**String Theory: Example Sheet 1**

**Malcolm Perry, Lent Term 2012**



**1.** Consider an electron in orbit around a neutron. Assume that electromagnetic dipole effects can be neglected and only the Newtonian gravitational force is relevant. What is the radius of Bohr orbit of the ground state? What object is this comparable to? Now re-evaluate your sense of the importance of quantum gravity.

**2a.** Consider the following action for a point particle

$$S = \frac{1}{2}\int d\tau \ \left(e^{-1}\dot{X}^a\dot{X}^b\eta_{ab} - em^2\right) \tag{1}$$

The dynamical fields are $e(\tau)$ and $X^a(\tau)$. Write down the equations of motion. Show that after substituting the equation of motion for $e$, one recovers the form of the point-particle action that we saw in the lectures.

**b.** Higher-dimensional objects are called *branes*. More precisely, if the object has $p$ spatial dimensions, it is called a $p$-brane. Thus a membrane is a 2-brane. The dynamics of a $p$-brane moving in Minkowski space is given by the Dirac action,

$$S = -T\int d^{p+1}\xi\sqrt{-\det\gamma} \tag{2}$$

Here $\xi^\mu$, $\mu = 0,\ldots,p$ are coordinates on the brane worldvolume, while $\gamma_{\mu\nu}$ is the pull-back of the Minkowski metric onto the brane.

$$\gamma_{\mu\nu} = \frac{\partial X^a}{\partial\xi^\mu}\frac{\partial X^b}{\partial\xi^\nu}\eta_{ab} \tag{3}$$

Show that this is equivalent to the Polyakov-type action with dynamical worldvolume metric $\gamma_{\mu\nu}$,

$$S = -\frac{T}{2}\int d^{p+1}\xi \ \sqrt{-\gamma}\left(\gamma^{\mu\nu}\partial_\mu X^a\partial_\nu X^b\eta_{ab} - (p-1)\right)$$

**3.** Show that the Polyakov action for the string is reparameterization invariant.

**4.** The Polyakov string is invariant under translational and Lorentz symmetries,

$$X^a \to \Lambda^a{}_b X^b + c^a$$

Show that, in conformal gauge, the Noether currents associated to these global symmetries on the worldsheet are

$$P_a^\alpha = T\partial^\mu X_a \quad \text{and} \quad J^\mu_{ab} = P^\mu_a X_b - P^\mu_b X_a$$

Write down the conserved charge associated with Lorentz transformations in terms of the modes of the string and interpret the result.

**5.** Assuming the commutation relations for $x^a$, $p^a$ and $\alpha^a_n$ in the mode expansion for the open string, compute the commutation relations for $X^a(\sigma)$ and $\Pi_b(\sigma')$.

**6** Consider the following trajectory of an open string

$$X^0 = B\tau,\ X^1 = B\cos\tau\ \cos\sigma,\ X^2 = B\sin\tau\ \cos\sigma,\ X^i = 0\, i > 2. \tag{4}$$

and assume the conformal gauge condition.

Show that this configuration describes a solution of the equations of motion for an open string with Neumann boundary conditions. Show also that the ends of the string are moving at the speed of light.

Assuming the energy of the string is given by $P^0$, show that the angular momentum $J$ obeys

$$\alpha' E^2 = |J|. \tag{5}$$

Show that the energy-momentum of the string vanishes.

**7** Assume that the classical trajectory of an open string is given by

$$X^0 = 3A\tau,\ X^1 = A\cos 3\tau\ \cos 3\sigma,\ X^2 = A\sin a\tau,\ \cos b\sigma. \tag{6}$$

What values of $a$ and $b$ are consistent with the vanishing of the string energy momentum tensor?

What the energy and angular momentum of these string configurations and what is the relation between them?

**8** The generators of angular momentum for the open string are given in terms of the mode expansion by

$$J^{ab} = x_0^a p^b - x_0^b p^a - i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha^a_{-n}\alpha^b_n - \alpha^b_{-n}\alpha^a_n). \tag{7}$$

Show that these generators obey the Poincare algebra.

**9** Show that the Virasoro generators commute with the generators of the Lorentz algebra.

**10** Consider an open string state given by

$$|\phi\rangle = (A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C(\alpha_0\cdot\alpha_{-1})^2)|0,k\rangle. \tag{8}$$

What are the conditions on $A, B$ and $C$ such that this state obeys the physical state and mass-shell conditions for $a = 1$ and arbitrary $D$. Explain what conclusions you can draw from your results

**11** Consider the states of the closed bosonic string in the critical dimension that describe the first massive excited state. Describe how they fit into multiplets of the little group $SO(25)$.