Mathematical Tripos Part III
String Theory. Lent 2018: Example Sheet 1

1 Verify that the relativistic point particle action

$$I = \int_{t_a}^{t_B} dt \left\{ \dot{X}^m P_m - \frac{1}{2} e \left(P^2 + m^2 \right) \right\}$$

is invariant (ignoring boundary terms) under the Diff_1 gauge transformations

$$\delta_\xi X = \xi \dot{X}\,, \qquad \delta_\xi P = \xi \dot{P}\,, \qquad \delta_\xi e = \partial_t \left(e\xi \right)\,,$$

where $\xi(t)$ is an arbitrary infinitesimal function. Now verify that it is also invariant (ignoring boundary terms) under the "canonical" gauge transformation

$$\delta_\alpha X = \alpha P\,, \qquad \delta_\alpha P = 0\,, \qquad \delta e = \dot{\alpha}\,,$$

where $\alpha(t)$ is an arbitrary infinitesimal function.

Show that any action functional $I[\psi, \phi]$ is invariant under the gauge transformation

$$\delta_f \psi = f \frac{\delta I}{\delta \phi}\,, \qquad \delta_f \phi = -f \frac{\delta I}{\delta \psi}\,,$$

for *arbitrary* infinitesimal function f. Such "trivial" gauge invariances have no physical effect. Use this observation to explain why the "Diff_1" and "canonical" gauge invariances of the point particle action are physically equivalent.

2 A free non-relativistic point particle of mass m is described by the action

$$I[\mathbf{x}] = \int dt \left\{ \frac{1}{2} m \left| \frac{d\mathbf{x}}{dt} \right|^2 \right\}. \qquad (\dagger)$$

Show, by elimination of variables, that an equivalent action is

$$I[t, \mathbf{x}, E, \mathbf{p}; e] = \int d\lambda \left\{ \dot{\mathbf{x}} \cdot \mathbf{p} - \dot{t}E + e\left(E - \frac{|\mathbf{p}|^2}{2m} \right) \right\} \qquad \left(\dot{\mathbf{x}} = \frac{d\mathbf{x}}{d\lambda} \right) \qquad (\star)$$

where λ is an arbitrary worldline parameter. Find the gauge transformations of the canonical variables $(\mathbf{x}, \mathbf{p})$ and (t, E) generated by the constraint function, and verify that the action $(\star)$ is invariant for an appropriate gauge transformation of e. Now show that this gauge invariance can be fixed by imposing the temporal gauge condition $t = \lambda$, and thereby recover the action $(\dagger)$.

Let $\mathbf{P}$ be the Noether charge corresponding to translation invariance $\mathbf{x} \to \mathbf{x} + \mathbf{a}$. Show that $\mathbf{P} = \mathbf{p}$. Verify that the action $(\star)$ is invariant (neglecting a time-boundary term) under the Galilean boost transformation

$$\delta_\mathbf{v} \mathbf{x} = \mathbf{v}t\,, \qquad \delta_\mathbf{v} \mathbf{p} = m\mathbf{v}\,, \qquad \delta_\mathbf{v} E = \mathbf{v} \cdot \mathbf{p}\,.$$

Deduce that the corresponding Noether charge is $\mathbf{B} = t\mathbf{p} - m\mathbf{x}$, and verify that $\dot{\mathbf{B}} = 0$ as a consequence of the equations of motion. Show that $\{P, B\}_{PB} = m$.

Apply Dirac's method of quantization with constraints to $(\star)$ to show that the wavefunction $\psi(t, \mathbf{x})$ of the quantum non-relativistic free particle satisfies the time-dependent Schroedinger equation

$$-\frac{1}{2m} \nabla^2 \psi = \frac{i}{\hbar} \partial_t \psi\,.$$

3 In the temporal gauge $x^0(t) = t$ the relativistic point particle has the action

$$I = \int dt \left\{ \dot{\vec{x}} \cdot \vec{p} - p^0 \right\} , \qquad p^0 = \sqrt{|\vec{p}|^2 + m^2} .$$

Verify that this is invariant under the infinitesimal transformation

$$\delta \vec{x} = \vec{\Lambda} t - \left(\vec{\Lambda} \cdot \vec{x} \right) \vec{p}/p_0 , \qquad \delta \vec{p} = \vec{\Lambda} p_0 ,$$

for constant vector parameter $\vec{\Lambda}$. Find the corresponding Noether charge. What is the signficance of this symmetry?

4 The Hamiltonian form of the action for a relativistic p-brane of tension T_p is

$$I = \int dt \int d^p\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left(P^2 + T_p^2 \det k \right) - u^i \partial_i X^m P_m \right\}$$

where k is the induced p-metric on the brane in coordinates σ^i $(i = 1, \ldots, p)$. By elimination of P_m and the Lagrange multipliers (e, u^i), verify that this action is equivalent to the Dirac action $S = -T_p \int dt \int d^p\sigma \sqrt{-\det g}$, where g is the induced worldvolume $(p+1)$-metric. [*Hint: use the identity* $\det g \equiv (g_{tt} - k^{ij} g_{ti}\, g_{tj}) \det k$.]

5 Explain why the following configuration describes a circular closed string of time-dependent radius $R(t) = R_0 |\cos(t/R)|$.

$$X^0 = t , \qquad X^1 = R_0 \cos\left(\frac{t}{R_0}\right) \cos\left(\frac{\sigma}{R_0}\right) , \qquad X^2 = R_0 \cos\left(\frac{t}{R_0}\right) \sin\left(\frac{\sigma}{R_0}\right) ,$$

with R_0 a positive constant and $X^3 = X^4 = \ldots = 0$. Compute the induced worldsheet metric and hence show (i) that the above configuration solves the NG equations of motion and (ii) that $2\pi R(t)$ is the proper length of the string. Write down an expression for the total energy E and show that $E = (2\pi R_0)T$. Deduce that the rest-energy density of the string is T.

6 Write down the Hamiltonian form of the NG string action. In the case of an open string with free ends, what are the boundary conditions on X? Why do the ends of the string move at the speed of light?

Explain why the following configuration describes a rigidly rotating straight string in which the ends of the string are at $\sigma = 0$ and $\sigma = \pi L/2$.

$$X^0 = t \qquad X^1 = \frac{L}{2} \sin\left(\frac{2t}{L}\right) \cos\left(\frac{2\sigma}{L}\right) , \qquad X^2 = \frac{L}{2} \cos\left(\frac{2t}{L}\right) \cos\left(\frac{2\sigma}{L}\right) ,$$

with $X^3 = X^4 = \ldots = 0$. Compute the induced worldsheet metric, and show that it is conformally flat. Use your result to show that L is the proper length of the string, and that the above configuration solves the Nambu-Goto equations of motion.

Write down expressions for the conserved total energy E and angular-momentum J of an open string of tension T with free ends, and use it to show that a rigidly rotating open string of proper length L has

$$E = \frac{L}{4\alpha'} , \qquad J = \frac{L^2}{16\alpha'} \qquad \left(\alpha' = \frac{1}{2\pi T} \right) .$$

Hence show that $J = \alpha' E^2$. [*In a "Regge plot" of J against E^2 the slope is α', so α' is called the "Regge slope parameter".*]