a state implies that $p^2 = -M^2$, where

$$M^2 = 2\pi T \left(N - a\right) \, . \tag{5.9}$$

The constant a is introduced to take care of the operator ordering ambiguity in passing from the classical to the quantum theory.

Now we proceed to analyse the string spectrum level by level. We shall use the standard notation

$$2\pi T = 1/\alpha' \tag{5.10}$$

- $N = 0$. There is one state, and hence a scalar, with $\alpha' M^2 = -a$.

- $N = 1$. There are now $(D-2)$ states, $\hat{\boldsymbol{\alpha}}_{-1}|0\rangle$ with $\alpha' M^2 = (1-a)$. The only way that these states could be part of a Lorentz-invariant theory is if they describe the polarization states of a *massless* vector (a massive vector has $(D-1)$ polarisation states). It is important to appreciate that Lorentz invariance of the *quantum* theory in light-cone gauge is not guaranteed because the Lorentz invariance of the classical action in light-cone gauge is not *manifest* (i.e. not linearly realised; the Lorentz transformations are non-linear in light-cone gauge).

In fact, we now see that a necessary condition for Lorentz invariance of the quantum theory is that we choose

$$a = 1\,. \tag{5.11}$$

This ensures that the $N = 1$ state is massless, but it also tells us that the $N = 0$ scalar is a tachyon; i.e. "particle" with *spacelike* D-momentum. This looks bad; we'll return to it later.

- $N = 2$. Given $a = 1$ the $N = 2$ states are massive, with $\alpha' M^2 = 1$. The states are those of (5.6), which are in the symmetric 2nd-rank tensor plus vector representation of $SO(D-2)$. These states form a symmetric traceless tensor of $SO(D-1)$ and hence describe a massive spin-2 particle[28].

We now know that the open string ground state is a scalar tachyon, and its first excited state is a massless vector, a "photon". All higher level states are massive, and so should be in $SO(D-2)$ representations that can be combined to form $SO(D-1)$ representations (i.e. representations of the rotation group). We have seen that this is true for $N = 2$ (in this case, exceptionally, the $SO(D-1)$ representation is irreducible) and it can be shown to be true for all $N \geq 2$.

The parameter $-a$ has a physical meaning as the total zero-point energy of the oscillator modes of the string. To see why, let's suppose that we had chosen the standard Weyl ordering of operators for a harmonic oscillator. This tells us to take $N \to \hat{N}_{\text{weyl}}$ with

$$\begin{aligned} \hat{N}_{\text{weyl}} &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\hat{\boldsymbol{\alpha}}_{-k}\cdot\hat{\boldsymbol{\alpha}}_k + \hat{\boldsymbol{\alpha}}_k\cdot\hat{\boldsymbol{\alpha}}_{-k}\right) = \hat{N} + \frac{1}{2}\sum_{k=1}^{\infty}\left[\hat{\boldsymbol{\alpha}}^m_{-k}, \hat{\boldsymbol{\alpha}}^n_k\right]\eta_{mn} \\ &= \hat{N} + \frac{(D-2)}{2}\sum_{k=1}^{\infty} k\,. \end{aligned} \tag{5.12}$$

The extra constant term is the sum over the zero-point contributions of the oscillators[29]. The eigenvalues of $\hat{N}_{\text{weyl}}$ are $N - a$ where (as before) N is the eigenvalue of $\hat{N}$, and

$$-\,a = \frac{(D-2)}{2}\sum_{k=1}^{\infty} k\,. \tag{5.13}$$

The sum on the RHS diverges, so it doesn't appear to make sense to use Weyl ordering, but Euler was of the opinion that $\sum_{k=1}^{\infty} k = -1/12$. That's because he was working on properties of the infinite series.

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}\,. \tag{5.14}$$

[28]A symmetric traceless tensor field of rank n is usually said to describe a particle of "spin n" even though "spin" is not sufficient to label states in space times of dimension $D > 4$.

[29]There are $(D-2)$ oscillators of angular frequency $|k|$ associated to the pair $(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k^\dagger)$).

When this sum converges, it defines a function (now known as the Riemann zeta function) that can be analytically continued to the entire complex s-plane except at $s = 1$, where it has a simple pole. In particular

$$\zeta(-1) = -\frac{1}{12}\,. \tag{5.15}$$

If we use this to justify Euler's intuition, then we conclude that

$$a = \frac{(D-2)}{24}\,. \tag{5.16}$$

Hence, since we know that Lorentz invariance requires $a = 1$, we also conclude that $D = 26$. Remarkably, it is indeed true that Lorentz invariance requires $D = 26$.

5.1.1 Critical dimension

Because Lorentz invariance is not manifest (i.e. linearly realized) in the light-cone gauge, there is no gauarantee that it will be preserved in the passage from the classical to the quantum theory. To check Lorentz invariance we have to compute the commutators of the quantum Lorentz charges $\hat{\mathcal{J}}^{mn}$. Since $\hat{\mathcal{J}} = \hat{L} + \hat{S}$ and it is easy to see that $[\hat{L}, \hat{S}] = 0$, we can focus on the spin $\hat{S}^{mn}$. As $SO(D-2)$ invariance is manifest in light-cone gauge, it cannot be broken; indeed, we saw that the states are all tensors of $SO(D-2)$. For the same reason, the commutation relations of $\hat{S}^{IJ}$ with itself and with $\hat{S}^{-I}$ must be those expected from the corresponding PB relations, which merely tell us that $\hat{S}^{IJ}$ is an $SO(D-2)$ tensor and that $\hat{S}^{-I}$ is an $SO(D-2)$ vectors; recall that S^{+I} is zero in light-cone gauge. The only commutation relation that is required by Lorentz invariance[30] but not guaranteed to hold in light-cone gauge is

$$\left[\hat{S}^{-I}, \hat{S}^{-J}\right] = 0 \qquad (?) \tag{5.17}$$

If the $\{,\}_{PB} \to -i[,]$ rule were to apply here then Lorentz invariance of the quantum string would follow, but it does not obviously apply in light-cone gauge because in this gauge $\hat{S}^{-I}$ is cubic in transverse oscillator operators. The product of two such operators is 6th-order in transverse oscillators, so the commutator is 4th-order, in principle, but as there is no quartic term classically there must be an operator ordering that leads to its cancellation in the quantum theory. However, any re-ordering to achieve this could generate a term quadratic in transverse operators, which would be inconsistent with Lorentz invariance.

From the classical expression (3.147) for α_k^- we have

$$\hat{\alpha}_n^- = -\frac{\pi T}{2p_-}\sum_{j\in\mathbb{Z}} \hat{\boldsymbol{\alpha}}_j \cdot \hat{\boldsymbol{\alpha}}_{n-j}\,, \tag{5.18}$$

[30]As can be verified by computing the same commutator for L; see Q.II.3.

which is unambiguous since there is no operator ordering ambiguity. Similarly, from the classical expression for S^{-I} in (3.145) we may guess that

$$\hat{S}^{-I} = -\sum_{k=1}^{\infty} \frac{i}{k} \left(\hat{\alpha}^-_{-k} \hat{\alpha}^I_k - \hat{\alpha}^I_{-k} \hat{\alpha}^-_k \right) . \tag{5.19}$$

There is an ordering ambiguity here because

$$\left[\hat{\alpha}^-_k , \hat{\boldsymbol{\alpha}}_\ell \right] = \frac{\sqrt{\pi T}}{p_-} \hat{\boldsymbol{\alpha}}_{k+\ell} \, , \tag{5.20}$$

but the order of operators must be as given in order that $\hat{S}^{-I}|0\rangle = 0$.

The commutation relation (5.20) is exactly as we should expect from the corresponding PB relation. The only other commutation relation that we need to compute the commutator (5.17)is $[\hat{\alpha}^-_k , \hat{\alpha}^-_\ell]$. From the PB relations of (3.147) we might expect

$$\left[\hat{\alpha}^-_k , \hat{\alpha}^-_\ell \right] = -\frac{\sqrt{\pi T}}{p_-} (k-\ell)\, \hat{\alpha}^-_{k+\ell} \qquad (?). \tag{5.21}$$

However, this commutator must be examined carefully; it must equal some expression quadratic in the transverse oscillator operators but if a re-ordering is needed to get agreement with the above guess then a constant term (times $1/p_-^2$) could be generated. In fact, one finds that

$$\left[\hat{\alpha}^-_k , \hat{\alpha}^-_\ell \right] = -\frac{\sqrt{\pi T}}{p_-} (k-\ell)\, \hat{\alpha}^-_{k+\ell} + \frac{2\pi T}{p_-^2} \left[\left(a - \frac{(D-2)}{24} \right) k + \frac{(D-2)}{24} k^3 \right] \delta_{k+\ell} \, , \tag{5.22}$$

where

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \, . \end{cases} \tag{5.23}$$

We see that there is indeed an additional non-classical term when $k + \ell = 0$.

Using the fact that $\sqrt{\pi T}\, \alpha^-_0 = p_+$, and the mass-shell constraint in the form

$$-2p_+ p_- = |\mathbf{p}|^2 + 2\pi T (N-a) \, , \tag{5.24}$$

we may rewrite the $\ell = -k$ commutation relation of (5.22) as

$$\left[\hat{\alpha}^-_k , \hat{\alpha}^-_{-k} \right] = \frac{k\pi T}{p_-^2} \left(\frac{|\mathbf{p}|^2}{\pi T} + 2(N-a) \right) + \frac{2\pi T}{p_-^2} \left[\left(a - \frac{(D-2)}{24} \right) k + \frac{(D-2)}{24} k^3 \right] \tag{5.25}$$

Noticing that the terms involving the constant a cancel, we can further rewrite this commutation relation as

$$\left[\hat{\alpha}^-_k , \hat{\alpha}^-_{-k} \right] = \frac{k\pi T}{p_-^2} \left(|\mathbf{p}|^2 + 2\pi T N \right) + \frac{2\pi T}{p_-^2} \left(\frac{D-2}{24} \right) \left(k^3 - k \right) . \tag{5.26}$$

This commutator is the essential one to verify, and we shall now do so.

- **Proof of (5.26)**. It suffices to consider $k > 0$ since the $k < 0$ case then follows by hermitian conjugation. In this case $\hat{\alpha}_k^- |0\rangle = 0$, so that

$$\langle 0| \left[\hat{\alpha}_k^-, \hat{\alpha}_{-k}^-\right] |0\rangle = ||\hat{\alpha}_k^- |0\rangle||^2 . \tag{5.27}$$

Observe also, again for $k > 0$, that

$$\hat{\alpha}_{-k}^- |0\rangle = -\frac{\sqrt{\pi T}}{p_-} \left(\boldsymbol{\alpha}_0 \cdot \hat{\boldsymbol{\alpha}}_{-k} + \frac{1}{2} \sum_{j=1}^{k-1} \hat{\boldsymbol{\alpha}}_{-j} \cdot \hat{\boldsymbol{\alpha}}_{j-k} \right) |0\rangle . \tag{5.28}$$

This means that

$$\begin{aligned} ||\hat{\alpha}_k^- |0\rangle||^2 &= \frac{\pi T}{p_-^2} \left[k|\boldsymbol{\alpha}_0|^2 + \frac{1}{4} \sum_{j=1}^{k-1} \sum_{\ell=1}^{k-1} \langle 0| \left(\hat{\boldsymbol{\alpha}}_{-j} \cdot \hat{\boldsymbol{\alpha}}_{j-k}\right) \left(\hat{\boldsymbol{\alpha}}_{-j} \cdot \hat{\boldsymbol{\alpha}}_{j-k}\right) |0\rangle \right] \\ &= \frac{k|\mathbf{p}|^2}{p_-^2} + \frac{\pi T}{2p_-^2} (D-2) \sum_{j=1}^{k-1} j(k-j) , \end{aligned} \tag{5.29}$$

where we have used $\sqrt{\pi T}\, \boldsymbol{\alpha}_0 = \mathbf{p}$. Now using the identities

$$\sum_{j=1}^{k-1} j \equiv \frac{1}{2} k(k-1) , \qquad \sum_{j=1}^{k-1} j^2 \equiv \frac{1}{6} k(k-1)(2k-1) , \tag{5.30}$$

and recalling (5.27), we deduce that

$$\langle 0| \left[\hat{\alpha}_k^-, \hat{\alpha}_{-k}^-\right] |0\rangle = \frac{k|\mathbf{p}|^2}{p_-^2} + \frac{\pi T}{p_-^2} \frac{(D-2)}{24} \left(k^3 - k\right) . \tag{5.31}$$

This is consistent with (5.26), since $N = 0$ in the ground state, and it gives the stated non-classical addition to the commutation relation (5.21) that is the naive expectation based on the classical PB result. When combined with the classical result, this quantum compuation therefore implies (5.26).

The result (5.22) is just a first step but it shows how the spacetime dimension D and the parameter a can enter into the final result, which is

$$\left[\hat{S}^{-I}, \hat{S}^{-J}\right] = \frac{4\pi T}{p_-^2} \sum_{k=1}^{\infty} \left(\left[\frac{(D-2)}{12} - 2 \right] k + \frac{1}{k} \left[2a - \frac{(D-2)}{12} \right] \right) \hat{\alpha}_{-k}^{[I} \hat{\alpha}_k^{J]} . \tag{5.32}$$

This is zero for $D > 3$ iff

$$a = 1 \quad \& \quad D = 26 . \tag{5.33}$$

We therefore confirm that Lorentz invariance requires $a = 1$, but we now see that it also requires $D = 26$; this is the *critical dimension* of the NG string.

5.1.2 Quantum closed string

Finally, we consider light-cone gauge quantization of the closed string. There are now two sets of oscillator operators, with commutation relations

$$\left[\hat{\alpha}^I_k, \hat{\alpha}^J_{-k}\right] = k\delta^{IJ} = \left[\hat{\tilde{\alpha}}^I_k, \hat{\tilde{\alpha}}^J_{-k}\right] . \tag{5.34}$$

The oscillator vacuum is now

$$|0\rangle = |0\rangle_R \otimes |0\rangle_L , \qquad \hat{\boldsymbol{\alpha}}_k|0\rangle_R = 0 \quad \& \quad \hat{\tilde{\boldsymbol{\alpha}}}_k|0\rangle_L = 0 \qquad \forall k > 0 . \tag{5.35}$$

We define the level number operators $\hat{N}$ and $\hat{\tilde{N}}$ such that they annihilate the oscillator vacuum. Again, we can choose a basis for the Fock space built on the oscillator vacuum for which these operators are diagonal, with eigenvalues N and $\tilde{N}$. In the space of states of definite p and definite $(N, \tilde{N})$ the mass-shell and level matching constraints are

$$p^2 + 4\pi T \left[(N - a_R) + \left(\tilde{N} - a_L\right)\right] = 0 \quad \& \quad (N - a_R) = \left(\tilde{N} - a_L\right) . \tag{5.36}$$

Since we want $|0\rangle$ to be a physical state we must choose $a_L = a_R = a$, and then we have

$$p^2 + 8\pi T\,(N - a) = 0 \quad \& \quad \tilde{N} = N . \tag{5.37}$$

This means that we can organise the states according to the level N, with $M^2 = 8\pi T(N - a)$. We must do this respecting the level-matching condition $\tilde{N} = N$. Let's look at the first few levels

- $N = 0$. There is one state, and hence a scalar, with $\alpha' M^2 = -4a$.
- $N = 1$. There are now $(D - 2) \times (D - 2)$ states

  $$\hat{\alpha}^I_{-1}|0\rangle_R \otimes \hat{\tilde{\alpha}}^J_{-1}|0\rangle_L \tag{5.38}$$

  We can split these into irreducible representations by taking the combinations

  $$\left[h_{IJ}(p) + \delta_{IJ}\phi(p) + b_{IJ}(p)\right] \hat{\alpha}^I_{-1}|0\rangle_R \otimes \hat{\tilde{\alpha}}^J_{-1}|0\rangle_L , \tag{5.39}$$

  where h_{IJ} is symmetric traceless tensor, b_{IJ} an antisymmetric tensor and ϕ a scalar. The only way that these could be part of a Lorentz-invariant theory is if h_{IJ} are the physical components of a *massless* spin-2 field because massive spin-2 would require a symmetric traceless tensor of the rotation group $SO(D-1)$. Then b_{IJ} must be the physical components of a massless antisymmetric tensor field, and ϕ a massless scalar (the dilaton).

  Since we require $M^2 = 0$ we must choose $a = 1$ again[31]. This means that the ground state is again a tachyon.

[31]This is usually given as $a = 2$ but that's due to a different definition of a for the closed string.

- $N = 2$. Since $a = 1$, the $N = 2$ states are massive, with $\alpha' M^2 = 4$. Recalling that the open string states at level 2 combined into a symmetric traceless tensor of $SO(D-1)$, we see that the level-2 states of the closed string will combine into those $SO(D-1)$ representations found in the product of two symmetric traceless $SO(D-1)$ tensors. This includes a 4th-order totally symmetric traceless tensor describing a massive particle of spin-4; there will be several lower spins too.

The most remarkable fact about these results is that the closed string spectrum contains a massless spin-2 particle, suggesting that a closed string theory will be a theory of quantum gravity. As for the open string, one finds that Lorentz invariance is preserved only if $D = 26$ (the calculation needed to prove this is a repeat of the open string calculation because the spin operator is a sum of a contribution from "left" oscillators and a contribution from "right" oscillators).

As for the open string, the closed string in its oscillator ground state is a tachyon. Formally, this is a "particle" of spacelike D-momentum but it is really an indication that the Minkowski ground state is unstable. The tachyon is absent in superstring theory, for which the critical dimension is $D = 10$, and there are various ways to compactify dimensions so as to arrive at more-or-less realistic models of gravity coupled to matter.

5.2 "Old covariant" quantization

Dirac's method of dealing with first-class constraints would appear to allow us to quantize the string in a way that preserves manifest Lorentz invariance. Let's consider the open string with free-end b.c.s. Recall that the action in terms of Fourier modes is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_{n\in\mathbb{Z}} \lambda_{-n} L_n \right\}, \qquad L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k \cdot \alpha_{n-k}\,, \quad (5.40)$$

and that $\{L_k, L_\ell\}_{PB} = -i\,(k-\ell)\,L_{k+\ell}$. Applying the $\{,\} \to -i[,]$ rule to the PBs of the canonical variables, we get the canonical commutation relations

$$[\hat{x}^m, \hat{p}_n] = i\delta^m_n\,, \qquad \left[\hat{\alpha}^m_k, \hat{\alpha}^n_{-k}\right] = k\eta^{mn}\,. \quad (5.41)$$

Now we define the oscillator vacuum $|0\rangle$ by

$$\hat{\alpha}^m_k |0\rangle = 0 \quad \forall k > 0 \qquad (m = 0, 1, \ldots, D-1). \quad (5.42)$$

The Fock space is built on $|0\rangle$ by the action of the creation operators $\hat{\alpha}^m_{-k}$, but this gives a space with many unphysical states since we now have D oscillators for each k, whereas we know (from light-cone gauge quantization) that $D-2$ suffice to construct the physical states.

Can we remove unphysical states by imposing the physical state conditions

$$\hat{L}_n|\text{phys}\rangle = 0 \quad \forall n \ ? \tag{5.43}$$

If this were possible then we would have achieved a Lorentz covariant quantization of the massless spin-1 particle at level 1 without the need for unphysical polarisation states, but this is not possible: the Lorentz-invariant Lorenz gauge $p \cdot A = 0$ reduces their number from D to $D-1$ but no Lorentz-invariant condition will reduce it to $D-2$. So we are going to run into a problem!

Notice that we do not encounter an operator ordering ambiguity when passing from the classical phase-space function L_n to the corresponding operator $\hat{L}_n$ except when $n = 0$, so the operator $\hat{L}_n$ is unambiguous for $n \neq 0$ and it is easy to see that

$$\hat{L}_n|0\rangle = 0\,, \qquad n > 0\,. \tag{5.44}$$

However, it is also easy to see [exercise: check these statements] that

$$\begin{aligned}
\hat{L}_{-1}|0\rangle &\equiv \frac{1}{2}\sum_k \hat{\alpha}_k \cdot \hat{\alpha}_{-1-k}|0\rangle = \hat{\alpha}_0 \cdot \hat{\alpha}_{-1}|0\rangle \\
\hat{L}_{-2}|0\rangle &\equiv \frac{1}{2}\sum_k \hat{\alpha}_k \cdot \hat{\alpha}_{-2-k}|0\rangle = \left(\hat{\alpha}_0 \cdot \hat{\alpha}_{-2} + \frac{1}{2}\hat{\alpha}_{-1}^2\right)|0\rangle\,,
\end{aligned} \tag{5.45}$$

so it looks as though not even $|0\rangle$ is physical. In fact, there are no states in the Fock space satisfying (5.43) because the algebra of the operators $\hat{L}_n$ has a quantum anomaly, which is such that the set of operators $\{\hat{L}_n; n \in \mathbb{Z}\}$ is not "first-class". That is what we shall now prove.

Since the $\hat{L}_n$ are quadratic in oscillator variables, the product of two of them is quartic but the commutator $[\hat{L}_m, \hat{L}_n]$ is again quadratic. That is what we expect from the PB, which is proportional to $\hat{L}_{m+n}$, but to get this from the expression that results from computing the commutator we may need to re-order operators, and that would produce a constant term. So, we are bound to find that

$$\left[\hat{L}_m, \hat{L}_n\right] = (m-n)\,\hat{L}_{m+n} + A_{mn} \tag{5.46}$$

for some constants A_{mn}. We can compute the commutator using the fact that

$$\left[\hat{L}_m, \hat{\alpha}_k\right] = -k\,\hat{\alpha}_{k+m}\,. \tag{5.47}$$

This can be verified directly but it also follows from the corresponding PB result because no ordering ambiguity is possible either on the LHS or the RHS. Using this, we find that

$$\begin{aligned}
\left[\hat{L}_m, \hat{L}_n\right] &= \frac{1}{2}\sum_k \left(\left[\hat{L}_m, \hat{\alpha}_k\right] \cdot \hat{\alpha}_{n-k} + \hat{\alpha}_k \cdot \left[\hat{L}_m, \hat{\alpha}_{n-k}\right]\right) \\
&= -\frac{1}{2}\sum_k k\,\hat{\alpha}_{k+m} \cdot \hat{\alpha}_{n-k} - \frac{1}{2}\sum_k (n-k)\,\hat{\alpha}_k \cdot \hat{\alpha}_{n+m-k}\,.
\end{aligned} \tag{5.48}$$

As long as $n+m \neq 0$ this expression is not affected by any change in the order of operators, so it must equal what one gets from an application of the $\{,\}_{PB} \to -i[,]$ rule. In other words, $A_{mn} = 0$ unless $m+n=0$. We can check this by using the fact that $\hat{\alpha}_{-k} = \hat{\alpha}_k^\dagger$, so that

$$\hat{\alpha}_k|0\rangle = 0 \quad \Leftrightarrow \quad \langle 0|\hat{\alpha}_{-k} = 0\,. \tag{5.49}$$

From this we see that for $m+n \neq 0$,

$$\langle 0|\hat{\alpha}_{k+m}\cdot\hat{\alpha}_{n-k}|0\rangle = 0 = \langle 0|\hat{\alpha}_k\cdot\hat{\alpha}_{n+m-k}|0\rangle \qquad (m+n\neq 0)\,, \tag{5.50}$$

and hence that $\langle 0|[\hat{L}_m,\hat{L}_n]|0\rangle = 0$ unless $m+n=0$. This tells us that

$$A_{mn} = A(m)\delta_{m+n}\,. \tag{5.51}$$

We now focus on the $m+n=0$ case, for which

$$\left[\hat{L}_m,\hat{L}_{-m}\right] = 2m\hat{L}_0 + A(m) \quad \Rightarrow \quad A(-m) = -A(m)\,. \tag{5.52}$$

Because of an operator ordering ambiguity, the operator $\hat{L}_0$ is only defined up to the addition of a constant, so what we find for $A(m)$ will obviously depend on how we define $\hat{L}_0$. We shall define it as

$$\hat{L}_0 = \frac{1}{2}\hat{\alpha}_0^2 + \hat{N}_{(cov)}\,, \qquad \hat{N}_{(cov)} \equiv \sum_{k=1}^{\infty}\hat{\alpha}_{-k}\cdot\hat{\alpha}_k\,, \tag{5.53}$$

The operator $\hat{N}_{(cov)}$ is a Lorentz covariant analog of the level number operator of the light-cone gauge. As in that case, it annihilates the oscillator vacuum and has eigenvalues that are non-negative integers; these are the level numbers, which we again call N

From now on we drop the hats on operators

We could now return to (5.48), set $n=-m$, and then complete the computation to find $A(m)$. This can be done, but it has to be done with great care to avoid illegitimate manipulations of infinite sums (see Q.III.1). Here we take an indirect route. First we use the Jacobi identity[32]

$$[L_k,[L_m,L_n]] + \text{cyclic permutations} \equiv 0\,, \tag{5.54}$$

[32]This is a consequence of the associativity of the product of operators; i.e. we use the fact that $(L_kL_m)L_n = L_k(L_mL_n)$. The antisymmetric Lie product of two operators defined by their commutator is *not* associative (the Jacobi identity tells us that) but what is relevant here is the product used to define the commutator, not the Lie product defined by the commutator.

to deduce that [Exercise]

$$[(m-n)A(k)+(n-k)A(m)+(k-m)A(n)]\,\delta_{m+n+k}=0\,. \tag{5.55}$$

Now set $k=1$ and $m=-n-1$ (so that $m+n+k=0$) to deduce that

$$A(n+1)=\frac{(n+2)A(n)-(2n+1)A(1)}{n-1}\qquad n\geq 2\,. \tag{5.56}$$

This is a recursion relation that determines $A(n)$ for $n\geq 3$ in terms of $A(1)$ and $A(2)$, so there are two independent solutions of the recursion relation. You may verify that $A(m)=m$ and $A(m)=m^3$ are solutions, so now we have

$$[L_m,L_{-m}]=2mL_0+c_1m+c_2m^3\,, \tag{5.57}$$

for some constants c_1 and c_2. Observing that ($m>0$)

$$\langle 0|\,[L_m,L_{-m}]\,|0\rangle=\langle 0|L_mL_{-m}|0\rangle=||L_{-m}|0\rangle||^2\,, \tag{5.58}$$

and that

$$\langle 0|L_0|0\rangle=\frac{1}{2}\alpha_0^2=\frac{p^2}{2\pi T}\,, \tag{5.59}$$

we deduce that

$$||L_{-m}|0\rangle||^2-\left(\frac{p^2}{\pi T}\right)m=c_1m+c_2m^3\,. \tag{5.60}$$

We can now get two equations for the two unknown constants (c_1,c_2) by evaluating $||L_{-m}|0\rangle||^2$ for $m=1$ and $m=2$. Using (5.45) we find that

$$||L_{-1}|0\rangle||^2=\frac{1}{\pi T}\langle 0|p\cdot\alpha_1\,p\cdot\alpha_{-1}|0\rangle=\frac{p_mp_n}{\pi T}\langle 0|\alpha_1^m\alpha_{-1}^n|0\rangle=\frac{p^2}{\pi T}\,, \tag{5.61}$$

and that

$$\begin{aligned}||L_{-2}|0\rangle||^2&=\langle 0|\left(\alpha_0\cdot\alpha_2+\frac{1}{2}\alpha_1^2\right)\left(\alpha_0\cdot\alpha_{-2}+\frac{1}{2}\alpha_{-1}^2\right)|0\rangle\\&=\frac{1}{\pi T}\langle 0|p\cdot\alpha_2\,p\cdot\alpha_{-2}|0\rangle+\frac{1}{4}\langle 0|\alpha_1^2\alpha_{-1}^2|0\rangle\\&=\frac{2p^2}{\pi T}+\frac{D}{2}\,,\end{aligned} \tag{5.62}$$

from which we see that

$$||L_{-1}|0\rangle||^2-\frac{p^2}{\pi T}=0\,,\qquad ||L_{-2}|0\rangle||^2-\frac{2p^2}{\pi T}=\frac{D}{2}\,, \tag{5.63}$$

and hence that

$$c_1+c_2=0\,,\qquad c_1+4c_2=\frac{D}{4}\qquad\Rightarrow\qquad c_2=-c_1=\frac{D}{12}\,. \tag{5.64}$$

Inserting this result into (5.57), we have

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}\left(m^3 - m\right) , \tag{5.65}$$

and hence that

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{D}{12}\left(m^3 - m\right) \delta_{m+n} , \tag{5.66}$$

where

$$\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 . \end{cases} \tag{5.67}$$

This is an example of the *Virasoro algebra*. In general, the Virasoro algebra takes the form

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\left(m^3 - m\right) \delta_{m+n} , \tag{5.68}$$

where c is the *central charge*. In the current context we get this algebra with $c = D$.

5.2.1 The Virasoro constraints

We have just seen that without breaking manifest Lorentz covariance, it is not possible to impose the physical state conditions that we would need to impose to eliminate all unphysical degrees of freedom. In view of our light-cone gauge results, this should not be too much of a surprise. We saw that the level-one states of the open string are the polarisation states of a massless spin-1 particle, but a *manifestly* covariant description of a massless spin-1 particle *necessarily* involves unphysical degrees of freedom.

This argument suggests that we should aim to impose weaker conditions, leaving unphysical states that will have to be dealt with in some other way. The strongest constraints that we can consistently impose are the Virasoro constraints:

$$(L_n - a\delta_n)\,|\Psi\rangle = 0 \quad \forall n \geq 0 . \tag{5.69}$$

Here we allow for the fact that the operator L_0 appearing here could differ by a constant from the one defined in (5.53). We shall refer to states satisfying these conditions as "Virasoro allowed". Notice that for any two Virasoro allowed states $|\Psi\rangle$ and $|\Psi'\rangle$,

$$\langle\Psi'|\,(L_n - a\delta_n)\,|\Psi\rangle = 0 \quad \forall n , \tag{5.70}$$

because it follows from $\alpha_{-k} = \alpha_k^\dagger$ that

$$L_{-n} = L_n^\dagger . \tag{5.71}$$

Notice too that the oscillator vacuum $|0\rangle$ is Virasoro allowed, because of (5.44), with $p^2 = 2\pi T a$; i.e. $\alpha' M^2 = -a$, just as we found in light-cone gauge. We may construct a basis for the oscillator Fock space by the successive actions of creation operators α_{-k} on $|0\rangle$. As for the light-cone case, we can organise the basis states according to their level number N, which is now the eigenvalue of the covariant level number operator. In a basis for which the D-momentum operator is also diagonal, with D-vector eigenvalues p, the mass-shell condition for a state $|\Psi_N\rangle$ at level N is

$$p^2 + M^2 = 0\,, \qquad \alpha' M^2 = N - a\,. \tag{5.72}$$

Let's consider the first few levels. The only $N = 0$ state in the oscillator Fock space is $|0\rangle$; this is a scalar with $\alpha' M^2 = -a$.

The general $N = 1$ state is

$$A_m(p)\,\alpha_{-1}^m|0\rangle\,, \qquad p^2 = 2\pi T(a-1)\,. \tag{5.73}$$

This is obviously annihilated by L_n for $n > 1$ and by L_1 if

$$0 = L_1(A\cdot\alpha_{-1})|0\rangle = A\cdot\alpha_0|0\rangle \quad \Rightarrow \quad p\cdot A = 0\,. \tag{5.74}$$

So the level-1 state (5.73) is Virasoro allowed iff $p \cdot A = 0$. Its norm-squared is

$$\| A\cdot\alpha_{-1}|0\rangle\|^2 = A_m A_n \langle 0|\alpha_1^m \alpha_{-1}^n|0\rangle = \eta^{mn}A_mA_n \equiv A^2\,. \tag{5.75}$$

This could be positive, negative or zero according to the value of the constant a:

- $a > 1$. In this case $M^2 < 0$, so p is spacelike. In a frame where $p = (0; p, \mathbf{0})$ the constraint $p \cdot A = 0$ is equivalent to $A_1 = 0$, so the general allowed level-1 state in this frame is

$$|\Psi_1\rangle = A_0\,\alpha_{-1}^0|0\rangle + \mathbf{A}\cdot\boldsymbol{\alpha}_{-1}|0\rangle\,, \qquad \||\Psi_1\rangle\|^2 = -A_0^2 + |\mathbf{A}|^2\,. \tag{5.76}$$

  The state with $\mathbf{A} = \mathbf{0}$ has negative norm; it is a "ghost". This implies a violation of unitarity (non-conservation of probability); we should not allow $a > 1$.

- $a < 1$. In this case $M^2 > 0$, so p is timelike. In a frame where $p = (p, \vec{0})$, the constraint $p\cdot A = 0$ implies that $A_0 = 0$, so the general allowed level-1 state in this frame is

$$|\Psi_1\rangle = \vec{A}\cdot\vec{\alpha}_{-1}|0\rangle\,, \qquad \||\Psi_1\rangle\|^2 = |\vec{A}|^2\,. \tag{5.77}$$

  Now, all non-zero allowed states have positive norm. There are $(D-1)$ independent such states, exactly the number required for a massive spin-1 particle. There is nothing unphysical about this but it doesn't agree with the light-cone gauge result. Covariant quantization with $a < 1$ breaks the conformal gauge invariance of the classical NG string, resulting in an additional level-1 state that was absent classically. This leads to difficulties in constructing an interacting theory[33].

[33] Attempts to do this go by the name of "non-critical string theory".

- $a = 1$. In this case $M^2 = 0$, so p is null, which agrees with the light-cone gauge result. In a frame where $p = (1; 1, \mathbf{0})$ the constraint $p \cdot A = 0$ implies that $A_0 = A_1$; equivalently $A_- = 0$. The general allowed level-1 state is then

  $$|\Psi_1\rangle = \left(A_+ \alpha_{-1}^+ + \mathbf{A} \cdot \boldsymbol{\alpha}_{-1}\right) |0\rangle \,. \tag{5.78}$$

  Now we get

  $$\| |\Psi_1\rangle \|^2 = \langle 0| \left(A_+ \alpha_1^+ + \mathbf{A} \cdot \boldsymbol{\alpha}_1\right) \left(A_+ \alpha_{-1}^+ + \mathbf{A} \cdot \boldsymbol{\alpha}_{-1}\right) |0\rangle = |\mathbf{A}|^2 \tag{5.79}$$

  There are no ghosts, but $\alpha_{-1}^+|0\rangle$ is a state of zero norm; i.e. a *null* state. In fact, it is orthogonal to all allowed states, *including itself*, as the equation above shows.

  Although the number of independent allowed level-1 states is $(D-1)$ we may identify any two states that differ by the addition of some multiple of the null state $\alpha_{-1}^+|0\rangle$; in other words we consider the equivalence class of allowed states defined by the equivalence relation

  $$\mathbf{A} \cdot \boldsymbol{\alpha}_{-1}|0\rangle \sim \mathbf{A} \cdot \boldsymbol{\alpha}_{-1}|0\rangle + A_+ \alpha_{-1}^+|0\rangle \tag{5.80}$$

  for any A_+. The dimension of the space of these equivalence classes is $(D-2)$ because the basis state $\alpha_{-1}^+|0\rangle$ is now equivalent to the zero state. If physical states are defined in this way we get agreement with the light-cone gauge.

  What we are finding here is essentially the Gupta-Bleuler quantization of electrodynamics (in D space-time dimensions).

Let's now look at the level-2 states assuming that the level-1 states are massless; i.e. assuming $a = 1$. These states have $\alpha' M^2 = 1$ and the general such state is

$$|\Psi_2\rangle = \left(A_{mn}\, \alpha_{-1}^m \alpha_{-1}^n + B_m\, \alpha_{-2}^m\right) |0\rangle \,. \tag{5.81}$$

This is trivially annihilated by L_k for $k > 2$. However, $L_1|\Psi_2\rangle = 0$ imposes the condition

$$B_n = -\alpha_0^m A_{mn} \,, \tag{5.82}$$

and $L_2|\Psi_2\rangle = 0$ imposes the further condition

$$\eta^{mn} A_{mn} = -2\alpha_0 \cdot B \,. \tag{5.83}$$

This means that only the traceless part of A_{mn} is algebraically independent, so the dimension of the Virasoro-allowed level-2 space is

$$\frac{1}{2} D(D+1) - 1 = \left[\frac{1}{2} D(D-1) - 1\right] + D \tag{5.84}$$

The dimension is D larger than the physical level-2 space that we found from light-cone gauge quantization (these formed the irreducible traceless symmetric rank-2 tensor of the $SO(D-1)$ rotation group). However, equivalence with the light-cone gauge results is still possible if there are sufficient null states, and no ghosts.

To analyse this we need to consider the norm-squared of $|\Psi_2\rangle$, which is

$$|||\Psi_2\rangle||^2 = 2A^{mn}A_{mn} + 2B^2 \,. \tag{5.85}$$

Then we need to consider the implications for this norm of (5.82) and (5.83). We will not carry out a complete analysis; the final result is that there are no ghosts only if $D \leq 26$ and then there are sufficient null states for equivalence with the light-cone gauge results iff $D = 26$.

It is simple to see that there are ghosts if $D > 26$, and null states if $D = 26$. Consider the special case of (5.81) for which

$$A^{mn} = \eta^{mn} + k_1 \alpha_0^m \alpha_0^n \,, \qquad B^m = k_2 \alpha_0^m \,. \tag{5.86}$$

This gives us the *Lorentz scalar* state

$$\left[\alpha_{-1}^2 + k_1 \left(\alpha_0 \cdot \alpha_{-1}\right)^2 + k_2 \alpha_0 \cdot \alpha_{-2}\right] |0\rangle \,. \tag{5.87}$$

The conditions (5.82) and (5.83) determine the constants (k_1, k_2) to be

$$k_1 = \frac{D+4}{10} \,, \qquad k_2 = \frac{D-1}{5} \,, \tag{5.88}$$

and then one finds that the norm-squared is (see Q.III.2)

$$-\frac{2}{25}\left(D-1\right)\left(D-26\right) \,. \tag{5.89}$$

This is negative for $D > 26$, so the state being considered is a ghost. To avoid ghosts we require $D \leq 26$.

For $D < 26$ the scalar state (5.87) has positive norm, and this implies that there is a physical scalar field at level 2, in disagreement with the result of light-cone gauge quantization. If fact, for $D < 26$ one finds not only an additional scalar but also an additional (massive) vector; these account for the increase by D in the dimension of the space of level-1 physical states in comparison to the light-cone gauge. Quantization has broken the gauge invariance of the classical theory, thereby introducing additional degrees of freedom that have no classical analog.

Finally, if $D = 26$ the scalar state (5.87) is null; it is also orthogonal to all other allowed states (as a further calculation shows). The same is true of the additional $(D-1)$-vector state, so if we define physical states as equivalence classes (in the way described for level 1) then we recover the level-2 results of the light-cone gauge.

It can be shown that provided $a = 1$ and $D = 26$, the results of the light-cone gauge are recovered in this way at all higher levels. This is the "no-ghost theorem" of string theory.

5.3 Conformal invariance and Vertex operators

The Virasoro constraints have the following interpretation: a physical state is a highest weight state for an irreducible representation of the Virasoro algebra of conformal weight a. As we have seen, Lorentz invariance of the quantum open string requires $a = 1$.

We shall focus on the closed string. In this case we have two sets of Virasoro conditions, so the physical states are highest weight states with weights $(1, 1)$. This statement can be transferred from states to operators. We shall seek those operators that create a given state in the string spectrum by their action on the following string state representing a string at rest in its ground state:

$$|p = 0\rangle \otimes |0\rangle \, . \tag{5.90}$$

These "vertex operators" should have conformal dimension $(1, 1)$, but what does this mean?

Let us first consider this question at the classical level, in the context of the conformal gauge action

$$I = T \int d\sigma^+ d\sigma^- \, \partial_- X \cdot \partial_+ X \, . \tag{5.91}$$

This action is invariant under conformal isometries of the worldsheet Minkowski metric, for which the algebra is $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$. The first Diff_1 acts on functions of σ^- and the second acts on functions of σ^+, but more generally these "functions" are one-component tensors that transform in the following way: if $A_\pm$ is a function of $\sigma^\pm$ and we consider new coordinates $\breve{\sigma}^\pm$ then the new "function" $\breve{A}_\pm$ of the new coordinates is

$$\breve{A}_\pm(\breve{\sigma}^\pm) = \left(\frac{d\sigma^\pm}{d\breve{\sigma}^\pm}\right)^{h_\pm} A_\pm(\sigma^\pm) \, , \tag{5.92}$$

for some number $h_\pm$ called the conformal dimension (or weight). More precisely, we should say that A_- has conformal dimensions $(h_-, 0)$ and A_+ has conformal dimensions $(0, h_+)$. Their product $A_- A_+$ has conformal dimensions (h_-, h_+).

The infinitesimal form of (5.92), found by setting $\breve{\sigma}^\pm = \sigma^\pm - \xi^\pm(\sigma^\pm)$, is

$$\delta_{\xi^\pm} A_\pm(\sigma^\pm) = \xi^\pm \partial_\pm A_\pm + h_\pm (\partial_\pm \xi^\pm) A_\pm \, . \tag{5.93}$$

The $h_\pm = 1$ case is special because in this case

$$\delta_{\xi^\pm} A_\pm = \partial_\pm \left(\xi^\pm A_\pm\right) \, , \tag{5.94}$$

and hence, since $\partial_\pm A_\mp = 0$, the product $A_+ A_-$ of conformal dimension $(1, 1)$ varies into a total derivative. This means that we can add $A_+ A_-$ as a perturbation to the conformal gauge Lagrangian without destroying the conformal invariance of the action; this is an important property that we will exploit later.

The transformation law (5.93) can be re-expressed as PB relations between Diff_1 generators and the Fourier coefficients of $A_\pm$ (see Q.III.4). For A_-, for example

$$\{L_m, A_n\}_{PB} = -i\,[m(h-1)-n]\,A_{m+n}\,, \qquad (h\equiv h_-) \tag{5.95}$$

The corresponding PB relation of $\tilde{L}_m$ with the Fourier coefficients $\tilde{A}_n$ of A_+ is the same but with $\tilde{h}\equiv h_+$ replacing h. As a check, recall that α_k are the Fourier components of $T\partial_- X$ (which is what $P - TX'$ becomes upon using the conformal gauge relation $P = T\dot{X}$). and also that

$$\{L_m, \alpha_n\}_{PB} = in\alpha_{n+m}\,, \tag{5.96}$$

which confirms that $h=1$ in this case. As a further check, recall that

$$\{L_m, L_n\}_{PB} = -i(m-n)L_{m+n}\,, \tag{5.97}$$

which tells us that $h = 2$ for Θ_{--} because $\{L_n\}$ are the Fourier coefficients of $\Theta_{--} = T(\partial_- X)^2$. This is expected because the conformal dimension of a product is, *classically*, the sum of the conformal dimensions of the factors of that product.

So far, we have discussed the classical meaning of conformal dimension for a class of worldsheet tensors. The quantum version is similar; for example, an operator $\hat{A}_-(\sigma_-)$ with Fourier coefficients $\hat{A}_n$ is a "conformal primary" operator of conformal dimension $h_- = h$ (and $\hat{A}_+(\sigma_+)$ with Fourier coefficients $\hat{\tilde{A}}_n$ is a "conformal primary" operator of conformal dimension $h_+ = \tilde{h}$) if

$$\begin{aligned}\left[\hat{L}_m, \hat{A}_n\right] &= [m(h-1)-n]\,\hat{A}_{m+n}\,,\\ \left[\tilde{L}_m, \tilde{A}_n\right] &= \left[m(\tilde{h}-1)-n\right]\tilde{A}_{m+n}\,.\end{aligned} \tag{5.98}$$

For example, $\partial_-\hat{X}$ is a conformal primary operator of conformal dimension $h=1$ because

$$\left[\hat{L}_m, \hat{\alpha}_m\right] = -n\hat{\alpha}_{n+m}\,. \tag{5.99}$$

However, the operator $\hat{\Theta}_{--}$ is no longer a conformal primary operator because of the central charge in the Virasoro algebra (its quantum conformal dimension is not defined).

What about X? For the closed string in the conformal gauge, the equations of motion have the solution

$$x(t) = x(0) + \left(\frac{p}{4\pi T}\right)t\,, \qquad \alpha_n(t) = e^{-int}\alpha_n(0)\,, \qquad \tilde{\alpha}_n(t) = e^{-int}\tilde{\alpha}_n(0)\,. \tag{5.100}$$

Using these relations in (3.103), we find that

$$X(t,\sigma) = X_R(\sigma^-) + X_L(\sigma^+)\,, \tag{5.101}$$

where

$$X_R(\sigma^-) = \frac{1}{2}\left[x(0) + \frac{p\sigma^-}{4\pi T}\right] + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}e^{-in\sigma^-}\alpha_n(0)\,,$$
$$X_L(\sigma^+) = \frac{1}{2}\left[x(0) + \frac{p\sigma^+}{4\pi T}\right] + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}e^{-in\sigma^+}\tilde{\alpha}_n(0)\,. \tag{5.102}$$

However, although X is periodic in σ, the functions $X_R(\sigma^-)$ and $X_L(\sigma^+)$ are *not* periodic (because of the term linear in $\sigma^\pm$). For this reason, the functions X_R and X_L are not conformal tensors on the cylindrical worldsheet of the closed string; they do not have the right transformation properties. However, this difficulty disappears if we consider $e^{ik\cdot X}$, for some Lorentz vector k:

$$e^{ik\cdot X} = e^{ik\cdot X_R(\sigma^-)}e^{ik\cdot X_L(\sigma^+)} \tag{5.103}$$

Both factors are periodic and hence well-defined conformal tensors on the cylindrical worldsheet; both of conformal dimension zero, so $e^{ik\cdot X}$ has conformal dimensions $(0,0)$. However, this is a *classical* result.

In the quantum theory we first need to define the operator that replaces $e^{ik\cdot X}$ by choosing an ordering of the operator products in $e^{ik\cdot\hat{X}}$. We first define (**now omitting hats on operators**)

$$V(k) = (V_<)V_0(V_>)\,, \qquad \tilde{V}(k) = (\tilde{V}_<)V_0(\tilde{V}_>)\,, \tag{5.104}$$

where

$$V_< = \exp\left(-\frac{1}{\sqrt{4\pi T}}\sum_{n<0}\frac{e^{-in\sigma^-}}{n}k\cdot\alpha_n(0)\right)\,,$$
$$V_> = \exp\left(-\frac{1}{\sqrt{4\pi T}}\sum_{n>0}\frac{e^{-in\sigma^-}}{n}k\cdot\alpha_n(0)\right)\,, \tag{5.105}$$

and similarly for $\tilde{V}_<$ and $\tilde{V}_>$ with σ^+ replacing σ^- and $\tilde{\alpha}(0)$ replacing $\alpha(0)$. We can now define an operator $\mathcal{V}(k)$ replacing the classical $e^{ik\cdot X}$ by

$$\mathcal{V}(k) = (\tilde{V}_<)V(k)(\tilde{V}_>) \equiv (V_<)\tilde{V}(k)(V_>)\,. \tag{5.106}$$

- **Proposition**: $\mathcal{V}$ has conformal dimensions $h_\pm = \frac{k^2}{8\pi T}$. The proof will be sketched shortly. It will be seen there that this is a purely quantum effect since the classical result is $h_\pm = 0$.

- **Corollary**: $\mathcal{V}(k)$ has conformal dimensions $(1,1)$ if $k^2 = 8\pi T$, which is precisely the value of k^2 for the closed string tachyon. This suggests that $\mathcal{V}(k)|p = 0\rangle\otimes|0\rangle$ could be related to the closed string tachyon state. We shall now investigate this.

The motivation behind the definition of $\mathcal{V}(k)$ is that all the annihilation operators are in $V_>$ and $\tilde{V}_>$, so that

$$V_>|0\rangle = |0\rangle\,, \qquad \tilde{V}_>|0\rangle = |0\rangle\,. \tag{5.107}$$

Moreover (exercise)

$$V_0|p=0\rangle = e^{ik\cdot\left[\hat{x}(0)+\frac{\hat{p}}{4\pi T}\right]} = e^{i\left(\frac{k^2}{8\pi T}\right)t}e^{ik\cdot x(0)}|p=0\rangle = e^{i\omega(k)t}|k\rangle\,, \tag{5.108}$$

where

$$\omega(k) = \frac{k^2}{8\pi T}\,. \tag{5.109}$$

We see that

$$\mathcal{V}(k)|p=0\rangle\otimes|0\rangle = e^{i\omega(k)t}|k\rangle\otimes(V_<)|0\rangle_R\otimes(\tilde{V}_<)|0\rangle_L\,. \tag{5.110}$$

This is not yet the tachyon state when $k^2 = 8\pi T$ because of the operator $V_<$, but if we replace t by $(1-i\epsilon)t$ for arbitrarily small positive number ϵ then $V_< \to 1$ as $t\to-\infty$. In other words, with this prescription the tachyon state $|k\rangle\otimes|0\rangle$ represents a tachyon in the far past, which then propagates into the far future, so that the worldsheet is a cyclinder that is infinite in both directions.

By taking $t\to(1-i\epsilon)t$ we have implicitly supposed that all worldsheet fields are restrictions of functions of a complex variable t to the real axis in the complex t-plane, and we have just infinitesimally rotated the line on which we initially restricted these fields. We could rotate by $-\pi/2$ instead of $-\epsilon$. This is equivalent to setting $t=-i\tau$ and then taking τ (the "imaginary time") to be real; this is analogous to "Wick-rotation" in QFT. If we do this then

$$\sigma^- = -iw\,,\quad \sigma^+ = -i\bar{w}\,;\qquad w=\tau-i\sigma\,, \tag{5.111}$$

We have now replaced the real coordinates on the cylindrical closed string worldsheet by a complex coordinate w, subject to the identification

$$w\sim w+2\pi i\,, \tag{5.112}$$

and the Minkowski worldsheet metric $-dt^2+d\sigma^2\equiv -d\sigma^-d\sigma^+$ has become the Euclidean worldsheet metric $d\tau^2+d\sigma^2\equiv dwd\bar{w}$. The conformal invariance of our conformal gauge string action will allow us to conformally map the cyclinder to the complex plane with complex coordinate

$$z=e^w=e^\tau e^{i\sigma}\,. \tag{5.113}$$

Notice that circles in the z-plane represent the closed string at a fixed "imaginary time" τ (it is actually real after the Wick rotation). Also the "far past" becomes a neighbourhood of the origin $z=0$. We can now write $V_<$ as

$$V_< = \exp\left\{-\frac{1}{\sqrt{4\pi T}}\sum_{n<0}\frac{z^{-n}}{n}k\cdot\alpha_n(0)\right\}\to 1 \quad \text{as } z\to 0\,, \tag{5.114}$$

and similarly for $\tilde{V}_<$. At $z=0$ we have $V_< = 1$ and $\tilde{V}_< = 1$, so

$$\mathcal{V}(k)|_{z=0}|p=0\rangle \otimes |0\rangle = e^{i\omega(k)t}|k\rangle \otimes |0\rangle \,. \tag{5.115}$$

This is a phase times the closed string tachyon state provided that $k^2 = 8\pi T$, but (by the above proposition) this is precisely the condition for $\mathcal{V}(k)$ to have conformal dimensions $(1,1)$.

The operator $\mathcal{V}(k)$ for $k^2 = 8\pi T$ is called the tachyon vertex operator. Now consider the operator (m, n are Lorentz indices here)

$$\partial_- X^m \partial_i X^n h_{mn}(k) \,, \qquad h_{mn}(k) = c_{mn}\mathcal{V}(k) \,. \tag{5.116}$$

for constants c_{mn}. As $\partial_- X^m \partial_+ X^n c_{mn}$ has conformal dimensions $(1,1)$ (which is also its classical conformal dimension) we should expect to have to choose k such that $\mathcal{V}(k)$ has conformal dimensions $(0,0)$. According to the proposition above, this requires $k^2 = 0$. An analysis similar to the one above for the tachyon shows that this operator acting on the state representing a string at rest in its ground state yields a level one state of the closed string provided that one takes $z \to 0$ and imposes the conditions

$$k^m h_{mn} = 0 \,, \tag{5.117}$$

which also follow by requiring that the level 1 state satisfy the Virasoro conditions (this is the closed string analog of the $k^m A_m = 0$ constraint that we we derived from the Virasoro conditions imposed on a level-1 state of the open string). Notice that in this level-1 case all operators in the level 1 vertex operator (which include the graviton, dilaton and notoph vertex operators) have their classical dimension.

More generally, *every* state of the string corresponds to the action, at $z=0$, of a vertex operator of conformal dimensions $(1,1)$ on the string ground state, and there are no other operators of conformal dimensions $(1,1)$. The *classical* conformal dimensions of vertex operators for states at level N is (N,N), so it is *only* at level-1 that the classical dimension matches the quantum dimension.

5.3.1 Computation of anomalous dimension of $\mathcal{V}(k)$

- Starting from the expressions of (5.105), show that

$$\begin{aligned} -i\partial_- V_< &= (V_<)\left[\frac{1}{\sqrt{4\pi T}}\sum_{n<0} e^{in\sigma} k\cdot\alpha_n\right] \,, \\ -i\partial V_> &= \left[\frac{1}{\sqrt{4\pi T}}\sum_{n>0} e^{in\sigma} k\cdot\alpha_n\right] V_> \end{aligned} \tag{5.118}$$

  and

$$-i\partial_- V_0 = \frac{1}{2}\left(V_0 \frac{k\cdot\alpha_0}{\sqrt{4\pi T}} + \frac{k\cdot\alpha_0}{\sqrt{4\pi T}} V_0\right) \,. \tag{5.119}$$

Thereby deduce that

$$-i\partial_- V(k) = \frac{1}{2\sqrt{4\pi T}} V_< \left[\left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_n \right) V_0 + V_0 \left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_n \right) \right] V_> \quad (5.120)$$

- Now show that

$$\begin{aligned} [L_m, V_>] &= \frac{1}{\sqrt{4\pi T}} \left(\sum_{n>0} e^{in\sigma} k\cdot\alpha_{m+n} \right) V_> \\ [L_m, V_0] &= \frac{k\cdot\alpha_m}{\sqrt{4\pi T}} V_0 \,. \end{aligned} \quad (5.121)$$

These are exactly the results expected from the corresponding PB relations, so they are essentially classical. Because of the need to re-order operators, there is a deviation from the classical result in the commutator

$$[L_m, V_<] = \frac{1}{\sqrt{4\pi T}} (V_<) \left(\sum_{n<0} e^{in\sigma} k\cdot\alpha_{m+n} \right) + (m-1) \left(\frac{k^2}{8\pi T} \right) e^{-im\sigma} V_< \quad (5.122)$$

The first term is classical and the last term is quantum.

Combining these results one finds that

$$[L_m, V(k)] = \frac{1}{\sqrt{4\pi T}} (V_<) \left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_{m+n} \right) V_0 V_> + (m-1) \left(\frac{k^2}{8\pi T} \right) e^{-im\sigma} V \quad (5.123)$$

We now use

$$\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_{m+n} = e^{-im\sigma} \left(\sum_{n>0} e^{in\sigma} k\cdot\alpha_n \right) \quad (5.124)$$

to get

$$\begin{aligned} [L_m, V(k)] &= e^{-im\sigma} \left[\frac{1}{\sqrt{4\pi T}} V_< \left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_n \right) V_0 V_> + (m-1) \left(\frac{k^2}{8\pi T} \right) V \right] \\ &= e^{-im\sigma} \left\{ \frac{1}{2\sqrt{4\pi T}} V_< \left[\left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_n \right) V_0 + V_0 \left(\sum_{n\in\mathbb{Z}} e^{in\sigma} k\cdot\alpha_n \right) \right] V_> \right. \\ &\qquad \left. m \left(\frac{k^2}{8\pi T} \right) V \right\} \\ &= -ie^{-im\sigma} \left[\partial_- + im \left(\frac{k^2}{8\pi T} \right) \right] V \,. \end{aligned} \quad (5.125)$$

- From the above result, and the observation that the Fourier coefficients of $V(k)$ are
$$V_n = \oint d\sigma e^{-in\sigma} V \,, \tag{5.126}$$
one deduces that
$$[L_m, V_n] = [m(h-1) - n]\, V_{m+n} \,, \qquad h = \frac{k^2}{8\pi T} \tag{5.127}$$
A similar calculation can be done for $\tilde{V}(k)$, so we conclude that the conformal dimensions of $\mathcal{V}(k)$ are as stated in the proposition.

5.3.2 Looking ahead

The worldsheet of a free string has been turned into a complex surface, with the topology of a cylinder, by "Wick-rotation", and this cylinder has been mapped to the complex z-plane. The origin of this plane represents an incoming string, in some quantum state. A basis of these states is provided by the level-N states of the free string, and a choice of any one of these basis states corresponds to "insertion" of the corresponding vertex operator at the origin of the z-plane. There is another insertion at $z = \infty$ corresponding to the outgoing particle, so the worldsheet is really the Riemann sphere with two marked points at which vertex operators are inserted, but we could insert more vertex operators at other points. In this way, we can compute amplitudes for scattering of string states from a computation of the ground state expectation value of a product of vertex operators at any point on the sphere. A computation of this type with 4 tachyon vertex operators (which is the simplest non-trivial case) yields the Virasoro-Shapiro amplitude for the scattering of two tachyons, but we are going to derive this later by a path-integral method.

For open strings, the Riemann-sphere Euclidean worldsheet is replaced by the unit disc, with the boundary being the union of the worldlines of the two ends of the string. The incoming and outgoing strings now correspond to insertions of open string vertex operators on this boundary. The simplest non-trivial amplitude is found from an insertion on the disc boundary of four open-string tachyon vertex operators, and this gives the Veneziano amplitude.

We shall not pursue this operator approach here. Instead we turn to path integrals, with the aim of arriving at the Virasoro-Shapiro amplitude in this way. But first we need to see how the path-integral formulation of QM applies to the relativistic particle.

6. Interlude: Path integrals and the point particle

Let $A(X)$ be the quantum-mechanical amplitude for a particle to go from the origin of Minkowski coordinates to some other point in Minkowski space-time with cartesian coordinates X. As shown by Feynman, $A(X)$ has a path-integral representation. In the case of a relativistic particle of mass m, with phase-space action $I[x,p;e]$ we have

$$A(X) = \int [de] \int [dxdp]\, e^{iI[x,p;e]}\,, \qquad x(0) = 0\,, \quad x(1) = X\,. \tag{6.1}$$

Here we are parametrising the path such that it takes *unit parameter time* to get from the space-time origin to the space-time point with coordinates X. The integrals have still to be defined, but we proceed formally for the moment.

An immediate problem is the gauge invariance of the action, which implies that we are integrating over too many variables. We can solve this problem by gauge fixing, but we don't want to break manifest Lorentz covariance. For the string we chose the conformal gauge; the particle analog is

$$e(t) = 1 \qquad (?) \tag{6.2}$$

However (as for the conformal gauge) one cannot make this choice everywhere on the worldline. This can be seen from the fact that $\int_0^1 edt$ is gauge invariant; in fact, it's proportional to the elapsed proper time, which could be *any positive number* whereas $\int_0^1 dt = 1$. We could choose to set $e(t) = 1$ everywhere except near $t = 0$, in which case the mass-shell constraint would become an initial condition (as discussed for the string in conformal gauge). Alternatively, we can set

$$e(t) = s \tag{6.3}$$

for a "variable constant" s. The action then becomes

$$I \to I_s[x,p] = \int_0^1 dt\, \dot{x}^m p_m - \frac{s}{2}\int_0^1 dt\, \left(p^2 + m^2\right)\,. \tag{6.4}$$

Variation of this action with respect to the constant s yields the integrated constraint $\int dt\,(p^2+m^2) = 0$, but the integrand is a constant of the motion so the equations of motion will imply that $p^2 + m^2 = 0$ everywhere on the worldline. We now have the simpler path integral[34]

$$A(X) = \int_0^\infty ds \int [dxdp]\, e^{iI_s[x,p]}\,. \tag{6.5}$$

This is not quite right (although close enough for some purposes) because we have not done the gauge fixing properly; we'll return to this point later.

We have to integrate over functions $x(t)$ and $p(t)$. Let us first write these functions as

$$\begin{aligned} x(t) &= Xt + q(t)\,, \quad q(0) = q(1) = 0 \\ p(t) &= P + \pi(t)\,, \quad \int_0^1 dt\, \pi(t) = 0\,, \end{aligned} \tag{6.6}$$

[34]We have assumed that $e > 0$ (by assuming that $s > 0$). We do this because the worldline time-reversal invariance of the action is an additional discrete diffeomorphism, which we fix by requiring $e > 0$.

so q satisfies simpler boundary conditions and P is the average D-momentum. We now find that

$$A(X) = \int_0^\infty ds \int d^D P\, e^{iX^m P_m - i\frac{s}{2}(P^2+m^2)} F(s) \tag{6.7}$$

where

$$F(s) = \int [dq d\pi] \exp\left\{ i \int_0^1 dt \left[\dot{q}^m \pi_m - \frac{1}{2} s \pi^2 \right] \right\} . \tag{6.8}$$

The functions q and π can be expressed as Fourier series:

$$q(t) = \sum_{k=1}^\infty \frac{x_k}{k\pi} \sin(k\pi t)\,, \qquad \pi(t) = \sum_{k=1}^\infty p_k \cos(k\pi t)\,. \tag{6.9}$$

This gives us

$$\begin{aligned} F(s) &= \text{const.} \prod_{k=1}^\infty \int d^D x_k \int d^D p_k \exp\left\{ \frac{i}{2} \left[x_k \cdot p_k - \frac{s}{2} p_k^2 \right] \right\} \\ &= \text{const.} \prod_{k=1}^\infty \int d^D u_k \int d^D v_k \exp\left\{ \frac{i}{2} \left[u_k \cdot v_k - \frac{1}{2} v_k^2 \right] \right\} \end{aligned} \tag{6.10}$$

where we have set $x_k = \sqrt{s}\, u_k$ and $p_k = v_k/\sqrt{s}$ to arrive at the second equality. This shows that $F(s)$ is actually independent of s. It is just an infinite constant, which we can absorb into the definition of the normalisation of $A(X)$, which is now

$$A(X) = \text{const.} \int d^D P\, e^{iX^m P_m} \int_0^\infty ds\, e^{-i\frac{s}{2}(P^2+m^2)} . \tag{6.11}$$

We are now going to view the s integral as a contour integral in the complex s-plane. We will rotate the contour to the negative imaginary axis and then set $s = -i\tilde{s}$ for real $\tilde{s}$. This gives us

$$\begin{aligned} A(X) &= \text{const.} \int d^D P e^{iX^m P_m} \int_0^\infty d\tilde{s}\, e^{-\frac{s}{2}(P^2+m^2)} \\ &= \text{const.} \int d^D P \frac{e^{iXP}}{P^2+m^2} , \end{aligned} \tag{6.12}$$

which is the Feynman propagator.

6.1 Faddeev-Popov determinant

The problem with the formula (6.1) is that, because of gauge-invariance, we are integrating over too many functions. We circumvented this problem by imposing the gauge condition $e = s$, effectively by inserting a delta functional $\delta[e - s]$ into the $\int[de]$ functional integral. But is this procedure correct?

We can express any function $e(t)$ as a gauge transform of $e = s$:

$$e(t) = e_s(t) \equiv s + \dot{\alpha}(t)\,, \qquad \alpha(1) = \alpha(0) \tag{6.13}$$

where the parameter $\alpha(t)$ is a map from the worldline to the gauge group (abelian and 1-dimensional in this case). The restriction on $\alpha(t)$ arises because we insist that $s = \int_0^1 dte(t)$. The problem with the formula (6.1) is that we are implicitly including a functional integral over $\alpha(t)$ that gauge invariance would allow us to omit if it were explicit. We can make it explicit by writing

$$\int [de] = \int_0^\infty ds \int [de_0\,[\alpha]] = \int_0^\infty ds \int [d\alpha]\,\Delta_{FP}\,, \tag{6.14}$$

where Δ_{FP} is the Jacobian for the change of variables from $e_0[\alpha](t)$ to $\alpha(t)$:

$$\Delta_{FP} = \det\left[\frac{\delta e_0[\alpha](t)}{\delta\alpha(t')}\right] = \det\left[\frac{\delta\dot\alpha(t)}{\delta\alpha(t')}\right] = \det\left[\delta'(t-t')\right] \tag{6.15}$$

This functional Jacobian is called the Faddeev-Popov determinant. Provided that the functional integrand is gauge invariant, the $[d\alpha]$ functional integral will produce only an infinite constant that can be absorbed into the undetermined normalization of the path integral for the amplitude $A[X]$, which now takes the corrected form

$$A(x) = \int_0^\infty ds\,\Delta_{FP}\int [dxdp]\,e^{iI_s[x,p]}\,. \tag{6.16}$$

This replaces (6.5), from which the Δ_{FP} factor was missing. That's why (6.5) was "not quite right" but in this simple case the Δ_{FP} factor is independent of any of the other variables that we have to integrate over, so it can only change the normalization of $A(X)$ which anyway depends on the detailed definitions of the path integral[35]. However, it is essential to take into account the FP determinant for generic gauge theories; in particular, it is very important to the path-integral quantization of the NG string, which we will get to soon.

6.2 Fadeev-Popov ghosts

Let (b_i, c^i) $(i = 1,\dots,n)$ be n pairs of *anticommuting* variables. This means that

$$\{b_i, b_j\} = 0\,, \quad \left\{b_i, c^j\right\} = 0\,, \quad \left\{c^i, c^j\right\} = 0 \quad \forall\, i,j = 1,\dots,n\,, \tag{6.17}$$

where $\{,\}$ means anticommutator: $\{A,B\} = AB + BA$. Any function of anticommuting variables has a terminating Taylor expansion because no one anticommuting variable can appear twice. Consider the $n = 1$ case

$$f(b,c) = f_0 + bf_1 + cf_{-1} + bc\tilde f_0\,, \tag{6.18}$$

where $(f_0, f_{\pm1}, \tilde f_0)$ are independent of both b and c. Then

$$\frac{\partial}{\partial b} f = f_1 + c\tilde f_0 \quad \Rightarrow \quad \frac{\partial}{\partial c}\frac{\partial}{\partial b} f = \tilde f_0\,. \tag{6.19}$$

[35] It is for this reason that the FP determinant can be ignored, for most purposes, in the path-integral approach to QED.

Essentially, a derivative with respect to b removes the part of f that is independent of b and then strips b off what is left. However, we should move b to the left of anything else before stripping it off; this is equivalent to the definition of the derivative as a "left derivative". Using this definition we have

$$\frac{\partial}{\partial c} f = f_{-1} - b\tilde{f}_0 \quad \Rightarrow \quad \frac{\partial}{\partial b}\frac{\partial}{\partial c} f = -\tilde{f}_0 \,. \tag{6.20}$$

There is minus sign relative to (6.19) because we had to move c to the left of b. This result shows that

$$\left\{ \frac{\partial}{\partial b}, \frac{\partial}{\partial c} \right\} = 0 \,. \tag{6.21}$$

That is, *partial derivatives with respect to anticommuting variables anti-commute.* In particular, since a function of anticommuting variables is necessarily linear in any one of them,

$$\left[\frac{\partial}{\partial b}\right]^2 = 0 \,, \qquad \left[\frac{\partial}{\partial c}\right]^2 = 0 \,. \tag{6.22}$$

We can also integrate over anticommuting variables. The (Berezin) integral over an anticommuting variable is defined to be the same as the partial derivative with respect to it. Consider, for example, the Gaussian integral

$$\int d^n b \, d^n c \; e^{b_i M^i{}_j c^j} = \left[\frac{\partial}{\partial b_n} \cdots \frac{\partial}{\partial b_1}\right] \left[\frac{\partial}{\partial c^n} \cdots \frac{\partial}{\partial c^1}\right] e^{b_i M^i{}_j c^j} \,. \tag{6.23}$$

If we expand the integrand in powers of bMc the expansion terminates at the nth term, which is also the only term that contributes to the integral because it is the only one to contain all b_i and all c^i. Because of the anti-commutativity of the partial derivatives, we then find that

$$\int d^n c \, d^n b \;\; e^{b_i M^i{}_j c_j} \;\propto\; \frac{1}{n!} \varepsilon_{i_1 \cdots i_n} M^{i_1}{}_{j_1} \cdots M^{i_n}{}_{j_n} \, \varepsilon^{j_1 \cdots j_n} = \det M \tag{6.24}$$

We can use a functional variant of this result to write the FP determinant as a Gaussian integral over anticommuting "worldline fields" $b(t)$ and $c(t)$:

$$\begin{aligned} \det\left[\delta(t-t')\partial_t\right] &= \int [db\, dc] \exp\left[\int dt \int dt' \, b(t') \left[\delta'(t-t')\right] c(t)\right] \\ &= \int [db\, dc] \exp\left[-\int dt\, b\dot{c}\right] \,. \end{aligned} \tag{6.25}$$

The anticommuting worldline fields are known collectively as the FP ghosts, although it is useful to distinguish between them by calling c the ghost and b the anti-ghost.

N.B. There is no relation between the FP ghosts and the ghosts that appear in the NG string spectrum for $D > 26$. The same word is being used for two entirely different things!

Using (6.25) in the expression (6.16) we arrive at the result

$$A(X) = \int_0^\infty ds \int [dxdp] \int [db\, dc]\, e^{iI_{qu}}\,, \tag{6.26}$$

where the "quantum" action is

$$I_{qu} = \int dt \left\{ \dot{x}^m p_m + ib\dot{c} - H_{qu} \right\}\,, \qquad H_{qu} = \frac{s}{2}\left(p^2 + m^2\right)\,. \tag{6.27}$$

We now have a mechanical system with an extended phase-space with additional, anticommuting, coordinates (b, c). The factor of i multiplying $b\dot{c}$ is needed by the convention that a product of two "real" anticommuting variables is "imaginary". Of course, an anticommuting number cannot really be real; it is "real" if we declare it to be unchanged by complex conjugation.

6.2.1 FP ghosts for systems with Lie algebra constraints

Now we consider how these ideas apply to the more general mechanical system with action

$$I[q,p;\lambda] = \int dt \left\{ \dot{q}^I p_I - \lambda^i \varphi_i \right\}\,, \qquad \{\varphi_i, \varphi_j\}_{PB} = f_{ij}{}^k \varphi_k\,. \tag{6.28}$$

For simplicity, we shall assume that the structure functions $f_{ij}{}^k$ are constants, in which case they are the structure constants of a Lie algebra, and hence subject to the constraint (following from the Lie algebra Jacobi identity)

$$f_{[ij}{}^p f_{k]p}{}^q = 0\,. \tag{6.29}$$

We shall suppose that we have chosen to gauge fix (perhaps only partially) by imposing the condition

$$\lambda^i = \bar{\lambda}^i \quad \text{(constants)}. \tag{6.30}$$

The FP operator is then found by varying the gauge transformation of the gauge-fixing function $(\lambda^i - \bar{\lambda}^i)$ with respect to the gauge parameter, so

$$\Delta_{FP} = \det\left(\frac{\delta_\epsilon \lambda^i(t')}{\delta \epsilon^j(t)}\right)\Bigg|_{\lambda^i = \bar{\lambda}^i}\,, \qquad \delta_\epsilon \lambda^i = \dot{\epsilon}^i - \epsilon^j \bar{\lambda}^k f_{kj}{}^i\,. \tag{6.31}$$

This gives

$$\Delta_{FP} = \det\left[\left(\delta^i_j \partial_t - \bar{\lambda}^k f_{kj}{}^i\right)\delta(t-t')\right] \quad \propto \quad \int [dbdc]\, e^{iI_{FP}[b,c]}\,, \tag{6.32}$$

where the FP action is

$$I_{FP}[b,c] = \int dt \left\{ ib_i \left[\dot{c}^i - c^j \bar{\lambda}^k f_{kj}{}^i\right]\right\}\,. \tag{6.33}$$

This must be added to the original action to get the "quantum action"

$$I_{qu} = \int dt \left\{ \dot{q}^I p_i + i b_i \dot{c}^i - H_{qu} \right\} , \qquad H_{qu} = \bar{\lambda}^k \Phi_k , \tag{6.34}$$

where

$$\Phi_k = \phi_k + \phi_k^{(\mathrm{gh})} , \qquad \phi_k^{(\mathrm{gh})} = i c^j f_{jk}{}^i b_i . \tag{6.35}$$

The reason for writing the "quantum Hamiltonian" this way will become clear shortly.

We now have an action for a mechanical system with an extended phase space (actually a *superspace*) for which some coordinates are anticommuting. Leaving aside the Hamiltonian, the action can be constructed from the following closed 2-form on this space:

$$\Omega = d\left(p_m dx^m + i b_i dc^i\right) = dp_m \wedge dx^m + i db_i \wedge dc^i . \tag{6.36}$$

This is an "orthosymplectic" 2-form because the anticommutativity of the FP ghosts means that[36]

$$db_i \wedge dc^i = dc^i \wedge db_i . \tag{6.37}$$

This leads to a Poisson bracket for the new canonical variables b_i and c^i that is *symmetric* rather than antisymmetric:

$$\left\{b_i, c^j\right\}_{PB} = \left\{c^j, b_i\right\}_{PB} = -i\delta_i^j . \tag{6.38}$$

The only other non-zero PB is the usual $\{q^I, p_J\}_{PB} = \delta_J^I$. Using these PB relations one finds that

$$\left\{\phi_i^{(\mathrm{gh})}, \phi_j^{(\mathrm{gh})}\right\}_{PB} = f_{ij}{}^k \phi_k^{(\mathrm{gh})} , \tag{6.39}$$

and hence that

$$\{\Phi_i, \Phi_j\}_{PB} = f_{ij}{}^k \Phi_k . \tag{6.40}$$

Thus, the new functions Φ_i, defined on the extended phase space, satisfy the same PB relations as the original constraint functions φ_i.

Recall that the gauge invariance is completely fixed when the equation $\delta_\epsilon \lambda^i = 0$ has no non-zero solutions for ϵ. In this case, the operator appearing in the FP ghost action will be invertible, and the b_i equation will imply that $c^i = 0$, and vice-versa; in other words, the FP ghosts can be trivially eliminated from the action. This is precisely why they are not needed in light-cone gauge. They are needed only when the gauge-fixing condition does *not* completely fix the gauge, leaving some residual gauge invariance and hence unphysical degrees of freedom; these are "cancelled out" (in a way made precise by the BRST formalism) by the FP ghosts. So let us now suppose that $\delta_\epsilon \lambda^i = 0$ allows non-zero solutions for the parameters ϵ^i. These residual gauge parameters satisfy

$$\dot{\epsilon}^i = \epsilon^k \bar{\lambda}^j f_{jk}{}^i . \tag{6.41}$$

[36] The usual minus sign coming from the antisymmetry of the wedge product of 1-forms is cancelled by the minus sign coming from changing the order of b and c.

The residual gauge transformation of the "quantum" Hamiltonian is

$$\delta_\epsilon H_{\rm qu} = \left\{\bar\lambda^j\Phi_j, \epsilon^k\Phi_k\right\}_{PB} = \epsilon^k\bar\lambda^j f_{jk}{}^i = \dot\epsilon^k\Phi_k\,, \tag{6.42}$$

where the equality uses (6.41). The variation of $-H_{\rm qu}$ therefore cancels the usual $\dot\epsilon^k\Phi_k$ contribution from the rest of the Lagrangian.

The residual gauge transformations of the (q,p) variables are found as before, and a similar calculation yields the residual gauge transformation of the FP ghost variables, e.g.

$$\delta_\epsilon c^i = \left\{c^i, \epsilon^k\Phi_k\right\}_{PB} = \epsilon^k\left\{c^i, \varphi_k^{(\rm gh)}\right\}_{PB} = \epsilon^k c^j f_{jk}{}^i\,. \tag{6.43}$$

Compare this with the transformation of λ^i (it is the same if one omits the $\dot\epsilon^i$ term). Similarly,

$$\delta_\epsilon b_i = \left\{b_i, \epsilon^k\Phi_k\right\}_{PB} = \epsilon^k f_{ik}{}^j b_j\,. \tag{6.44}$$

Notice that, in general, b_i and c^i transform differently (as will be the case for the string).

7. Path integrals and the NG string

We now aim to use the conformal gauge in a path-integral approach to quantisation of the closed NG string. Recall that the conformal gauge for the NG string is $\lambda^\pm = 1$, and that in this gauge the gauge variation of $\lambda^\pm$ is

$$\delta_\xi\lambda^\pm = 2\partial_\mp\xi^\pm\,. \tag{7.1}$$

The FP determinant is therefore

$$\Delta_{FP} = \det\begin{pmatrix} 2\partial_+\left[\delta(t-t')\delta(\sigma-\sigma')\right] & 0 \\ 0 & 2\partial_-\left[\delta(t-t')\delta(\sigma-\sigma')\right]\end{pmatrix}\,. \tag{7.2}$$

Following the steps spelled out for the particle we arrive at the following FP-ghost contribution to the action:

$$I_{FP} = 2i\int dt\oint d\sigma\left\{b\,\partial_+ c + \tilde b\,\partial_-\tilde c\right\}\,. \tag{7.3}$$

Adding the FP action to the usual phase-space action, we get the "quantum" action for the closed NG string in conformal gauge

$$\begin{aligned} I_{qu}[X,P;b,c;\tilde b,\tilde c] &= \int dt\oint d\sigma\left\{\dot X^m P_m + ib\dot c + i\tilde b\dot{\tilde c} - \mathcal{H}_{qu}\right\}\,, \\ \mathcal{H}_{qu} &= \frac{P^2}{2T} + \frac{T}{2}(X')^2 - i\left(bc' - \tilde b\tilde c'\right)\,. \end{aligned} \tag{7.4}$$

From this action we can read off the PB relations; in particular

$$\{b(\sigma), c(\sigma')\}_{PB} = -i\delta(\sigma - \sigma') = \{\tilde{b}(\sigma), \tilde{c}(\sigma')\}_{PB} \,. \tag{7.5}$$

As explained previously for the generic mechanical model with (Lie algebra) first-class constraints, this action is automatically invariant under the residual gauge invariance that survives the conformal gauge; in other words, it is conformal invariant, with the ghosts $(c, \tilde{c})$ transforming in the same way as the parameters under a composition of two residual gauge transformations; this tells us that $(c, \tilde{c})$ will transform as the light-cone components of a vector field.

$$\delta_\xi c = \xi^- \partial_- c \; - \left(\partial_- \xi^-\right) c \,, \qquad \delta_\xi \tilde{c} = \xi^+ \partial_+ \tilde{c} - \left(\partial_+ \xi^+\right) \tilde{c} \,. \tag{7.6}$$

Invariance of the action then requires $(b, \tilde{b})$ to transform as the light-cone components of a (symmetric traceless) "quadratic differential ":

$$\delta_\xi b = \xi^- \partial_- b \; + 2\left(\partial_- \xi^-\right) b \,, \qquad \delta_\xi \tilde{b} = \xi^+ \partial_+ \tilde{b} + 2\left(\partial_+ \xi^+\right) \tilde{b} \,. \tag{7.7}$$

The Noether charges are the Fourier coefficients of the two non-zero light-cone components of the FP-ghost stress tensor

$$L_m^{(gh)} = \oint d\sigma \, e^{-im\sigma} \Theta_{--}^{(gh)} \,, \qquad \tilde{L}_m^{(gh)} = \oint d\sigma \, e^{im\sigma} \Theta_{++}^{(gh)} \,. \tag{7.8}$$

These stress tensor components are

$$\begin{aligned} \Theta_{--}^{(gh)} &= i\left[2b\,\partial_- c - c\,\partial_- b\right] = -i\left[2b\,c' + b'c\right] \,, \\ \Theta_{++}^{(gh)} &= i\left[2\tilde{b}\,\partial_+ \tilde{c} - \tilde{c}\,\partial_+ \tilde{b}\right] = i\left[2\tilde{b}\,\tilde{c}' + \tilde{b}'\,\tilde{c}\right] \,, \end{aligned} \tag{7.9}$$

where the FP ghost equations of motion have been used for the second equalities.

We now pass to the Fourier-mode form of the action. In addition to the Fourier series for $P \pm TX'$, we will need the Fourier series expansions

$$\begin{aligned} c &= \sum_{k \in \mathbb{Z}} e^{ik\sigma} c_k \,, & b &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ik\sigma} b_k \,, \\ \tilde{c} &= \sum_{k \in \mathbb{Z}} e^{-ik\sigma} \tilde{c}_k \,, & \tilde{b} &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-ik\sigma} \tilde{b}_k \,. \end{aligned} \tag{7.10}$$

This gives us

$$L_m^{(gh)} = \sum_{k \in \mathbb{Z}} (m + k)\, b_{m-k} c_k \,, \qquad \tilde{L}_m^{(gh)} = \sum_{k \in \mathbb{Z}} (m + k)\, \tilde{b}_{m-k} \tilde{c}_k \,. \tag{7.11}$$

For $m = 0$ we have

$$\begin{aligned} L_0^{(gh)} &= N_{(gh)} \equiv \sum_{k=1}^{\infty} k\left(b_{-k} c_k + c_{-k} b_k\right) \,, \\ \tilde{L}_0^{(gh)} &= \tilde{N}_{(gh)} \equiv \sum_{k=1}^{\infty} k(\tilde{b}_{-k} \tilde{c}_k + \tilde{c}_{-k} \tilde{b}_k) \,. \end{aligned} \tag{7.12}$$

As we shall see, $N_{(gh)}$ and $\tilde{N}_{(gh)}$ will have the interpretation as (covariant) ghost level numbers.

The "quantum" action in terms of Fourier modes is

$$I_{qu} = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \left(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k} \right) + \sum_{n \in \mathbb{Z}} i \left(b_{-n} \dot{c}_n + \tilde{b}_{-n} \dot{\tilde{c}}_n \right) - H_{qu} \right\} ,$$
$$H_{qu} = \mathcal{L}_0 + \tilde{\mathcal{L}}_0 \tag{7.13}$$

where

$$\mathcal{L}_0 = L_0 + N_{(gh)} = \frac{1}{2}\alpha_0^2 + N_{qu} , \qquad N_{qu} = N + N_{(gh)} ,$$
$$\tilde{\mathcal{L}}_0 = \tilde{L}_0 + N_{(gh)} = \frac{1}{2}\alpha_0^2 + \tilde{N}_{qu} , \qquad \tilde{N}_{qu} = \tilde{N} + \tilde{N}_{(gh)} . \tag{7.14}$$

We can now read off the PBs of the Fourier modes. For the new, anticommuting, variables we have[37]

$$\{c_n, b_{-n}\}_{PB} = -i , \qquad \left\{\tilde{c}_n, \tilde{b}_{-n}\right\}_{PB} = -i , \qquad (n \in \mathbb{Z}). \tag{7.15}$$

Notice that $n = 0$ is included, although the (anti)ghost zero modes (b_0, c_0) and $(\tilde{b}_0, \tilde{c}_0)$ do not appear in the Hamiltonian. Using these PB relations, one finds that

$$\left\{L_m^{(gh)}, c_n\right\}_{PB} = i\,(2m+n)\,c_{n+m} , \qquad \left\{L_m^{(gh)}, b_n\right\}_{PB} = -i\,(m-n)\,b_{m+n} . \tag{7.16}$$

This leads to

$$\left\{L_m^{(gh)}, L_n^{(gh)}\right\}_{PB} = -i\,(m-n)\,L_{m+n}^{(gh)} , \tag{7.17}$$

and similarly for $\tilde{L}_m^{(gh)}$.

The total "quantum" conformal charges are

$$\mathcal{L}_m = L_m + L_m^{(gh)} , \qquad \tilde{\mathcal{L}}_m = \tilde{L}_m + \tilde{L}_m^{(gh)} . \tag{7.18}$$

These are the Fourier modes of the non-zero components of the energy-momentum stress tensor of the quantum action (7.4), and their algebra is that of Witt $\oplus$ Witt:

$$\{\mathcal{L}_m, \mathcal{L}_n\}_{PB} = -i\,(m-n)\,\mathcal{L}_{m+n}$$
$$\{\mathcal{L}_m, \tilde{\mathcal{L}}_n\}_{PB} = 0$$
$$\{\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n\}_{PB} = -i\,(m-n)\,\tilde{\mathcal{L}}_{m+n} . \tag{7.19}$$

The PBs of the $\mathcal{L}_m$ with the Fourier modes of the various fields determine the transformations of these fields under the residual conformal invariance of the conformal gauge.

[37] These PB relations are equivalent to (7.5).

Recall that $A^{(h)}(\sigma^-)$ has conformal dimension $(h,0)$ if its Fourier coefficients $A_n^{(h)}$ satisfy

$$\{\mathcal{L}_m, A_n^{(h)}\}_{PB} = -i\,[m\,(h-1) - n]\,A_{n+m}^{(h)}\,, \tag{7.20}$$

and similarly for $A^{(\tilde{h})}(\sigma^+)$ of conformal dimension $(0,\tilde{h})$. For instance, the PB relations

$$\begin{aligned} \{\mathcal{L}_m, \alpha_k\}_{PB} &= ik\alpha_{k+m}\,, \\ \{\mathcal{L}_m, c_k\}_{PB} &= i\,(2m+k)\,c_{k+m}\,, \\ \{\mathcal{L}_m, b_k\}_{PB} &= -i\,(m-k)\,b_{k+m}\,. \end{aligned} \tag{7.21}$$

tell us that $\partial_- X$ has conformal dimension $(1,0)$ (as we knew). Let's use the following notation for this:

$$[\partial_- X] = (1,0)\,, \qquad [\partial_+ X] = (0,1)\,, \tag{7.22}$$

which are the conformal dimensions of light-cone components a worldsheet one-form. Compare this with

$$[c] = (-1,0)\,, \qquad [\tilde{c}] = (0,-1)\,, \tag{7.23}$$

which are the conformal dimensions of the light-cone components of a worldsheet vector. Finally,

$$[b] = (2,0)\,, \qquad \left[\tilde{b}\right] = (0,2)\,. \tag{7.24}$$

These are the conformal dimensions of the light-cone components of a (symmetric traceless) quadratic differential.

After elimination of P from the action (7.4), we get the generalisation of the conformal gauge action (3.86):

$$I_{qu} = \int d^2\sigma\, \left\{2T\,\partial_+ X\cdot\partial_- X + i\left(b\,\partial_+ c + \tilde{b}\,\partial_-\tilde{c}\right)\right\}\,. \tag{7.25}$$

From the above results for conformal dimensions, and taking into account that acting with ∂_- raises the conformal dimension by $(1,0)$ and acting with ∂_+ raises it by $(0,1)$, we see that *all terms in the Lagrangian have conformal dimension* $(1,1)$, as required for conformal invariance of the action

This is all classical, but it carries over to the quantum theory with the same changes that we encountered previously: central charge in the Virasoro algebra and anomalous dimensions for vertex operators.

7.0.1 Critical dimension again

For the anticommuting FP ghosts, with Fourier modes satisfying the *symmetric* PB relations (7.5), we adapt the usual $\{,\}_{PB} \to -i[,]$ rule to $\{,\}_{PB} \to -i\{,\}$. This gives us the canonical *anticommutation* relations (we omit the hats on operators)

$$\{c_n, b_{-n}\} = 1\,, \qquad \left\{\tilde{c}_n, \tilde{b}_{-n}\right\} = 1 \qquad n\in\mathbb{Z}\,. \tag{7.26}$$

We define the (anti)ghost oscillator vacuum to be the state

$$|0\rangle_{gh} = |0\rangle_R^{gh} \otimes |0\rangle_L^{gh}\,, \tag{7.27}$$

such that

$$c_n|0\rangle_R^{gh} = 0\,, \quad \tilde{c}_n|0\rangle_L^{gh} = 0\,, \qquad n > 0\,, \tag{7.28}$$

and

$$b_n|0\rangle_R^{gh} = 0\,, \quad \tilde{b}_n|0\rangle_L^{gh}\,, \qquad n \geq 0\,. \tag{7.29}$$

Notice the asymmetry for $n = 0$. This is because $\{c_0, b_0\} = 1$, and we choose b_0 to be the annihilation operator.

We can act on $|0\rangle^{gh}$ with the (anti)ghost creation operators, $(c_{-n}, \tilde{c}_{-n})$ for $n \geq 0$ and $(b_{-n}, \tilde{b}_{-n})$ for $n > 0$, to get states in an (anti)ghost Fock space; The full oscillator vacuum is now the tensor product state[38]

$$|0) = |0\rangle \otimes |0\rangle_{gh}\,, \tag{7.30}$$

where $|0\rangle$ is the oscillator vacuum of the "old covariant" quantization procedure; i.e. $|0\rangle = |0\rangle_R \otimes |0\rangle_L$.

For $m \neq 0$ there is no ordering ambiguity in the operator versions of $L_m^{(gh)}$ and $\tilde{L}_m^{(gh)}$ of (7.11). For $m = 0$ we take the ordering to be as given in (7.12). We then have

$$L_m^{(gh)}|0\rangle_R^{gh} = 0\,, \qquad \tilde{L}_m^{gh}|0\rangle_L^{gh} = 0\,, \qquad m \geq 0\,. \tag{7.31}$$

We know that the classical $L_m^{(gh)}$ satisfy a Witt algebra with respect to PBs, so the same logic that we used previously for the L_m tells us that the quantum $L_m^{(gh)}$ satisfy a Virasoro algebra with some central charge. In other words,

$$\left[L_m^{(gh)}, L_n^{(gh)}\right] = (m-n)\left(L_{m+n}^{(gh)} - a\delta_{m+n}\right) + \frac{c_{gh}}{12}\left(m^3 - m\right)\delta_{m+n} \tag{7.32}$$

for some constants a and c_{gh}. A similar statement applies to $\tilde{L}_m^{(gh)}$ so we focus now on the $L_m^{(gh)}$ operators.

The constants a and $c_{\rm gh}$ can be found as follows. First we observe that

$$\begin{aligned} \|L_{-m}^{(gh)}|0\rangle^{gh}\|^2 &= {}^{gh}\langle 0|L_m^{(gh)} L_{-m}^{(gh)}|0\rangle^{gh} = {}^{gh}\langle 0|\left[L_m^{(gh)}, L_{-m}^{(gh)}\right]|0\rangle^{gh} \\ &= -2ma + \frac{c_{gh}}{12}\left(m^3 - m\right)\,, \end{aligned} \tag{7.33}$$

where the last equality follows from (7.39) and the fact that $L_0^{gh}|0\rangle^{gh} = 0$. Choosing $m = 1$ and then $m = 2$ we deduce that

$$\|L_{-1}^{(gh)}|0\rangle^{gh}\|^2 = -2a\,, \qquad \|L_{-2}^{(gh)}|0\rangle^{gh}\|^2 = -4a + \frac{c_{gh}}{2}\,. \tag{7.34}$$

[38]We will use the notation $|\ )$ to indicate a state in the space obtained by taking the tensor product of the Fock space built on the oscillator vacuum $|0\rangle$ with the Fock space built on the ghost oscillator vacuum $|0\rangle_{gh}$.

We can now compute the LHSs directly using

$$L_{-1}^{(gh)}|0\rangle = \sum_{k\in\mathbb{Z}}(k-1)b_{-1-k}c_k|0\rangle^{gh} = -\left(b_{-1}c_0 + 2b_0c_{-1}\right)|0\rangle\,, \tag{7.35}$$
$$L_{-2}^{(gh)}|0\rangle = \sum_{k\in\mathbb{Z}}(k-2)b_{-2-k}c_k|0\rangle^{gh} = -\left(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2}\right)|0\rangle\,.$$

For example[39]

$$\begin{aligned}\left|\left|L_{-1}^{(gh)}|0\rangle\right|\right|^2 &= \langle 0|\left(c_0b_1 + 2c_1b_0\right)\left(b_{-1}c_0 + 2b_0c_{-1}\right)|0\rangle \\ &= -2\langle 0|\left(c_0b_0b_1c_{-1} + b_0c_0c_1b_{-1}\right)|0\rangle \quad (\text{using } b_0^2 = c_0^2 = 0) \\ &= -2\langle 0|\left(c_0b_0\left\{b_1, c_{-1}\right\} + \left\{c_1, b_{-1}\right\}b_0c_0\right)|0\rangle \quad (\text{using } b_1|0\rangle = c_1|0\rangle = 0) \\ &= -2\langle 0|\left\{c_0, b_0\right\}|0\rangle = -2\,, \end{aligned} \tag{7.36}$$

from which we conclude that $a = 1$. Similarly,

$$\begin{aligned}\left|\left|L_{-2}^{(gh)}|0\rangle\right|\right|^2 &= \langle 0|\left(2c_0b_2 + 3c_1b_1 + 4c_2b_0\right)\left(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2}\right)|0\rangle \\ &= -8\langle 0|\left(c_0b_0b_2c_{-2} + b_0c_0c_2b_{-2}\right)|0\rangle - 9\langle 0|c_1b_{-1}b_1c_{-1}|0\rangle \\ &= -8\langle 0|\left(c_0b_0\left\{b_2, c_{-2}\right\} + b_0c_0\left\{c_2, b_{-2}\right\}\right)|0\rangle - 9\langle 0|\left\{c_1, b_{-1}\right\}\left\{b_1, c_{-1}\right\}|0\rangle \\ &= -8\langle 0|(\left\{c_0, b_0\right\}|0\rangle - 9 = -17\,, \end{aligned} \tag{7.37}$$

from which we conclude that

$$-4 + \frac{c_{gh}}{2} = -17 \quad\Rightarrow\quad c_{gh} = -26\,. \tag{7.38}$$

Now we know that

$$\left[L_m^{(gh)}, L_n^{(gh)}\right] = (m-n)\left(L_{m+n}^{(gh)} - \delta_{m+n}\right) - \frac{26}{12}\left(m^3 - m\right)\delta_{m+n}\,. \tag{7.39}$$

But we also know that

$$\left[L_m, L_n\right] = (m-n)\,L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m+n}\,. \tag{7.40}$$

Combining these results, we find that the operators $\mathcal{L}_m = L_m + L_m^{(gh)}$ have the commutation relations

$$\left[\mathcal{L}_m, \mathcal{L}_n\right] = (m-n)\left(\mathcal{L}_{m+n} - \delta_{m+n}\right) + \frac{(D-26)}{12}\left(m^3 - m\right)\,. \tag{7.41}$$

The central charge of the Virasoro algebra is now $D - 26$. Since the conformal symmetry of the conformal gauge action is really a gauge invariance "in disguise", as

[39] Here we assume that c_0 and b_0 are hermitian since the classical anticommuting variables are "real". This in spite of the fact that these operators satisfy $c_0^2 = b_0^2 = 0$. Can an operator that squares to zero be hermitian? Yes, for a particular choice of inner product on the state space.

discussed previously, we should expect to have to impose physical state constraints $[\mathcal{L}_m - \delta_m]|\text{phys}) = 0$, and this is only possible when the central charge of the Virasoro algebra spanned by the $\mathcal{L}_m$ is zero, and this requires $D = 26$. In this spacetime dimension, and only in this dimension, the FP ghosts must cancel the conformal anomaly[40].

7.1 Virasoro-Shapiro amplitude

Now we shall compute the amplitude for the scattering of N tachyons using the path integral. It is not very physical to consider the "scattering of tachyons" but this is the simplest amplitude to compute, and it illustrates a number of important features of string theory.

We work in the semi-classical approximation in which the worldsheet is assumed to be a Riemann sphere with N punctures, at which we insert a tachyon vertex operator. This approximation allows two simplifications. One is that the FP ghosts contribute only to the overall normalisation, so we now omit them. The other is that there are no modular parameters to sum over because the Riemann sphere is unique up to conformal equivalence.

Let's suppose that the ith tachyon has D-momentum p_i, and that its vertex operator is inserted at the point $z = z_i$ on the Riemann sphere. In the path-integral version of the computation, we have to insert $e^{ip\cdot X(z_i)}$ into the path integral and then integrate over z_i. For N such insertions we get the amplitude

$$A(p_1, \dots, p_N) = \int [dX]\, e^{-I_E} \prod_{i=1}^{N} \int_{RS} d^2 z_i \; e^{ip_i \cdot X_i} \,, \tag{7.42}$$

where $X_i = X(z_i, \bar{z}_i)$. This amplitude is a function of the D-momenta of all the tachyons. Amplitudes involving any of the particles in the string spectrum can be calculated in a similar way but the calculation is more complicated because the vertex operators are more complicated.

We can rewrite the tachyon scattering amplitude as

$$A(p_1, \dots, p_N) = \prod_{i=1}^{N} \int_{RS} d^2 z_i \int [dX]\, e^{-I_E + i\sum_{j=1}^{N} p_j \cdot X_j} \,. \tag{7.43}$$

Now we observe that

$$-I_E + i\sum_{j=1}^{N} p_j \cdot X_j = -T \int_{RS} d^2 z \left\{ \partial X \cdot \bar{\partial} X - \frac{i}{T}\left[\sum_{j=1}^{N} \delta^2(z - z_j) p_j\right] \cdot X \right\}. \tag{7.44}$$

[40] In fact, the physical state condition, in this Lorentz covariant formulation with FP ghosts, is the stronger condition that $\hat{Q}_{BRST}|\text{phys}\rangle = 0$, where $\hat{Q}_{BRST}$ is the operator version of the Noether charge associated to a kind of supersymmetry that interchanges the original variables with the FP ghosts. Consistency requires that $\hat{Q}^2_{BRST} = 0$, and this turns out to be true only if $D = 26$.

where the delta function is defined such that[41] $\int d^2z\,\delta^2(z-z_i)f(z) = f(z_i)$. Integrating by parts, we can replace $\partial X \cdot \bar\partial X = -X \cdot \bar\partial\partial X$ since ∂ is acting on functions defined on the Riemann sphere, which has no boundary. This gives us

$$-I_E + i\sum_{j=1}^{N} p_j \cdot X_j = T\int_{RS} d^2z \left\{ X \cdot \left[\bar\partial\partial X + \frac{i}{T}\sum_{j=1}^{N} \delta^2(z-z_j)p_j \right] \right\} . \qquad (7.45)$$

The idea now is to complete the square in X but to do this we need to invert $\bar\partial\partial = \frac{1}{4}\nabla^2$ and there is a problem with this because ∇^2 has a zero eigenvalue on the sphere. The eigenfunction is the constant function, i.e. $X(z) = X_0$, so we should write

$$\int [dX] = \int d^D X_0 \int [dX]' , \qquad (7.46)$$

where $[dX]'$ is an integral over all functions *except* to the constant function[42]. Isolating the X_0-dependence we now have

$$\begin{aligned} A(p_1,\dots,p_N) &= \left[\int d^D X_0\, e^{i(\sum_j p_j)\cdot X_0} \right] \hat A(p_1,\dots,p_N) \\ &\propto \delta\left(\sum_{j=1}^{N} p_j \right) \hat A(p_1,\dots,p_N) , \end{aligned} \qquad (7.47)$$

where the path integral for $\hat A$ excludes the integration over the constant function. The delta-function prefactor imposes conservation of the total D-momentum.

We can now invert $\bar\partial\partial$; the inverse is the 2D Green function, which we define such that[43]

$$\bar\partial\partial G(z-z_i) = \delta^2(z-z_i) \quad \Rightarrow \quad G(z) = \frac{1}{\pi}\ln|z| . \qquad (7.48)$$

Setting

$$X(z) = Y(z) - \frac{i}{2T}\sum_{i=1}^{N} G(z-z_i)p_i , \qquad (7.49)$$

we have[44] $[dX]' = [dY]'$, and [Exercise]

$$-I_E + i\sum_{j=1}^{N} p_j \cdot X_j = T\int_{RS} d^2z\; Y \cdot \bar\partial\partial Y \;+\; \frac{1}{4T}\sum_i\sum_j G(z_i-z_j)p_i\cdot p_j . \qquad (7.50)$$

[41]Notice that $d^2z = 2dxdy \;\Rightarrow\; 2\delta^2(z) = \delta(x)\delta(y)$.

[42]More precisely, orthogonal to the constant function but we pass over the specification of the norm needed to define orthogonality.

[43]The formula for G can be verified by using the formulas $\bar\partial z^{-1} = 2\pi\delta^2(z)$ and the fact that $2\delta^2(z) = \delta(x)\delta(y)$ for $z = x+iy$.

[44]A shift in the integration variable has no effect because we integrate over all values of the (non-constant) functions X.

The terms in the double sum are infinite when $i = j$, but also independent of the momenta, so these terms can be omitted; they can only affect the overall normalisation. The Gaussian $[dY]'$ path integral also contributes only to the overall normalisation. Using the specific form of the Green function, we are then left with

$$\hat{A}(p_1, \ldots, p_N) \propto \prod_{i=1}^{N} \int_{RS} d^2 z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}}, \qquad \alpha_{ij} = \frac{p_i \cdot p_j}{2\pi T}. \tag{7.51}$$

We ignored the FP ghosts in arriving at this formula on the grounds that integration over them contributes only to the overall normalisation. Let's reassess this claim, focusing on the (b, c) ghosts; the problem is that the action for (b, c) will be zero for any solution of either $\bar{\partial} c = 0$ or $\bar{\partial} b = 0$ that is defined everywhere on the RS, so the Berezin integral over it will give a zero amplitude. As it happens, there are no globally defined solutions of $\bar{\partial} b = 0$ on the RS but there are three independent globally defined solutions of $\bar{\partial} c = 0$, corresponding to the conformal Killing vector fields that generate the $Sl(2; \mathbb{C})$ group of conformal isometries of the RS. [See Q.IV.1]. Integrating over these functions gives a zero amplitude, apparently, but actually there is a compensating infinity in our result (7.51). This amplitude is $Sl(2; C)$-invariant, which means that there is an implicit integration over the group $Sl(2; C)$ resulting in a factor equal to the volume of $Sl(2; C)$, but this volume is infinite.

Let's verify the $Sl(2; \mathbb{C})$ invariance of $\hat{A}$ as given in (7.51). The $Sl(2; \mathbb{C})$ transformation of z is

$$z \to z' = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{7.52}$$

Using this one finds that

$$z_i' - z_j' = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}, \qquad d^2 z' = \frac{d^2 z}{|cz + d|^4}, \tag{7.53}$$

and hence that

$$\prod_{i=1}^{N} d^2 z_i' \prod_{j<k} |z_j' - z_k'|^{\alpha_{jk}} = \left[\prod_{i=1}^{N} d^2 z_i \prod_{j<k} |z_j - z_k|^{\alpha_{jk}}\right] \left[\prod_{i=1}^{N} |cz_i + d|^{-4 - \sum_j' \alpha_{ij}}\right], \tag{7.54}$$

where

$$\begin{aligned} \sum_j{}' \alpha_{ij} &= \sum_{j=1}^{N} \alpha_{ij} - \alpha_{ii} \qquad (i = 1, \ldots, N) \\ &= \frac{1}{2\pi T} p_i \cdot \left(\sum_{j=1}^{N} p_j\right) - \frac{p_i^2}{2\pi T} \\ &= -\frac{p_i^2}{2\pi T} \qquad \text{(by momentum conservation).} \end{aligned} \tag{7.55}$$

We see from this that the amplitude is $Sl(2;\mathbb{C})$ invariant only if

$$-4 + \frac{p_i^2}{2\pi T} = 0 \quad \Leftrightarrow \quad p_i^2 = 8\pi T\,. \tag{7.56}$$

This is the mass-shell condition for the tachyonic ground state of the string!

The $Sl(2;\mathbb{C})$ invariance of the amplitude (7.51) means that there are three too many integrals. We could fix this problem by the gauge choice

$$f_i \equiv z_i - u_i = 0 \quad i = 1,2,3. \tag{7.57}$$

Inserting $\delta^2(f_1)\delta^2(f_2)\delta^2(f_3)$ into the integrand of (7.51) removes three of the integrals, but this would not be the correct thing to do. The problem is that the $Sl(2;\mathbb{C})$ invariance of the amplitude implies that it is proportional to the volume of $Sl(2;\mathbb{C})$, which is infinite because the group is non-compact, but this factor does not appear *explicitly* in the expression (7.51) for the amplitude.

We know how to solve this problem. When we fix the positions of the first three points by insertion of delta functions, we must also include a Fadeev-Popov determinant. The infinitesimal form of the transformation (7.52) is[45]

$$\delta z = \alpha_0 + \alpha_1 z + \alpha_2 z^2\,. \tag{7.58}$$

For $f_i = z_i - u_i$ we have $\delta f_i = \delta z_i$ and hence

$$\det\left[\frac{\partial(\delta f_i)}{\partial \alpha_j}\right] = \left\|\begin{matrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{matrix}\right\| = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)\,. \tag{7.59}$$

Because the variables are complex (we are inserting three 2D delta functions), the FP determinant is the modulus squared of this, so

$$\Delta_{FP} = |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2\,. \tag{7.60}$$

Following the earlier argument for gauge fixing the particle action, the insertion of the delta functions with the FP determinant allows us to factor out the (infinite) volume Ω of $Sl(2;\mathbb{C})$; dividing by this volume we then get

$$\Omega^{-1}\hat{A}(p_1,\dots,p_N) \;\propto\; \prod_{i=1}^{N} \int d^2 z_i\, \delta^2(f_1)\delta^2(f_2)\delta^2(f_3)\Delta_{FP} \prod_{j<k} |z_j - z_k|^{\alpha_{jk}}\,. \tag{7.61}$$

This can be checked as follows. Multiply both sides by $|(u_1-u_2)(u_2-u_3)(u_3-u_1)|^{-2}$ and integrate over (u_1,u_2,u_3). On the RHS the u integrals can be done using the

[45]This shows that $(\partial, z\partial, z^2\partial)$ are the globally defined conformal Killing vector fields.

delta functions, the Δ_{FP} factor is then cancelled and we recover the expression (7.51). On the LHS the integral cancels the factor of Ω^{-1} because, formally,

$$\Omega = \int \frac{d^2u_1 d^2u_2 d^2u_3}{|(u_1 - u_2)(u_2 - u_3)^2(u_3 - u_1)|^2} . \tag{7.62}$$

This integral is infinite but the integrand is the $Sl(2;\mathbb{C})$ invariant measure on the $Sl(2;\mathbb{C})$ group manifold, parametrised by three complex coordinates on which $Sl(2;\mathbb{C})$ acts by the fractional linear transformation (7.52).

We may now do the (z_1, z_2, z_3) integrals of (7.61) to get

$$\begin{aligned} \Omega^{-1}\hat{A}(p_1, \ldots, p_N) \;\propto\; & |u_1 - u_2|^{2+\alpha_{12}}|u_2 - u_3|^{2+\alpha_{23}}|u_3 - u_1|^{2+\alpha_{13}} \\ \times \prod_{i=4}^{N} \int d^2z_i & \prod_{i=4}^{N} |u_1 - z_i|^{\alpha_{1i}}|u_2 - z_i|^{\alpha_{2i}}|u_3 - z_i|^{\alpha_{3i}} \prod_{4\leq j<k} |z_i - z_j|^{\alpha_{jk}} . \end{aligned} \tag{7.63}$$

This can be simplified enormously by the choice

$$u_3 = 1 , \qquad u_2 = 0 , \qquad u_1 \to \infty . \tag{7.64}$$

In this limit we get a factor of

$$|u_1|^{4-\alpha_{11}+\sum_i \alpha_{1i}} = 1 , \tag{7.65}$$

where the equality follows upon using both the mass-shell condition and momentum conservation. The remaining terms give the Virasoro-Shapiro amplitude

$$\hat{A}_{VS}(p_1, \ldots, p_N) \propto \prod_{i=4}^{N} \int d^2z_i \prod_{i=4}^{N} |z_i|^{\alpha_{2i}}|z_i - 1|^{\alpha_{3i}} \prod_{3<j<k} |z_j - z_k|^{\alpha_{jk}} . \tag{7.66}$$

The result for $N = 3$ is a constant, which can be interpreted as a coupling constant. For $N = 4$ we have the Virasoro amplitude

$$\hat{A}(p_1, p_2, p_3, p_4) = \int d^2z\, |z|^{\alpha_{24}}|z - 1|^{\alpha_{34}} . \tag{7.67}$$

7.1.1 The Virasoro amplitude

Consider the elastic scattering of two identical particles of mass m. In the rest frame, the incoming particles have D-momenta

$$p_1 = (E, \vec{p}) , \qquad p_2 = (E, -\vec{p}) . \tag{7.68}$$

The outgoing particles have D-momenta

$$-p_3 = (E, \vec{p}\,') , \qquad -p_4 = (E, -\vec{p}\,') . \tag{7.69}$$

Notice that $p_1 + p_2 + p_3 + p_4 = 0$, as required by D-momentum conservation. In addition, since $p_i^2 = -m^2$ for $i = 1, 2, 3, 4$, we have

$$|\vec{p}|^2 = |\vec{p}\,'|^2 = E^2 - m^2\,. \tag{7.70}$$

The scattering angle θ_s is defined by

$$\cos\theta_s = \frac{\vec{p}\cdot\vec{p}\,'}{E^2 - m^2}\,. \tag{7.71}$$

We may trade the frame-dependent variables $(E, \cos\theta_s)$ for the Lorentz scalar Mandelstam variables

$$\begin{aligned} s_M &= -\left(p_1 + p_2\right)^2 = 4E^2 \\ t_M &= -\left(p_1 + p_3\right)^2 = -2(E^2 - m^2)\left(1 - \cos\theta_s\right) \\ u_M &= -\left(p_1 + p_4\right)^2 = -2(E^2 - m^2)\left(1 + \cos\theta_s\right) \end{aligned} \tag{7.72}$$

These are not all independent because

$$s_M + t_M + u_M = 4m^2\,. \tag{7.73}$$

The variable s_M is the square of the centre of mass energy. For fixed s_M, the variable t_M determines the scattering angle.

In the context of closed string tachyon scattering, for which $m^2 = -8\pi T$, it is convenient to use the rescaled Mandelstam variables[46]

$$(s, t, u) = \frac{1}{8\pi T}(s_M, t_M, u_M)\,. \tag{7.74}$$

This gives us

$$\begin{aligned} s &= -\frac{1}{8\pi T}\left(p_1 + p_2\right)^2 = -2 - \frac{1}{2}\alpha_{12} \\ t &= -\frac{1}{8\pi T}\left(p_1 + p_3\right)^2 = -2 - \frac{1}{2}\alpha_{13} \\ u &= -\frac{1}{8\pi T}\left(p_1 + p_4\right)^2 = -2 - \frac{1}{2}\alpha_{14} \end{aligned} \tag{7.75}$$

Using this and momentum conservation yields

$$\begin{aligned} \alpha_{34} &= \alpha_{12} = -4 - 2s\,, \\ \alpha_{24} &= \alpha_{13} = -4 - 2t\,, \\ \alpha_{23} &= \alpha_{14} = -4 - 2u\,, \end{aligned} \tag{7.76}$$

and

$$s + t + u = -4\,. \tag{7.77}$$

[46]Equivalently, we can choose units for which $8\pi T = 1$.

We can also write s and t as

$$s = \frac{E^2}{2\pi T}\,, \quad t = -2\left(1+\frac{s}{4}\right)(1-\cos\theta_s)\;. \tag{7.78}$$

Using (7.76) we can rewrite the Virasoro amplitude of (7.67) as

$$\hat{A} = \int d^2z\,|z|^{2\alpha}|z-1|^{2\beta}\,, \qquad \alpha = -2-t\,, \qquad \beta = -2-s\,. \tag{7.79}$$

Next, using the identity

$$\frac{1}{\pi}\int d^2z\,|z|^{2\alpha}|z-1|^{2\beta} \equiv \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-1)}{\Gamma(\alpha+\beta+2)\Gamma(-\alpha)\Gamma(-\beta)}\,, \tag{7.80}$$

we arrive at the *Virasoro amplitude* for the scattering of two closed string tachyons:

$$A(s,t) \;\propto\; \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(-1-u)}{\Gamma(u+2)\Gamma(s+2)\Gamma(t+2)} \qquad (u=-4-s-t). \tag{7.81}$$

Here, $\Gamma(z)$ is Euler's Gamma function, with the properties

$$z\Gamma(z) = \Gamma(z+1)\,, \qquad \Gamma(n) = (n-1)! \quad \text{for } n\in\mathbb{Z}^+\,. \tag{7.82}$$

It also has an analytic continuation to a meromorphic function on the complex z-plane with no zeros and simple poles at $z=-n$ for $n\geq 0$, with residues $(-1)^n/n!$.

For fixed t the Virasoro amplitude A becomes a function of s with simple poles at

$$s = -1, 0, 1, 2, \ldots \tag{7.83}$$

These poles correspond to resonances, i.e. to other particles in the spectrum (stable particles, in fact, because the poles are on the real axis in the complex s-plane). The position of the pole on the real axis gives the mass-squared of the particle in units of $8\pi T$. The pole at $s=-1$ is the tachyon itself; in other words, the tachyon can be considered as a bound state of two other tachyons. The pole at $s=0$ implies the existence of a massless particle, or particles. The residue of this pole is

$$-\frac{\Gamma(-1-t)\Gamma(3+t)}{\Gamma(-2-t)\Gamma(t+2)} = t^2-4\,. \tag{7.84}$$

This is a quadratic function of t and hence of $\cos\theta_s$, which implies that there must be a massless particle of spin 2 (but none of higher spin). The residue of the pole at $s=n$ is a polynomial in t of order $2(n+1)$, so that $2(n+1)$ is the maximum spin of particles in the spectrum with mass-squared $n\times(8\pi T)$. In a plot of J_{max} against s, such particles appear at integer values of J_{max} on a straight line with slope $\alpha'/2$ and intercept 2 (value of J_{max} at $s=0$). This is the leading Regge trajectory. All other particles in the spectrum appear on parallel "daughter" trajectories in the

(J, s) plane (e.g. the massless spin-zero particle in the spectrum is the first one on the trajectory with zero intercept. In fact, the entire string spectrum can be found in this way!

If we had computed the amplitude for scattering gravitons instead of tachyons then we would have found the tachyon as a resonance. This shows that it is not consistent to simply omit the tachyon from the spectrum.

Another feature of the Virasoro amplitude is its $s \leftrightarrow t$ symmetry. Poles in A as a function of s at fixed t therefore reappear as poles in A as a function of t at fixed s. These correspond to the exchange of a particle. In particular, a massless spin-2 particle is exchanged, and arguments imply that such a particle must be the quantum associated to the gravitational force, so a theory of interacting closed strings is a theory of quantum gravity.

The Virasoro amplitude for closed strings was preceded by the Veneziano amplitude for open strings[47]

$$A(s,t) = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)}\,, \tag{7.85}$$

where now

$$s = -\frac{1}{2\pi T}\left(p_1 + p_2\right)^2\,, \qquad t = -\frac{1}{2\pi T}\left(p_1 + p_3\right)^2\,. \tag{7.86}$$

This amplitude also has poles at $s = -1, 0, 1, 2, \ldots$, but the maximum spin for $s = n$ is now $J_{\max} = n + 1$, and the leading Regge trajectory has slope α' and intercept 1 (this is the constant a that equals the zero point energy in the light-cone gauge quantization of the open string).

7.2 String theory at 1-loop: taming UV divergences

We will now take a brief look at what happens at one string-loop. In this case amplitudes are found from the path integral by considering vertex operator insertions at points on flat torus, which we can define by a doubly periodic identification in the complex z-plane. Without loss of generality (because we are free to rescale z and change its sign) we can choose

$$z \sim z + 1\,, \qquad z \sim z + \tau \qquad \mathcal{I}m\,\tau > 0\,. \tag{7.87}$$

Any further analytic transformation that preserves the first of these identifications will leave τ unchanged[48]. However, not all values of τ in the upper half plane define

[47] Veneziano did not compute it from string theory (which did not then exist) but just proposed it on the basis of its properties.

[48] By a *non-analytic* coordinate transformation we can put the equivalence relations into the form $z \sim z + n + im$ for integers (n, m), but the metric is then conformal to $|dz + \mu d\bar{z}|^2$, and μ now parametrises the conformally inequivalent metrics on the torus with standard identifications.

inequivalent tori because the identifications are obviously unchanged by the translation

$$T: \quad \tau \to \tau + 1\,. \tag{7.88}$$

They are also unchanged by the inversion

$$S: \quad \tau \to -\frac{1}{\tau}\,. \tag{7.89}$$

To see this multiply the equivalence relations of (7.87) by $-1/\tau$ and then rescale $z \to -\tau z$ to get[49]

$$z \sim z - \frac{1}{\tau}\,, \qquad z \sim z - 1\,. \tag{7.90}$$

Composition of the S and T maps generates elements of the group $PSl(2;\mathbb{Z})$ which acts on τ by a fractional linear transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2;\mathbb{Z}) \tag{7.91}$$

Two matrices of $Sl(2;\mathbb{Z})$ that differ by a sign have the same action on τ, so $PSl(2;\mathbb{Z}) \cong Sl(2;\mathbb{Z})/\{\pm 1\}$. Inequivalent tori are parametrised by complex numbers τ lying in a fundamental domain of $PSl(2;\mathbb{Z})$ in the complex τ-plane.

In the path integral representation of the one string-loop amplitudes, we have to sum over all inequivalent tori (Euclidean worldsheets of genus 1). This leads to an integral over τ:

$$A \propto \int_F d^2\tau \ldots \tag{7.92}$$

where F is any fundamental domain of $PSl(2;\mathbb{Z})$; it is convenient to choose it to be the one in which we may take $\mathcal{I}m\,\tau \to \infty$ on the imaginary axis. In this limit the torus becomes long and thin and it starts to look like a one-loop Feynman diagram (with vertices at various points if we had vertex operators at points on the torus). In this *infra-red* limit we can interpret $\mathcal{I}m\,\tau$ as the modular parameter s of a particle worldline. In the particle case we would have

$$A \propto \int_0^\infty ds \ldots \tag{7.93}$$

and we typically get divergent results from the part of the integral where $s \to 0$. These are *ultra-violet* divergences. They are absent in string theory because the domain of integration F does not include the origin of the τ-plane.

Moral There are no UV divergences in string theory. We have discussed only one string-loop but the result is general. This can be understood in other ways. For

[49]The relation $z \sim z + 1$ is equivalent to $z \sim z - 1$.

example, the force due to exchange of the massless states of the closed string includes the force of gravity because it includes a spin-2 field. Whereas this leads to unacceptable UV behaviour in GR, the UV divergences are cut off in string theory at the string length scale

$$\ell_s \sim \sqrt{\alpha'} \tag{7.94}$$

because at this scale the exchange of the massive string states becomes as important as the graviton exchange.

7.3 The dilaton and the string-loop expansion

The way that the dilaton field $\Phi(X)$ couples to the string in its Polyakov formulation is through the scalar curvature of the independent worldsheet metric $\gamma_{\mu\nu}$. In two dimensions the Riemann curvature tensor is entirely determined by its double trace, the Ricci scalar $R(\gamma)$, but this allows us to add to the Euclidean NG action the term

$$I_\Phi = \frac{1}{4\pi}\int d^2z\, \Phi(X)\sqrt{\gamma}\, R(\gamma)\,. \tag{7.95}$$

Here are some features of this term:

- If $\Phi = \phi_0$, a constant then

$$I_\Phi = \phi_0\, \chi\,, \qquad \chi = \frac{1}{4\pi}\int d^2z\, \sqrt{\gamma}\, R(\gamma)\,. \tag{7.96}$$

  The integral χ is a topological invariant of the worldsheet, called the Euler number. For a compact orientable Riemann surface without boundary (which we'll abbreviate to "Riemann surface" in what follows) the Euler number is related to the genus g (the number of doughnut-type "holes") by the formula

$$\chi = 2\,(1-g)\ . \tag{7.97}$$

- In conformal gauge, we can write the line element for the (Euclidean signature) metric γ as $ds^2(\gamma) = 2e^\sigma dz d\bar{z}$, i.e. a conformal factor e^σ (an arbitrary function of z and $\bar{z}$) times the Euclidean metric. We then find that $\sqrt{\gamma}\, R(\gamma) = 2\nabla^2\sigma$ and hence, after integrating by parts,

$$I_\Phi = \frac{1}{2\pi}\int d^2z\ \sigma\, \partial X^m \bar{\partial} X^n \partial_n\partial_m \Phi\,. \tag{7.98}$$

  This dependence on σ shows that I_Φ is *not* conformal invariant, unless Φ is constant. This is allowed because I_Φ, being independent of the string tension T, comes with an additional factor of α' relative to the NG action (we have to consider the lack of conformal invariance of I_Φ at the same time that we consider possible conformal anomalies).

These properties suggest that we write

$$\Phi = \phi_0 + \phi(X)\,, \tag{7.99}$$

where $\phi(X)$ is zero in the vacuum; i.e. the constant ϕ_0 is the "vacuum expectation value" of $\Phi(X)$. Then there will appear a factor in the path integral of the form

$$e^{-\phi_0\chi} = \left(g_s^2\right)^{g-1}\,, \qquad g_s \equiv e^{\phi_0}\,. \tag{7.100}$$

For $g = 0$ this tells us that the Riemann sphere contribution to scattering amplitudes is weighted by a factor of $1/g_s^2$. If we use these amplitudes to construct an effective field theory action S from which we could read off the amplitudes directly (by looking at the various interaction terms) then this action will come with a factor of $1/g_s^2$ (we can then absorb all other dimensionless factors into a redefinition of g_s, i.e. of ϕ_0). If we focus on the amplitudes for scattering of massless particles then we find that the effective action for the massless fields $\{h_{mn}, b_{mn}, \phi\}$ is

$$S = \frac{1}{g_s^2 \ell_s^{(D-2)}} \int d^D x\, \sqrt{-\det g}\, e^{-2\phi} \left[2\Lambda + R(g) - \frac{1}{3}H^2 + 4\,(\partial\phi)^2 + \mathcal{O}(\alpha')\right]\,, \tag{7.101}$$

where g is the spacetime metric ($g_{mn} = \eta_{mn} + h_{mn}$), $H = db$ is the 3-form field strength for the 2-form potential b, and

$$\Lambda = \frac{(D-26)}{3\alpha'}\,, \tag{7.102}$$

although the perturbative expansion in powers of α' is justifiable only if $\Lambda = 0$, i.e. if $D = 26$. Some other features of this space-time action are

- The exact result for S will involve a series of all order in α' since the coupling of the background fields to the string introduces interactions into the 2D QFT defined by the string worldsheet action I.

- Coupling the string only to background fields associated to the massless particles yields a renormalizable 2D QFT. This is equivalent to the statement that it is consistent to truncate the full equations of motion of the background fields associated to all particles in the string spectrum by setting to zero the background fields of the massive particles. In other words, the massless fields do not act as sources for the massive fields. In contrast any massive field will act as a source for more massive fields, implying that they must all be included once one is included. This statement is equivalent to the non-renormalizability of the 2D QFT with higher-derivative interactions.

- The integrand involves a factor of $e^{-2\phi}$. This is because the action must be such that $\phi_0 \equiv \ln g_s$ and $\phi(X)$ must appear only through the combination $\Phi = \phi_0 + \phi(X)$.

In the effective spacetime action, g_s^2 plays the role of $\hbar$. This suggests that we have been considering so far only the leading term in a semi-classical expansion. This is true because we have still to consider Riemann surfaces with genus $g > 0$, and a string amplitude at genus g is weighted, according to to (7.100), by a factor of $(g_s^2)^{g-1}$, i.e. a factor of $(g_s^2)^g$ relative to the zero-loop amplitude. This confirms that the string-loop expansion is a semi-classical expansion in powers of g_s^2. Taking into account all string loops gives us a *double expansion* of the effective field theory[50]

$$S = \frac{1}{g_s^2 \ell_s^{(D-2)}} \int d^D x \sqrt{-\det g} \sum_{g=0}^{\infty} g_s^{2g} \, e^{2(g-1)\phi} \, L_g \, , \qquad L_g = \sum_{l=0}^{\infty} \ell_s^{2l} L_g^{(l)} \, . \tag{7.103}$$

In effect, the expansion in powers of ℓ_s comes from first-quantisation of the string, and the expansion in powers of g_s comes from second-quantisation. How can we quantise twice? Is there not a single $\hbar$? The situation is actually not so different from that of the point particle. When we first-quantise we get a Klein-Gordon equation but with a mass $m/\hbar$; we then relabel this as m so that it becomes the mass parameter of the classical field equation, and then we quantise again. For the string, first quantization would have led to $\alpha'\hbar$ as the expansion parameter if we had not set $\hbar = 1$; if we relabel this as α' then $\hbar$ appears only in the combination $g_s^2\hbar$.

To lowest order in α' we have what looks like GR coupled to an antisymmetric tensor and a scalar. The D-dimensional Newton gravitational constant G_D is

$$G_D \propto g_s^2 \ell_s^{(D-2)} \, . \tag{7.104}$$

Consistency of the string-loop expansion (in powers of g_s^2) relies on this formula. Particles in the string spectrum have masses proportional to $1/\ell_s$, independent of g_s, so their contribution to the gravitational potential in D dimensions is proportional to g_s^2, and hence zero at zero string coupling. This means that the strings of free ($g_s = 0$) string theory do not back-react on the space-time metric; the metric is changed by the presence of strings only within perturbation theory. If this had not been the case it would not have been consistent to start (as we did) by considering a string in Minkowski spacetime.

Why is g_s called the string coupling constant? Consider a g string-loop vacuum to vacuum diagram with the appearance of a chain of g tori connected by long "throats", and think of it as a "fattened" Feyman diagram in which a chain of g loops connected by lines; where each line meets a loop we have a 3-point vertex. As

[50] There is a lot of freedom in the form of the Lagrangians $L_g^{(l)}$ beyond leading order. Recall that the construction of S involves a prior determination of scattering amplitudes of the level-1 fields, which we then arrange to replicate from a local spacetime Lagrangian. Since the amplitudes are all "on-shell" they actually determine only the field equations for the background fields, and then only up to field redefinitions. Even with all this freedom it is not obvious why it should be possible to replicate the string theory scattering amplitudes in this way, although this has been checked to low orders in the expansion and there are general arguments that purport to prove it.

there are $(g-1)$ lines that link the loops, and each of the two ends of each line ends at a vertex, we have a total of $2(g-1)$ vertices. If we associate a coupling constant to each vertex, call it g_s, we see that this particular diagram comes with a factor of $(g_s^2)^{g-1}$, which agrees with our earlier result.

Is there a g-loop Riemann surface with the appearance just postulated. Yes, there is. For $g > 0$ there is no longer a unique flat metric, as we have already seen for $g = 1$. For $g \geq 2$ there is a $3(g-1)$-parameter family of conformally inequivalent flat metrics; these parameters are called "moduli" (see Q.IV.2). This number can be understood intuitively from the "chain of tori" diagram if we associate one parameter with each propagator. For g loops we have, in addition to the $(g-1)$ links, $(g-2)$ "interior" loops with 2-propagators each, and two "end of chain" loops with one propagator each. The total number of propagators is therefore

$$(g-1) + 2(g-2) + 2 = 3(g-1)\,. \tag{7.105}$$

This is also, and not coincidentally, the dimension of the space of quadratic differentials on a Riemann surface of genus $g \geq 2$ (see Q.IV.2).

8. Interlude: The spinning particle

A point particle with non-zero spin can be accommodated by including additional anticommuting coordinates. Consider the non-relativistic particle action

$$I = \int dt \left\{ \dot{\vec{x}} \cdot \vec{p} + \frac{i}{2} \vec{\psi} \cdot \dot{\vec{\psi}} - \frac{|\vec{p}|^2}{2m} \right\}\,, \tag{8.1}$$

where the 3-vector variables $\vec{\psi}$ are anti-commuting and "real". The action is invariant under space rotations; the infinitesimal transformations are

$$\delta_\omega \vec{x} = \vec{\omega} \times \vec{x}\,, \qquad \delta_\omega \vec{p} = \vec{\omega} \times \vec{p}\,, \qquad \delta_\omega \vec{\psi} = \vec{\omega} \times \vec{\psi}\,. \tag{8.2}$$

The Noether charge is the angular momentum $\vec{J} = \vec{L} + \vec{S}$, where[51]

$$\vec{L} = \vec{x} \times \vec{p}\,, \qquad \vec{S} = -i \vec{\psi} \times \vec{\psi}\,. \tag{8.3}$$

The Poisson brackets of the anticommuting variables are

$$\left\{ \psi^a, \psi^b \right\}_{PB} = -i \delta^{ab}\,. \tag{8.4}$$

[51] $\vec{\psi} \times \vec{\psi}$ is non-zero because $\vec{\psi}$ is anticommuting.

- **Check**: For consistency with our earlier conventions for FP ghosts, the Lagrangian $L = i\psi^*\dot{\psi}$ for complex anticommuting ψ gives the Poisson bracket relations $\{\psi^*, \psi\}_{PB} = -i$. Now set $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$ to get

$$\psi_1 = \frac{1}{\sqrt{2}}\left(\psi + \bar{\psi}\right) , \quad \psi_2 = -\frac{i}{\sqrt{2}}\left(\psi - \bar{\psi}\right) . \tag{8.5}$$

  This gives us the following Lagrangian and corresponding PB relations

$$L = \frac{i}{2}\sum_{i=1}^{2}\psi_i\dot{\psi}_i , \qquad \to \quad \{\psi_i, \psi_j\}_{PB} = -i\delta_{ij} . \tag{8.6}$$

  This explains why the factor of 1/2 appears in the $\vec{\psi}\cdot\dot{\vec{\psi}}$ term in (8.1).

Upon quantization we get canonical anticommutation relations that can be realised by Pauli matrices:

$$\left\{\hat{\psi}^a, \hat{\psi}^b\right\} = \delta^{ab} \quad \Rightarrow \quad \hat{\psi}^a = \sqrt{\frac{\hbar}{2}}\,\sigma^a . \tag{8.7}$$

The spin vector operator is then

$$\hat{S}^a = -\frac{i}{2}\varepsilon^{abc}\hat{\psi}^b\hat{\psi}^c = -i\frac{\hbar}{4}\varepsilon_{abc}\sigma_b\sigma_c = \frac{\hbar}{2}\sigma_a \tag{8.8}$$

and hence

$$\hbar^{-2}\left|\hat{S}^a\right|^2 = \frac{3}{4} \qquad \left(= 2s + 1 \text{ for } s = \frac{1}{2}\right) \tag{8.9}$$

which shows that the action (8.1) describes a spin-$\frac{1}{2}$ particle.

8.1 Relativistic spin-1/2

The relativistic generalization, in D spacetime dimensions, is simplest for a massless particle[52]. The spin can be accommodated by including an anticommuting D-vector coordinate ψ^m. The action is

$$I = \int dt \left\{\dot{x}^m p_m + \frac{i}{2}\eta_{mn}\,\psi^m\dot{\psi}^n - \frac{1}{2}e\,p^2 - i\chi\,\psi\cdot p\right\} . \tag{8.10}$$

We read off from this action the Poisson brackets

$$\{x^m, p_n\}_{PB} = \delta^m_n , \qquad \{\psi^m, \psi^n\}_{PB} = -i\eta^{mn} . \tag{8.11}$$

[52]The massive case can be found by starting with the massless particle action (8.10) in $(D+1)$ dimensions and reducing to D dimensions with $P_D = m$.

Using this we find that

$$\{\psi \cdot p, \psi \cdot p\}_{PB} = -ip^2 , \qquad \{p^2, \psi \cdot p\}_{PB} = 0 , \tag{8.12}$$

which shows that the constraints are first-class and hence that they both generate gauge invariances. These are

$$\delta x^m = \alpha p^m + i\epsilon\psi^m , \qquad \delta\psi^m = -\epsilon p^m , \tag{8.13}$$

where ϵ is an infinitesimal anticommuting parameter. The action is invariant if the Lagrange multipliers transform as

$$\delta e = \dot{\alpha} , \qquad \delta\chi = \dot{\epsilon} . \tag{8.14}$$

To be precise, one finds that

$$\delta I = \frac{1}{2}\left[\alpha\, p^2 + i\epsilon\, \psi \cdot p\right]_{t_A}^{t_B} , \tag{8.15}$$

which is zero if α and ϵ are zero at $t = t_A$ and $t = t_B$.

The simplest way to see that this action does indeed describe a massless spin-$\frac{1}{2}$ relativistic particle is to quantize *à la* Dirac. Recall that we first quantize as if there were no constraints, which gives us the (anti)commutation relations

$$[x^m, p_n] = i\delta_m^n , \qquad \{\psi^m, \psi^n\} = \eta^{mn} . \tag{8.16}$$

These can be realised on a spinor wave function $\Psi(x)$ by

$$p_m = -i\partial_m , \qquad \psi^m = \frac{1}{\sqrt{2}}\Gamma^m , \tag{8.17}$$

where Γ^m are the $2^{[D]/2} \times 2^{[D]/2}$ spacetime Dirac matrices. Next, we impose the constraints as physical state conditions. We have only to impose the condition resulting from the constraint $\psi \cdot p = 0$ because this implies the mass-shell condition; the result is the massless Dirac equation

$$\Gamma^m \partial_m \Psi(x) = 0 . \tag{8.18}$$

If $D = 5, 6, 7$ mod 8 then Ψ must be complex, i.e. a "Dirac spinor", but if $D = 2, 3, 4, 8, 9$ mod 8 then we may choose it to be real[53]. For any D we can define

$$\Gamma_{D+1} = \Gamma^0\Gamma^1 \cdots \Gamma^D , \tag{8.19}$$

[53] This can be interpreted as a "gauging" of the time-reversal invariance of the particle action. In general, time reversal is represented in QM by an anti-unitary operator K, with the property that $K^2 = \pm 1$. When $K^2 = 1$, as is the case for the spinning particle when $D = 2, 3, 4, 8, 9$ mod 8, we can impose the condition $K\Psi = \pm\Psi$ as a new physical state condition. This is a reality condition because K involves taking the complex conjugate. When $K^2 = -1$, as is the case for the spinning particle when $D = 5, 6, 7$ mod 8, it is not consistent to impose a reality condition so Ψ is necessarily complex; this is a simple illustration of Kramer's degeneracy in QM.

but this matrix is ± 1 for odd D; for even D it anticommutes with each of the D Dirac matrices and its square is either $+1$ or -1. If $\Gamma_{D+1}^2 = 1$ we can define a chiral (anti-chiral) spinor as an eigenspinor of Γ_{D+1}. For $D = 2$ mod 8, but not otherwise, a real spinor can also be chiral. This fact is of importance for superstring theory.

9. Spinning strings

We can find a 'spinning string" by following the example of the spinning particle. We need to take "square roots" of the constraint functions $\mathcal{H}_\pm$, and to do this we need to introduce anticommuting worldsheet fields $\psi_\pm$. For the closed string, this gives us an action of the form

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m + \frac{i}{2}\left(\psi_+\dot{\psi}_+ + \psi_-\dot{\psi}_-\right) \right. \\ \left. -\lambda^-\mathcal{H}_- - \lambda^+\mathcal{H}_+ - i\chi^-\mathcal{Q}_- - i\chi^+\mathcal{Q}_+ \right\} . \qquad (9.1)$$

where $\chi^\pm$ are Lagrange multipliers. The constraint functions are

$$\mathcal{H}_\pm = \frac{1}{4T}\left(P \pm TX'\right)^2 \pm \frac{i}{2}\psi_\pm \cdot \psi'_\pm \, , \qquad \mathcal{Q}_\pm = \frac{1}{2\sqrt{T}}\left(P \pm TX'\right) \cdot \psi_\pm \, . \qquad (9.2)$$

This action reduces to the Hamiltonian form of the NG action when all anti-commuting variables are omitted. This means that all classical solutions of the NG string are solutions of the spinning string, but the new anti-commuting variables make a significant difference to the quantum theory.

We can read off the non-zero PB relations from the above action. In particular,

$$\left\{\psi_\pm^m(\sigma), \psi_\pm^n(\sigma')\right\}_{PB} = -i\eta^{mn}\delta(\sigma - \sigma') \, . \qquad (9.3)$$

A calculation using the PBs of the canonical variables shows that the non-zero PBs of the constraint functions are

$$\begin{aligned} \{\mathcal{Q}_\pm(\sigma), \mathcal{Q}_\pm(\sigma')\}_{PB} &= -i\,\mathcal{H}_\pm(\sigma)\,\delta(\sigma - \sigma') \\ \{\mathcal{Q}_\pm(\sigma), \mathcal{H}_\pm(\sigma')\}_{PB} &= \pm\left[\frac{1}{2}Q_\pm(\sigma) + Q_\pm(\sigma')\right]\delta'(\sigma - \sigma') \\ \{\mathcal{H}_\pm(\sigma), \mathcal{H}_\pm(\sigma')\}_{PB} &= \pm\left[\mathcal{H}_\pm(\sigma) + \mathcal{H}_\pm(\sigma')\right]\delta'(\sigma - \sigma') \, . \end{aligned} \qquad (9.4)$$

Let's verify the first of these:

- First, you should verify that

$$\left\{\left(P \pm TX'\right)^m(\sigma), \left(P \pm TX'\right)^n(\sigma')\right\}_{PB} = \pm 2T\eta^{mn}\delta'(\sigma - \sigma') \, . \qquad (9.5)$$

Using this and the canonical PB relation (9.3), we find that

$$
\begin{aligned}
\{\mathcal{Q}_\pm(\sigma), \mathcal{Q}_\pm(\sigma')\}_{PB} = -i\Big[& \frac{1}{4T}\left(P \pm TX'\right)(\sigma)\cdot\left(P \pm TX'\right)(\sigma')\delta(\sigma-\sigma') \\
& \pm\frac{i}{2}\psi_\pm(\sigma)\cdot\psi_\pm(\sigma')\delta'(\sigma-\sigma')\Big]
\end{aligned} \tag{9.6}
$$

Now we use the fact that $\psi_\pm^2 \equiv 0$ (because of the anticommutativity of $\psi_\pm$) to deduce that

$$
\begin{aligned}
0 &= \partial_{\sigma'}\left[\psi_\pm(\sigma)\cdot\psi_\pm(\sigma)\delta(\sigma-\sigma')\right] = \partial_{\sigma'}\left[\psi_\pm(\sigma)\cdot\psi_\pm(\sigma')\delta(\sigma-\sigma')\right] \\
&= \psi_\pm(\sigma)\cdot\psi'_\pm(\sigma')\delta(\sigma-\sigma') - \psi_\pm(\sigma)\cdot\psi_\pm(\sigma')\delta'(\sigma-\sigma'),
\end{aligned} \tag{9.7}
$$

and hence that

$$
\pm\frac{i}{2}\psi_\pm(\sigma)\cdot\psi_\pm(\sigma')\delta'(\sigma-\sigma') = \psi_\pm(\sigma)\cdot\psi'_\pm(\sigma')\delta(\sigma-\sigma'). \tag{9.8}
$$

Using this in (9.6) we deduce that $\{\mathcal{Q}_\pm(\sigma), \mathcal{Q}_\pm(\sigma')\}_{PB} = -i\,\mathcal{H}_\pm(\sigma)\,\delta(\sigma-\sigma')$.

From (9.4) we see that the constraints are all first-class, so they generate gauge invariances of the canonical variables via their PBs with $\xi^\pm\mathcal{H}_\pm + i\epsilon^\pm\mathcal{Q}_\pm$ for parameters $\xi^\pm(t,\sigma)$ and anticommuting parameters $\epsilon^\pm(t,\sigma)$. The gauge transformations are

$$
\begin{aligned}
\delta X &= \frac{1}{2T}\xi^-\left(P - TX'\right) + \frac{1}{2T}\xi^+\left(P + TX'\right) + \frac{i}{2\sqrt{T}}\left(\epsilon^+\psi_+ + \epsilon^-\psi_-\right) \\
\delta P &= -\frac{1}{2}\Big[\xi^-\left(P - TX'\right)\Big]' + \frac{1}{2}\left[\xi^+\left(P + TX'\right)\right]' + \frac{i\sqrt{T}}{2}\left(\epsilon^+\psi_+ - \epsilon^-\psi_-\right)' \\
\delta\psi_\pm &= \pm\frac{1}{2}\xi^\pm\psi'_\pm \pm \frac{1}{2}\left(\xi^\pm\psi_\pm\right)' - \frac{1}{2\sqrt{T}}\left(P \pm TX'\right)\epsilon^\pm.
\end{aligned} \tag{9.9}
$$

One then finds that the action is invariant provided the Lagrange multipliers are assigned the gauge transformations

$$
\begin{aligned}
&\delta_\xi\lambda^\pm = \dot\xi^\pm \mp \lambda^\pm\left(\xi^\pm\right)' \pm \xi^\pm\left(\lambda^\pm\right)', \qquad \delta_\xi\chi^\pm = \mp\frac{1}{2}\left(\xi^\pm\right)'\chi^\pm \pm \xi^\pm\left(\chi^\pm\right)', \\
&\delta_\epsilon\lambda^\pm = \frac{i}{\sqrt{T}}\chi^\pm\epsilon^\pm, \qquad \delta_\epsilon\chi^\pm = \dot\epsilon^\pm \mp \lambda^\pm\left(\epsilon^\pm\right)' \pm \frac{1}{2}\left(\lambda^\pm\right)'\epsilon^\pm.
\end{aligned} \tag{9.10}
$$

9.1 Conformal gauge and superconformal symmetry

The conformal gauge for the spinning string is

$$
\lambda^\pm = 1, \qquad \chi^\pm = 0. \tag{9.11}
$$

This leaves residual gauge invariances with parameters restricted by

$$
\xi^\pm = \xi^\pm(\sigma^\pm), \qquad \epsilon^\pm = \epsilon^\pm(\sigma^\pm). \tag{9.12}
$$

Using the conformal gauge conditions in (9.1) yields

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m + i\left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-\right) - \frac{1}{2T}\left[P^2 + (TX')^2\right] \right\} . \quad (9.13)$$

After elimination of P, by its equation of motion $P = T\dot{X}$, this becomes

$$I = 2T \int dt \oint d\sigma \left\{ \partial_+ X \cdot \partial_- X + \frac{i}{2T}\left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-\right) \right\} . \quad (9.14)$$

The residual gauge invariance is a symmetry of this action, with transformations[54]

$$\begin{aligned} \delta X &= \left(\xi^- \partial_- + \xi^+ \partial_+\right) X + \frac{i}{2\sqrt{T}}\left(\epsilon^- \psi_- + \epsilon^+ \psi_+\right) , \\ \delta \psi_\pm &= \xi^\pm \partial_\pm \psi_\pm + \frac{1}{2}\left(\partial_\pm \xi^\pm\right) \psi_\pm - \sqrt{T} \partial_\pm X \, \epsilon^\pm . \end{aligned} \quad (9.15)$$

The Noether charges are precisely $\mathcal{H}_\pm$ and $\mathcal{Q}_\pm$.

- Let's check invariance of the integral of $\psi_+ \cdot \partial_- \psi_+$ with respect to the ξ-transformation of ψ_+. Since $\delta(\psi_+ \cdot \partial_- \psi_+) = 2\delta_\xi \psi_+ \cdot \partial_- \psi_+ + \partial_-(\cdots)$ and we can *ignore total derivatives*, we have

$$\begin{aligned} \delta_\xi \left(\frac{1}{2}\psi_+ \cdot \partial_- \psi_+\right) &= \left(\xi^+ \partial_+ \psi_+\right) \cdot \partial_- \psi_+ + \frac{1}{2}\left(\partial_+ \xi^+\right) \psi_+ \cdot \partial_- \psi_+ \\ &= -\frac{1}{2}\left(\partial_+ \xi^+\right) \psi_+ \cdot \partial_- \psi_+ - \xi^+ \psi_+ \cdot \partial_- \partial_+ \psi_+ \\ &= -\frac{1}{2}\partial_+\left(\xi^+ \psi_+\right) \cdot \partial_- \psi_+ + \frac{1}{2}\xi^+ \partial_- \psi_+ \cdot \partial_+ \psi_+ \\ &= 0 , \end{aligned} \quad (9.16)$$

where the penultimate line comes from integrating by parts and using $\partial_- \xi^+ = 0$ to deduce (ignoring total derivatives) that

$$\begin{aligned} -\xi^+ \psi_+ \cdot \partial_- \partial_+ \psi_+ &= \left(\frac{1}{2} + \frac{1}{2}\right) \xi^+ \partial_- \psi_+ \cdot \partial_+ \psi_+ \\ &= -\frac{1}{2}\xi^+ \partial_+ \psi_+ \cdot \partial_- \psi_+ + \frac{1}{2}\xi^+ \partial_- \psi_+ \cdot \partial_+ \psi_+ , \end{aligned} \quad (9.17)$$

and the last line comes from the similar deduction (again using $\partial_- \xi^+ = 0$ and ignoring total derivatives) that

$$-\frac{1}{2}\partial_+\left(\xi^+ \psi_+\right) \cdot \partial_- \psi_+ = -\frac{1}{2}\xi^+ \partial_- \psi_+ \cdot \partial_+ \psi_+ . \quad (9.18)$$

[54] To check the agreement with (9.9) you must use the fact that the conformal gauge action has a trivial gauge invariance with transformation $\delta\psi_\pm = f^\pm \partial_\mp \psi_\pm$ for any functions $f^\pm$.

Each component of the Lorentz D-vector $\psi_\pm$ is also a worldsheet lightcone component of a real 2D spinor. To demonstrate this, we will work backwards from the 2D Dirac action:

- **2D Dirac action**. In 2D the Dirac matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ can be chosen to be real. For example,

$$\gamma^0 = i\sigma_2\,, \quad \gamma^1 = \sigma_1 \qquad (\Rightarrow \; \gamma^0\gamma^1 = \sigma_3). \tag{9.19}$$

  This means that that it is consistent to suppose that a 2D Dirac spinor Ψ is real[55]. The Dirac Lagrangian for a real *anticommuting* spinor Ψ is

$$L_{Dirac} = \frac{i}{2}\Psi^T\gamma_0\gamma^\mu\partial_\mu\Psi\,. \tag{9.20}$$

  Notice that the matrices $\gamma^0\gamma_\mu$ are symmetric; the Lagrangian would be a total derivative if Ψ were a commuting spinor! We include the factor of i because this is needed for "reality" if we adopt the usual convention for taking the complex conjugate of products of anticommuting variables.

  We may choose a light-cone basis for the Dirac matrices

$$\gamma^\pm = \gamma^0 \pm \gamma^1 \quad \Rightarrow \quad \gamma^\mu\partial_\mu = \gamma^+\partial_+ + \gamma^-\partial_-\,. \tag{9.21}$$

  Noticing that

$$\gamma_0\gamma^\pm = (1 \mp \sigma_3) \quad \Rightarrow \quad \frac{1}{2}\gamma_0\left(\gamma^+\partial_+ + \gamma^-\partial_-\right) = \begin{pmatrix} \partial_- & 0 \\ 0 & \partial_+ \end{pmatrix}, \tag{9.22}$$

  we have

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad \Rightarrow \quad L_{Dirac} = i\left(\Psi_+\partial_-\Psi_+ + \Psi_-\partial_+\Psi_-\right)\,. \tag{9.23}$$

The action for the closed spinning string in conformal gauge is therefore the action for a free massless 2D field theory for D real scalar fields and D real spinor fields. This action has a $(1,1)$ superconformal symmetry generated by the Fourier modes of the Noether charges $\mathcal{Q}_\pm$ and $\mathcal{H}_\pm$. Of course, the string theory constraints are equivalent to $\mathcal{Q}_\pm = 0$ and $\mathcal{H}_\pm = 0$, implying that the superconformal invariance is actually a gauge invariance of the spinning string theory

[55] A spinor that is real in a real basis for the Dirac matrices becomes a Majorana spinor in a general basis.

9.2 Open spinning string: free ends

The phase-space action for the open spinning string, of parameter length π, is

$$I = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m + \frac{i}{2}\left(\psi_+ \cdot \dot{\psi}_+ + \psi_- \cdot \dot{\psi}_-\right) \right. \\ \left. -\lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+ - i\chi^- \mathcal{Q}_- - i\chi^+ \mathcal{Q}_+ \right\} . \tag{9.24}$$

We must choose the boundary conditions at the string endpoints such that the action is stationary at solutions of the equations of motion, which means that the boundary term arising from a general variation must vanish. The relevant terms in the action giving rise to *new* boundary terms are

$$I_{\rm relevant} = \int dt \int_0^\pi \frac{i}{2} \left\{ \lambda^- \psi_- \cdot \psi_-' - \lambda^+ \psi_+ \cdot \psi_+' + \sqrt{T}\left[\chi^- \psi_- - \chi^+ \psi_+\right] \cdot X' \right\} . \tag{9.25}$$

and the new boundary terms are

$$\delta I|_{\rm on-shell} = \frac{i}{2} \int dt \left[Te\left(\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+\right) + \sqrt{T}\left(\chi^- \psi_- - \chi^+ \psi_+\right) \cdot \delta X \right]_0^\pi . \tag{9.26}$$

To simplify things, we'll assume that the previous open string NG b.c.s still apply, so that we now need to find b.c.s for the anticommuting variables such that the new boundary terms are zero. We shall restrict attention to the spinning string with free ends, in which case the δX term is zero only if the D-vector coefficient is zero. The other term must be zero too so we arrive at the conclusion that

$$\left(\chi^- \psi_-^m - \chi^+ \psi_+^m\right)_{\rm ends} = 0 , \qquad \left(\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+\right)_{\rm ends} = 0 , \tag{9.27}$$

The least restrictive solution to these requirements is to impose, *at each end separately*, the boundary conditions

$$\psi_+^m|_{\rm end} = \pm\, \psi_-^m|_{\rm end} \quad \& \quad \chi^+|_{\rm end} = \pm\, \chi^-|_{\rm end} . \tag{9.28}$$

The sign at any given end is not significant because we are free to redefine $\psi_\pm \to \pm\psi_\pm$, so we may choose

$$\psi_+|_{\sigma=0} = \psi_-|_{\sigma=0} . \tag{9.29}$$

However, the *relative* sign is significant so we now have two cases to consider:

- **Ramond sector**. *Same sign boundary conditions*: $\psi_+|_{\sigma=\pi} = \psi_-|_{\sigma=\pi}$. In this case the Fourier series expansions of $\psi_\pm$ are

$$\psi_\pm(t,\sigma) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{\mp ik\sigma}\, d_k(t) , \tag{9.30}$$

where the d_k are *anticommuting* coefficient functions of t. This obviously satisfies the boundary condition (9.29) at $\sigma = 0$, and the same boundary condition is satisfied at $\sigma = \pi$ because $e^{-ik\pi} = e^{ik\pi}$ for integer k.

There is a similar expansion for the parameters $\epsilon^{\pm}$. In particular, in the conformal gauge, where $\epsilon^{\pm}$ is a function only of $\sigma^{\pm}$, we have

$$\epsilon^{\pm}(\sigma^{\pm}) = \epsilon_0^{\pm} + \sum_{k\neq 0} e^{ik\sigma^{\pm}} \epsilon_k \tag{9.31}$$

The constants $\epsilon_0^{\pm}$ are the parameters for a 2D $(1,1)$ supersymmetry (one left-handed and one right-handed spinor charge).

- **Neveu-Schwarz sector**. *Opposite sign boundary conditions*: $\psi_+|_{\sigma=\pi} = -\psi_-|_{\sigma=\pi}$. In this case

$$\psi_{\pm}(t,\sigma) = \frac{1}{\sqrt{2\pi}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{\mp ir\sigma}\, b_r(t)\,. \tag{9.32}$$

The NS boundary condition is satisfied because $e^{-ir\pi} = -e^{ir\pi}$ for $r \in \mathbb{Z} + \frac{1}{2}$.

It might now seem that we have two different types of spinning string. However, consistency ultimately requires that we include both the Ramond and the Neveu-Schwarz strings as two "sectors" of a single RNS string. We shall now examine these two sectors separately, and obtain the light-cone gauge action for both.

9.2.1 Ramond sector

For the Fourier series expansion (9.30) we have

$$\frac{i}{2}\int_0^{\pi} d\sigma \left\{\psi_- \cdot \dot\psi_- + \psi_+ \cdot \psi_+\right\} = \frac{i}{2}\sum_{k\in\mathbb{Z}} d_{-k}\cdot \dot d_k = \frac{i}{2} d_0 \cdot \dot d_0 + i\sum_{k=1}^{\infty} d_{-k}\dot d_k\,. \tag{9.33}$$

The full Ramond string action in Fourier space therefore takes the form

$$\begin{aligned} I_R = \int dt \Big\{ & \dot x^m p_m + \frac{i}{2} d_0\cdot \dot d_0 + i\sum_{k=1}^{\infty}\left(\frac{1}{k}\alpha_{-k}\cdot\dot\alpha_k + d_{-n}\cdot \dot d_n\right) \\ & - \sum_{n\in\mathbb{Z}} \left(\lambda_{-n} L_n + i\chi_{-n} F_n\right) \Big\}\,, \end{aligned} \tag{9.34}$$

where L_n and F_n are Fourier coefficients of the *periodic* functions $\mathcal{H}(\sigma)$ and $\mathcal{Q}(\sigma)$ constructed from $\mathcal{H}_{\pm}$ and $\mathcal{Q}_{\pm}$ by the "doubling trick" that we used previously[56]:

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \left[\alpha_{n-k}\cdot\alpha_k + k\, d_{k-n}\cdot d_k\right]\,, \qquad F_n = \sum_{k\in\mathbb{Z}} \alpha_k \cdot d_{n-k}\,. \tag{9.35}$$

[56]For example, $\mathcal{Q}(\sigma) = \mathcal{Q}_-(\sigma)$ for $0 \leq \sigma \leq \pi$ and $\mathcal{Q}(\sigma) = \mathcal{Q}_+(\sigma)$ for $\pi \leq \sigma \leq 2\pi$.

The canonical PB relations are

$$\left\{\alpha_k^m, \alpha_{-k}^n\right\}_{PB} = -ik\,\eta^{mn}\,, \qquad \left\{d_{-k}^m, d_k^n\right\}_{PB} = -i\eta^{mn}\,. \tag{9.36}$$

Using these relations, one can show that

$$\begin{aligned} \{L_m, L_n\}_{PB} &= -i\,(m-n)\,L_{m+n}\,, \\ \{L_m, F_n\}_{PB} &= -i\left(\frac{m}{2}-n\right)F_{m+n}\,, \\ \{F_m, F_n\}_{PB} &= -2iL_{m+n}\,. \end{aligned} \tag{9.37}$$

Notice that this implies that the F_n are (classically) the Fourier components of an operator[57] of conformal dimension 3/2.

As for the NG string, the constraints generate gauge invariances via Poisson brackets, and the combination

$$\sum_{n\in\mathbb{Z}}\left(\xi_{-n}L_n + i\epsilon_{-n}F_n\right)\,, \tag{9.38}$$

generates the gauge transformations

$$\begin{aligned} \delta\alpha_n &= -in\left(\xi_n\alpha_0 + i\epsilon_n d_0\right) - in\sum_{k\neq n}\left(\xi_k\alpha_{n-k} + i\epsilon_k d_{n-k}\right)\,, \\ \delta d_n &= \epsilon_n\alpha_0 - in\xi_n d_0 + \sum_{k\neq n}\left(\epsilon_k\alpha_{n-k} - in\xi_k d_{n-k}\right)\,. \end{aligned} \tag{9.39}$$

We can fix all but the zero-mode gauge invariance with parameter α_0 by the gauge-fixing conditions

$$\alpha_k^+ = 0 \quad (k\neq 0)\,, \qquad d_k^+ = 0 \quad (\forall k)\,. \tag{9.40}$$

A gauge transformation of these conditions yields, assuming that $\alpha_0^+ \neq 0$, the equations $\xi_k = 0$ for $k\neq 0$ and $\epsilon_k = 0$ for all k. We may then solve the equations $L_n = 0$ ($n\neq 0$) and $F_n = 0$ (all n) to get expressions for α_n^- ($n\neq 0$) and d_n^- (all n). These expressions are needed to get the Lorentz generators in terms of transverse canonical variables, but they are not needed for the action, which is

$$I_R = \int dt\left\{\dot{x}^m p_m + \frac{i}{2}\mathbf{d}_0\cdot\dot{\mathbf{d}}_0 + \sum_{k=1}^{\infty}\left(\frac{i}{k}\boldsymbol{\alpha}_{-k}\cdot\dot{\boldsymbol{\alpha}}_k + i\mathbf{d}_{-k}\cdot\dot{\mathbf{d}}_k\right) - \lambda_0 L_0\right\}\,, \tag{9.41}$$

where, now, recalling that $\alpha_0 = p/\sqrt{\pi T}$ for an open string,

$$L_0 = \frac{p^2}{2\pi T} + \sum_{k=1}^{\infty}\left(\boldsymbol{\alpha}_{-k}\cdot\boldsymbol{\alpha}_k + k\,\mathbf{d}_{-k}\cdot\mathbf{d}_k\right)\,. \tag{9.42}$$

[57]Light-cone component of a worlsheet supersymmetry current.

From this we see that the mass-shell constraint is $p^2 + \mathcal{M}^2 = 0$ with

$$\mathcal{M}^2 = 2\pi T N_R \,, \qquad N_R = N_b + N_f \,, \tag{9.43}$$

where

$$N_b = \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n \,, \qquad N_f = \sum_{k=1}^{\infty} k\, \mathbf{d}_{-k} \cdot \mathbf{d}_k \,. \tag{9.44}$$

In the quantum theory, the transverse oscillator variables have the canonical (anti)commutation relations

$$\left[\alpha^I_k, \alpha^J_{-k}\right] = k\delta^{IJ} \,, \qquad \left\{d^I_{-k}, d^J_k\right\} = \delta^{IJ} \qquad\qquad k \geq 0 \,. \tag{9.45}$$

We define the oscillator vacuum in the usual way:

$$\alpha^m_k|0\rangle = 0 \quad \& \quad d^m_k|0\rangle = 0 \,, \qquad k > 0. \tag{9.46}$$

We also choose the operator ordering in the operator versions of N_b and N_f as given in (9.66), so that both annihilate the oscillator vacuum. Other states in the Fock space are found by acting with the creation operators $\{\alpha_{-k}, d_{-k}; k > 0\}$. Acting with α_{-k} raises the eigenvalue of N_b by k and acting with d_{-k} raises the eigenvalue of N_f by k, so the eigenvalues of the level number N are non-negative integers and we can organise the spectrum according to this level number.

As for the NG string in light-cone gauge, Lorentz invariance of the quantum theory is not guaranteed; this has to be checked by a computation of the commutation relations of the operators representing the Lorentz generators. Whereas this calculation yields the result that $D = 26$ for the NG string, it yields the result $D = 10$ for the Ramond string, so we now restrict to $D = 10$.

Recall that requiring Lorentz invariance of the quantum NG string in light-cone gauge also fixed the intercept parameter a. For the Ramond string we find $a = 0$. This is to be expected from its interpretation as the oscillator zero-point energy because every contribution from a Bose oscillator is now cancelled by a contribution from a Fermi oscillator. This means that the mass-squared at level $N_R = N_f + N_b$ is $(2\pi T)N_R$. In particular, *the* $N_R = 0$ *states are massless*; it is "states" and not "state" because the oscillator vacuum is degenerate; the operators d_0 commute with N_R but satisfy the anti-commutation relations

$$\left\{d^I_0, d^J_0\right\} = \delta^{IJ} \quad \Rightarrow \quad d^I_0 \to \frac{1}{\sqrt{2}}\gamma^I \,, \tag{9.47}$$

where γ^I are the 16×16 Dirac matrices, which we may choose to be real. The 16×16 matrices

$$S^{IJ}_0 = \frac{1}{2}\gamma^{[I}\gamma^{J]} \tag{9.48}$$

obey the commutation relations of the Lie algebra of Spin(8), which is the same as the Lie algebra of $SO(8)$. A special feature of Spin(8) is that its real 16-component spinor representation is reducible. Observe that the matrix

$$\gamma_9 = \gamma^1\gamma^2 \cdots \gamma^8 \tag{9.49}$$

has the properties

$$\gamma_9^2 = 1 \qquad \{\gamma_9, \gamma^I\} = 0\,. \tag{9.50}$$

The latter property implies that γ_9 has zero trace, so we can choose a basis for which

$$\gamma_9 = \begin{pmatrix} \mathbb{I}_8 & 0 \\ 0 & -\mathbb{I}_8 \end{pmatrix}\,. \tag{9.51}$$

This basis is consistent with reality of the matrices γ^I, so a real 16-component spinor of Spin(8) is the sum of two 8-dimensional representations: the eigenspinors of γ_9 with eigenvalues ± 1; equivalently a chiral and an anti-chiral spinor. In math-speak they are the $\mathbf{8}_s$ (spinor) and $\mathbf{8}_c$ (conjugate spinor) of Spin(8).

The restriction to $D = 10$ is confirmed when we quantise the Ramond string covariantly. In the path integral approach, we will now have new FP ghosts, usually denoted by (β, γ), arising from the conformal gauge fixing of the gauge symmetries generated by $\mathcal{Q}_\pm$. These FP ghosts are *commuting* because the gauge parameters are anticommuting, and this flips the sign of their contribution to the central charge. From Q.III.6 we know that an anticommuting (b, c) system for which $[b] = J$ and $[c] = 1 - J$ (or vice versa since the formula is invariant under $J \to 1 - J$) has central charge

$$c = -2(6J^2 - 6J + 1) \qquad (\Rightarrow \quad c = -26 \text{ for } J = 2). \tag{9.52}$$

In the standard (b, c) case we have $J = 2$ (because the L_m are Fourier components of a worldsheet field (2D stress tensor) whose classical conformal dimension is 2) and this gives $c = -26$. In the (β, γ) case we have $J = 3/2$ because F_n are Fourier components of a worldsheet field (2D supercurrent) whose classical conformal dimension is 3/2. Allowing for the change of sign due to the opposite "statistics" we deduce that the (β, γ) ghost system has central charge

$$c = +2(6J^2 - 6J + 1)|_{J=\frac{3}{2}} = 11\,. \tag{9.53}$$

The total ghost central charge has to cancel against the non-ghost contribution. We know that the worldsheet scalar fields contribute $c = D$. As we are discussing the open string, for which $\psi_\pm$ share the same Fourier coefficients, we need only consider ψ_+, say. Its action is essentially of the (b, c) form but with $J = \frac{1}{2}$ (since $J = 1 - J$) and also with 1/2 the number of variables since both b and c are replaced by ψ_+. This gives $c = 1/2$, and hence $c = D/2$ for D-vector ψ_+. Therefore, the central charge cancels when

$$D + \frac{1}{2}D - 26 + 11 = 0 \qquad \Rightarrow\ D = 10\,. \tag{9.54}$$

To summarise: the quantum Ramond string is Lorentz invariant in light-cone gauge only if $D = 10$, and the $N_R = 0$ state is then massless and degenerate. These massless states transform as the $\mathbf{8}_s \oplus \mathbf{8}_c$ representation of the transverse rotation group. All $N_R > 0$ states are massive.

9.2.2 Neveu-Schwarz sector

For the Fourier series expansion (9.32) we have

$$\frac{i}{2}\int_0^\pi d\sigma \left\{\psi_- \cdot \dot{\psi}_- + \psi_+ \cdot \psi_+\right\} = \frac{i}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_{-r}\cdot \dot{b}_r = i\sum_{r=\frac{1}{2}}^{\infty} b_{-r}\dot{b}_r\,, \tag{9.55}$$

The full NS string action in Fourier space is

$$\begin{aligned} I_{NS} = & \int dt \Big\{ \dot{x}^m p_m + \sum_{k=1}^{\infty}\frac{i}{k}\alpha_{-k}\cdot\dot{\alpha}_k + \sum_{r=\frac{1}{2}}^{\infty} ib_{-r}\cdot\dot{b}_r \\ & - \sum_{n\in\mathbb{Z}}\lambda_{-n}L_n - i\sum_{r\in\mathbb{Z}+\frac{1}{2}}\chi_{-r}G_r\Big\}\,, \end{aligned} \tag{9.56}$$

where

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\alpha_{n-k}\cdot\alpha_k + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\,b_{n-r}\cdot b_r \qquad (n\in\mathbb{Z}) \\ G_r &= \sum_{k\in\mathbb{Z}}\alpha_k\cdot b_{r-k} \qquad \left(r\in\mathbb{Z}+\frac{1}{2}\right)\,. \end{aligned} \tag{9.57}$$

The canonical PB relations are

$$\{\alpha_k^m,\alpha_{-k}^n\}_{PB} = -ik\,\eta^{mn}\,, \qquad \{b_{-r}^m, b_r^n\}_{PB} = -i\eta^{mn}\,. \tag{9.58}$$

The PB algebra of the constraint functions is

$$\begin{aligned} \{L_m,L_n\}_{PB} &= -i\,(m-n)\,L_{m+n}\,, \\ \{L_m,G_r\}_{PB} &= -i\left(\frac{m}{2}-r\right)G_{m+r}\,, \\ \{G_r,G_s\}_{PB} &= -2iL_{r+s}\,. \end{aligned} \tag{9.59}$$

The gauge transformations generated by the linear combination of constraint functions

$$\sum_{n\in\mathbb{Z}}\xi_{-n}L_n + \sum_{r\in\mathbb{Z}+\frac{1}{2}} i\epsilon_{-r}G_r\,, \tag{9.60}$$

are

$$\begin{aligned} \delta\alpha_n &= -in\xi_0\alpha_n - in\sum_{m\neq n}\xi_m\alpha_{n-m} + n\sum_r \epsilon_r b_{n-r} \\ \delta b_r &= \alpha_0\epsilon_r + \sum_{s\neq r}\alpha_{r-s}\epsilon_s - ir\sum_m \xi_m b_{r-m}\,. \end{aligned} \tag{9.61}$$

Invariance under these gauge transformations may be fixed by an extension of the light-cone gauge condition, as for the Ramond string:

$$\alpha_n^+ = 0 \quad (n \neq 0) \quad \& \quad b_r^+ = 0 \quad (\forall r)\,. \tag{9.62}$$

Let's check this: a gauge variation of these conditions yields

$$0 = -in\alpha_0^+ \xi_n\,, \quad (n \neq 0) \qquad \& \quad 0 = \alpha_0^+ \epsilon_r\,. \tag{9.63}$$

This tells us that the gauge invariance is completely fixed except for the ξ_0 transformation (assuming that α_0^+ is non-zero). Having fixed the gauge in this way, we may now solve $L_n = 0$ for α_n^- $(n \neq 0)$ and $G_r = 0$ for b_r^-. The resulting expressions for these variables are needed only for the Lorentz generators, not for the gauge-fixed action, which is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \boldsymbol{\alpha}_{-k} \cdot \dot{\boldsymbol{\alpha}}_k + i \sum_{r=\frac{1}{2}}^{\infty} \mathbf{b}_{-r} \cdot \dot{\mathbf{b}}_r - \lambda_0 L_0 \right\}\,, \tag{9.64}$$

where now, recalling that $\alpha_0 = p/\sqrt{\pi T}$ for open strings,

$$L_0 = \frac{p^2}{2\pi T} + \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n + \sum_{r=\frac{1}{2}}^{\infty} r\, \mathbf{b}_{-r} \cdot \mathbf{b}_r\,. \tag{9.65}$$

From this we see that the mass-shell constraint is $p^2 + \mathcal{M}^2 = 0$, with

$$\mathcal{M}^2 = 2\pi T \left(N_b + N_f\right)\,; \qquad N_b = \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n\,, \quad N_f = \sum_{r=\frac{1}{2}}^{\infty} r \mathbf{b}_{-r} \cdot \mathbf{b}_r\,. \tag{9.66}$$

In the quantum theory, the oscillator variables have the canonical (anti)-commutation relations

$$\begin{aligned} \left[\alpha_k^I, \alpha_{-k}^J\right] &= k\delta^{IJ}\,, \qquad k = 1, 2, \ldots \\ \left\{b_{-r}^I, b_r^J\right\} &= \delta^{IJ}\,, \qquad r = \frac{1}{2}, \frac{3}{2}, \ldots \end{aligned} \tag{9.67}$$

and the oscillator vacuum $|0\rangle$ is defined by

$$\alpha_k |0\rangle\,, \quad k > 0\,, \qquad b_r |0\rangle = 0\,, \quad r > 0\,. \tag{9.68}$$

The operators N_b and N_f both annihilate the oscillator vacuum for the ordering as given in (9.66), so in a basis where they are diagonal,

$$M^2 = 2\pi T \left(N_{NS} - a\right)\,, \qquad N = N_b + N_f\,, \tag{9.69}$$

where the constant a is now introduced to allow for the ambiguity due to operator ordering. We should no longer expect a bose-fermi cancellation of the zero-point energies because of the different "moding" of the bose and fermi oscillators. In contrast to the Ramond string, the oscillator ground state is non-degenerate, so it corresponds to a scalar particle, which will be a tachyon if $a > 0$.

Let's now look at the first excited states. The operator N_b has integer eigenvalues $0, 1, 2, \ldots$ but the operator N_f has eigenvalues $\frac{1}{2}, \frac{3}{2}, \ldots$, so $2N_{RS}$ is a non-negative integer, and *the first excited states have* $N_{NS} = \frac{1}{2}$. These states are

$$b^I_{-\frac{1}{2}}|0\rangle\,, \quad I = 1, \ldots, D-2\,. \tag{9.70}$$

By the same argument that we used at level-1 of the NG string, these must be the polarization states of a massless vector if the quantum theory is to preserve Lorentz invariance. This means that we must now choose

$$a = \frac{1}{2}\,. \tag{9.71}$$

On the other hand, the zero point energy $-a$ is formally given by

$$-a = \frac{1}{2}(D-2)\left[\sum_{n=0}^{\infty}(n+1) - \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\right]\,. \tag{9.72}$$

The relative sign is due to the opposite sign contribution of bosonic and fermionic oscillators. We can perform the sum using the fact that the generalized ζ-function,

$$\zeta(s,q) = \sum_{n=0}^{\infty}(n+q)^{-s}\,, \tag{9.73}$$

has a unique value at $s = -1$:

$$\zeta(-1,q) = -\frac{1}{12}\left(6q^2 - 6q + 1\right)\,. \tag{9.74}$$

This gives

$$-a = \frac{1}{2}(D-2)\left[-\frac{1}{12} - \frac{1}{24}\right] = -\frac{(D-2)}{16}\,. \tag{9.75}$$

Using this formula, we see that $a = \frac{1}{2}$ implies that $D = 10$. This value is confirmed by a computation of the Lorentz commutators. The theory is Lorentz invariant only if $a = \frac{1}{2}$ and $D = 10$.

To summarize: Lorentz invariance of the NS string requires $D = 10$ and then we have a scalar tachyon with $\alpha' M^2 = -\frac{1}{2}$ at level $N_{NS} = 0$ and a massless vector at level $N_{NS} = 1/2$ with physical polarizations in the $\mathbf{8}_v$ (vector) representatiuon of the transverse Spin(8) rotation group. All $N_{NS} \geq 1$ states are massive.

10. Superstrings

Open string interactions always allow closed string states to appear as virtual particles in loops of the string loop expansion, so the quantum theory is inconsistent unless we include closed strings. For closed strings, the choice between R and NS boundary conditions becomes the choice between worldsheet "fermions" are periodic or anti-periodic in σ; in the context of a path-integral formulation of the quantum theory, for which we use the conformal gauge action (in which $\psi_\pm$ are the components of a worldsheet spinor) this choice amounts to a choice of spin structure on the Riemann surface worldsheet. Consistency of the quantum theory forces a sum over spin structures and this forces us to combine the R and NS strings.

The NS sector of the spinning string still has a tachyon. At the free-string level there would be nothing to stop us from simply discarding the tachyon state; we could have already done that for the NG string. However, we cannot expect arbitrary truncations of the spectrum to be consistent with interactions; anything we throw out will usually reappear in loops in the quantum theory, making the truncation inconsistent. The only way to guarantee consistency of some truncation that removes the tachyon is by means of a symmetry. If there is a symmetry that excludes the tachyon, then its exclusion will be consistent if we can introduce interactions consistent with the symmetry. Nothing of that kind is available for the NG string, or for the NS-sector of the spinning string, but once we combine the NS sector with the R sector, a possibility presents itself: spacetime supersymmetry.

10.1 RNS Superstring

Let's take a closer look at the low-lying states of the combined R and NS spinning strings, organised according to the value of $\alpha' M^2$ (recall that for *integer* $\alpha' M^2$ both R and NS states contribute, with $N_{NS} = N_R + \frac{1}{2}$):

$$\begin{array}{lcc}
\alpha' M^2 = -\frac{1}{2}: & |0\rangle_{NS} & \mathbf{1} \\
\alpha' M^2 = 0: & \left\{|0\rangle_R\,;\ b^I_{-\frac{1}{2}}|0\rangle_{NS}\right\} & (\mathbf{8}_s \oplus \mathbf{8}_c) \oplus \mathbf{8}_v \\
\alpha' M^2 = \frac{1}{2}: & \left\{\alpha^I_{-1}|0\rangle_{NS}\,,\ b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}}|0\rangle_{NS}\right\} & \mathbf{8}_v \oplus \mathbf{28} \\
\alpha' M^2 = 1: & \left\{\alpha^I_{-1}|0\rangle_R,\ d^I_{-1}|0\rangle_R\right\} & (\mathbf{8}_v \oplus \mathbf{8}_v) \otimes (\mathbf{8}_s \oplus \mathbf{8}_c) \\
 & \left\{\alpha^I_{-1} b^J_{-\frac{1}{2}}|0\rangle_{NS},\ b^I_{-\frac{3}{2}}|0\rangle_{NS}\,,b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}} b^K_{-\frac{1}{2}}|0\rangle_{NS}\right\} & (\mathbf{8}_v \otimes \mathbf{8}_v) \oplus \mathbf{8}_v \oplus \mathbf{56}_v
\end{array} \tag{10.1}$$

A useful lemma for Spin(8) tensor products is

- 8×8 **lemma**. Let $i = v, s, c$ label the three 8-dimensional representations of Spin(8). Then

$$\begin{aligned}
\mathbf{8}_i \otimes \mathbf{8}_i &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_i \\
\mathbf{8}_i \otimes \mathbf{8}_j &= \mathbf{8}_k \oplus \mathbf{56}_k \qquad (i, j, k \quad \text{cyclic})\,.
\end{aligned} \tag{10.2}$$

The $\mathbf{35}_v$ is a 2nd-rank symmetric traceless tensor, the $\mathbf{28}$ is a 2nd-rank antsymmetric tensor, and the $\mathbf{56}_v$ is a 3rd-rank antisymmetric tensor. The $\mathbf{35}_s$ ($\mathbf{35}_c$) is an (anti-)self-dual 4th-rank antisymmetric tensor. The $\mathbf{56}_s$ is a chiral vector-spinor and the $\mathbf{56}_c$ is an anti-chiral vector-spinor.

Now we define a notion of "G-parity" and declare that all physical states must be G-parity even. First we define worldsheet bosons (commuting worldsheet fields) to have even G-parity and worldsheet fermions (anticommuting worldsheet fields) to have odd G-parity. So far G-parity is determined by Grassmann parity, but the way this works for d_0 is that if the chiral component of $|0\rangle_R$ (the $\mathbf{8}_s$) is assigned even G-parity then the anti-chiral component (the $\mathbf{8}_c$) must be assigned odd G-parity; we'll call the chiral component $|0\rangle_{R^+}$ and the anti-chiral component $|0\rangle_{R^-}$. The NS oscillator vacuum is assigned odd G-parity (this removes the tachyon). The G-parity assignments of the oscillator vacua are therefore

$$\begin{cases} \mathbf{G-even}: & |0\rangle_{R^+} \\ \mathbf{G-odd}: & |0\rangle_{R^-} \ \& \ |0\rangle_{NS} \end{cases} \tag{10.3}$$

The requirement of even G-parity is called the Gliozzi-Scherk-Olive (GSO) projection, and the combined R and NS spinning strings with this projection is called the *RNS superstring*.

To see why it's called a "superstring" we look at the low-mass even G-parity states

$$\begin{array}{llr} \alpha' M^2 = 0: & \left\{|0\rangle_{R^+}\,;\, b^I_{-\frac{1}{2}}|0\rangle_{NS}\right\} & \mathbf{8}_s \oplus \mathbf{8}_v \\ \alpha' M^2 = 1: & \left\{\alpha^I_{-1}|0\rangle_{R^+},\, d^I_{-1}|0\rangle_{R^-}\right\} & (\mathbf{8}_v\otimes\mathbf{8}_s)\oplus(\mathbf{8}_v\otimes\mathbf{8}_c) \\ & \left\{\alpha^I_{-1}b^J_{-\frac{1}{2}}|0\rangle_{NS},\, b^I_{-\frac{3}{2}}|0\rangle_{NS}\,,\, b^I_{-\frac{1}{2}}b^J_{-\frac{1}{2}}b^K_{-\frac{1}{2}}|0\rangle_{NS}\right\} & (\mathbf{8}_v\otimes\mathbf{8}_v)\oplus\mathbf{8}_v\oplus\mathbf{56}_v \end{array} \tag{10.4}$$

We have 8+8 massless states. The $\mathbf{8}_v$ are the states of a $D=10$ photon and the $\mathbf{8}_s$ are the states of a $D=10$ Majorana (read that as "real") chiral spinor. These make up the fields of the Maxwell supermultiplet of $D=10$, i.e. *spacetime*, supersymmetry.

At the next level we have massive states.

$$\left\{\begin{array}{lll} \text{Spacetime fermions} & (\mathbf{8}_v\otimes\mathbf{8}_s)\oplus(\mathbf{8}_v\otimes\mathbf{8}_c) & =\mathbf{128} \\ \text{Spacetime bosons} & (\mathbf{8}_v\otimes\mathbf{8}_v)\oplus\mathbf{8}_v\oplus\mathbf{56}_v & =\mathbf{44}\oplus\mathbf{128} \end{array}\right\} \tag{10.5}$$

where the last column gives the Spin(9) representations. This is the physical light-cone content of a $D=10$ massive spin-2 supermultiplet. In fact, at every level the spectrum is a supermultiplet of $D=10$ spacetime supersymmetry. There is a simple way to see why, found by Green and Schwarz.

10.2 The Green-Schwarz superstring*

There is a way to reformulate the RNS superstring that makes manifest its spacetime supersymmetry. It relies on the triality property of the Spin(8) algebra. The representation theory for Spin(8) is unchanged by a permutation of its three 8-dimensional representations. Consider the Ramond string in light-cone gauge: if we make the replacement

$$\left\{d_k^I; I = 1, \ldots, 8\right\} \quad \rightarrow \quad \{\theta_k^\alpha; \alpha = 1, \ldots, 8\} \,, \tag{10.6}$$

where the θ_k^α are an $\mathbf{8}_c$ of anticommuting variables then we have only renamed and relabeled the anticommuting light-cone gauge variables, giving them a new spacetime interpretation (as an anti-chiral spinor of the transverse rotation group). Quantizing this (light-cone gauge) Green-Schwarz superstring leads to exactly the same results that we found previously for the Ramond string but with a permutation of the Spin(8) representations

$$\mathbf{8}_v \rightarrow \mathbf{8}_s \qquad \mathbf{8}_s \rightarrow \mathbf{8}_c \,, \qquad \mathbf{8}_c \rightarrow \mathbf{8}_v \,. \tag{10.7}$$

The ground state of the GS string has zero mass, like the Ramond string, but the Spin(8) representations at this level are

$$\mathbf{8}_v \oplus \mathbf{8}_s \,. \tag{10.8}$$

This coincides with the ground states of the RNS superstring. In other words, the result of imposing the GSO projection on the RNS string is reproduced by imposing Ramond boundary conditions on the GS superstring, discarding its NS sector. This obviously ensures bose-fermi matching at all levels[58]. To show that one gets supermultiplets at all levels requires more work, which we are not going to do but let's look at the first excited states:

$$\alpha_{-1}^I|0\rangle_{GS} \,, \qquad \theta_{-1}^\alpha|0\rangle_{GS} \,. \tag{10.9}$$

This gives us states with $\alpha' M^2 = 1$ in the Spin(8) representations

$$\begin{aligned}(\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) &= [(\mathbf{1} \oplus \mathbf{8}_v \oplus \mathbf{28}) \oplus (\mathbf{35}_v \oplus \mathbf{56}_v)] \oplus [(\mathbf{8}_s \oplus \mathbf{56}_c) \oplus (\mathbf{8}_c \oplus \mathbf{56}_s)] \\ &= [\mathbf{44} \oplus \mathbf{84}] \oplus \mathbf{128} \,, \end{aligned} \tag{10.10}$$

where the representations in the second line are those of Spin(9). Those in the square bracket are the bosons: the **44** is a symmetric traceless tensor, which describes a massive spin-2 particle, and the **84** is a third-order antisymmetric tensor. All fermions are in the **128**, which describes a massive spin-3/2 particle; all together we have the massive spin-2 multiplet of $\mathcal{N} = 1$ D=10 supersymmetry.

[58] The tensor product of $\mathbf{8}_c \oplus \mathbf{8}_v$ with any spin(8) tensor T is $(\mathbf{8}_c \otimes T) \oplus (\mathbf{8}_v \otimes T)$, which exhibits a bose-fermi matching, and the same is true if T is a tensor-spinor.

10.3 Closed superstrings*

As for the NG string, the physical states at each level of the closed superstring are just tensor products of two copies of the physical states of the open superstring at that level. In other words, the spin(8) representation content of the *massless* states will be the tensor product of two copies of the spin(8) representations of the massless states of the open string. For the latter we had to choose between the two possibilities of (**??**); the choice didn't matter there but now we have a *relative* choice to make because, for the closed GS superstring we have two sets of oscillators, one for the left-movers and one for the right-movers, and this means that we get two distinct closed superstring theories according to the relative choice of spin(8) chirality for the anticommuting variables $\theta_\pm$

- **IIA**. Opposite spin(8) chirality: $(\theta_-^\alpha, \quad \theta_+^{\dot\alpha})$. We use $\dot\alpha = 1, \dots, 8$ for the anti-chiral $\mathbf{8}_c$ spinor.
- **IIB**. Same spin(8) chirality: $(\theta_-^\alpha, \quad \theta_+^\alpha)$.

As Hamlet put it so eloquently: *IIB or not IIB, that is the question.*

Using this lemma we find the following results, which we organise according to their RNS origin:

- **IIA string**. The Spin(8) representation content of massless states is

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) = \begin{cases} \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v & \text{NS} - \text{NS} \\ \mathbf{8}_v \oplus \mathbf{56}_v & \text{R} - \text{R} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \text{R} - \text{NS} \\ \mathbf{8}_c \oplus \mathbf{56}_c & \text{NS} - \text{R} \end{cases} \tag{10.11}$$

  We get a spinor ground state for the Ramond open string, so the fermions of the closed superstring come from its R-NS and NS-R sectors. Notice that these give spinorial spin(8) states of *opposite* chirality. The states in the R-R sector are bi-spinors, which are equivalent to antisymmetric tensors; for the IIA superstring we get a vector A_I and a third-order antisymmetric tensor A_{IJK}.

- **IIB string**.The Spin(8) representation content of massless states is

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) = \begin{cases} \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v & \text{NS} - \text{NS} \\ \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_c & \text{R} - \text{R} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \text{R} - \text{NS} \\ \mathbf{8}_s \oplus \mathbf{56}_s & \text{NS} - \text{R} \end{cases} \tag{10.12}$$

  The spinorial states from the R-NS sector now have the same chirality as those from the NS-R sector. The R-R states are now a scalar A, a 2nd-order anti-symmetric tensor A_{IJ} and a 4th-order self-dual anti-symmetric tensor A_{IJKL};

the self-duality means that

$$A_{IJKL} = \frac{1}{4!}\epsilon_{IJKLMNPQ}A_{MNPQ} \tag{10.13}$$

where ϵ is the alternating spin(8) invariant tensor.

In both IIA and IIB cases we get the same NS-NS massless states, which are also the same as those of the closed NG string; as we saw in that case, the $\mathbf{35}_v$ can be interpreted as the physical polarisation states of a massless spin-2 particle.

All remaining massless bosonic states come from the R-R sector. For example, for the IIA superstring, the full set of tensorial spin(8) representations, coming from the combined NS-NS and R-R sectors, is

$$(\mathbf{1} \oplus \mathbf{8}_v \oplus \mathbf{35}_i) \oplus (\mathbf{288}_v \oplus \mathbf{56}_v) = \mathbf{44} \oplus \mathbf{84}\,, \tag{10.14}$$

where the second equality gives the spin(9) representations. They are the same that we found at the first massive level of the open superstring. In that context the spin(8) representations had to combine into spin(9) representations for consistency with Lorentz invariance. That's not the case here because we are now dealing with the *massless* particles in the IIA superstring spectrum; in the massless particle context the spin(9) representations are what would be required for Lorentz invariance in *eleven* spacetime dimensions, i.e. D=11. In fact, this is also true for the massless fermions of the IIA superstring.

11. M-Theory*

Here are a few facts about the fermions: each of the $\mathbf{8}_s \oplus \mathbf{56}_s$ and $\mathbf{8}_c \oplus \mathbf{56}_c$ states are the physical polarisation states of a massless D=10 spin-3/2 particle, either chiral or anti-chiral. Consistency of the interactions of massless spin-3/2 particles *requires* supersymmetry, so their presence in the massless spectrum of the closed spinning string is a simple way of seeing why the GSO projection is necessary for consistency[59]. It follows that the effective D=10 spacetime action for the massless states of either the IIA or the IIB superstring is an $\mathcal{N} = 2$ $D = 10$ supergravity theory. There are two of them, according to whether the two spin-3/2 fields have the same (IIB) or opposite (IIA) chirality.

The maximal spacetime dimension for which a supergravity theory exists is D=11, and dimensional reduction of the unique D=11 supergravity theory to D=10 yields the IIA supergravity theory, which is the effective low-energy theory for the massless states of the IIA superstring. For a long time this was seen as just a coincidence, since superstring theory appeared to require D=10. Another "coincidence" is

[59]There is no analogous argument for the open spinning string but quantum consistency of any open string theory requires the inclusion of closed strings.

that the Green-Schwarz construction of a Lorentz covariant and manifestly spacetime supersymmetric action for the IIA and IIB superstrings, also applies to membranes in D=11 (but not to any other p-branes, for any $p > 0$, for any $D > 11$), and its dimensional reduction yields the IIA GS superstring action.

In string theory one can compute, in principle, the amplitude for scattering of any particles in the string spectrum to arbitrary order in a string-loop expansion, with each term being UV finite. However, this expansion is a divergent one; we cannot sum the series, even in principle. This is also typically the case in QFT but the perturbation expansions of QFT are usually derived from an action, and some QFT's can be defined non-perturbatively, e.g. as a continuum limit of a lattice version. String theory is different because all amplitudes are found "on-shell", and a spacetime action constructed order by order from these amplitudes has no more information in it than the computed amplitudes. String theory just gives us a perturbation series; it does not tell us what it is that is being perturbed. The completed non-perturbative theory could be something completely different. The fact that D=10 is the critical dimension of superstring shows only that $D \geq 10$ because some dimensions could be invisible in perturbation theory.

Fortunately, the constraints due to maximal supersymmetry are so strong that the effective spacetime field theory for the massless particles of the superstring contains a lot of information about non-perturbative string theory, sufficient to show that the five distinct superstring theories[60] are unified by some 11-dimensional theory, known as M-theory, and that this theory includes D=11 supergravity. Unfortunately, we don't really know what this theory is, so this is a good place to finish this course on String Theory.

[60]In addition to the IIA and IIB closed superstring theories we have the Type I open superstring (this is the string theory that results from inclusion of the additional features needed to get quantum consistency of interacting open superstrings) and two heterotic superstring theories (for which the worldsheet action has only $(1,0)$ D=2 supersymmetry).