Mathematical Tripos Part III
String Theory. Lent 2018: Example Sheet 2

1 Show that the NG equations of motion for a closed string are satisfied by

$$X^0 = t\,, \qquad \vec{X} = \frac{1}{2}\left[\vec{A}(\sigma + t) + \vec{B}(\sigma - t)\right] ,$$

where the $(D-1)$-vector functions $\vec{A}$ and $\vec{B}$ are such that $\left|\vec{A}'\right|^2 = \left|\vec{B}'\right|^2 = 1$. Show that the induced worldsheet metric is

$$g_{\mu\nu} = \frac{1}{2}\left(1 + \vec{A}' \cdot \vec{B}'\right) \eta_{\mu\nu}\,.$$

Using the fact that

$$\oint d\sigma\, \vec{A}' = 0\,, \qquad \oint d\sigma\, \vec{B}' = 0\,,$$

deduce that the vector functions $\vec{A}'$ and $-\vec{B}'$ trace out (as $\sigma \to \sigma + 2\pi$) two closed curves on the unit $(D-2)$-sphere with centre of gravity at the origin of $(D-1)$-space. Use this to argue that intersections of these curves are inevitable when $D = 4$, the case of relevance to cosmic strings. Show that $\vec{X}' = 0$ at such intersections, i.e. at points on the string for which $\vec{A}' = -\vec{B}'$. These points are *cusps*. Why must cusps move at the speed of light?

2 The null (tensionless) closed string has the action

$$I = \int dt \oint d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e\, P^2 - u\, X' \cdot P \right\} .$$

Assuming periodicity with period 2π, and using the Fourier series expansions

$$X = \sum_k e^{-ik\sigma} X_k(t)\,, \quad P = \frac{1}{2\pi}\sum_k e^{-ik\sigma} P_k(t)\,, \quad e = 2\pi \sum_k e^{-ik\sigma} e_k(t)\,, \quad u = \sum_k e^{-ik\sigma} u_k(t)\,,$$

show that the Fourier space action is

$$I = \int dt \sum_k \left\{ \dot{X}_{-k} \cdot P_k - e_{-k} K_k - u_{-k} L_k \right\} , \qquad K_k = \frac{1}{2}\sum_n P_n \cdot P_{k-n}\,, \qquad L_k = -i \sum_n n X_n \cdot P_{k-n}\,.$$

What are the non-zero Poisson Brackets of the phase-space variables? Use your result to show that

$$\{L_j, L_k\}_{PB} = -i\,(j-k)\,L_{j+k}\,, \qquad \{L_j, K_k\}_{PB} = -i\,(j-k)\,K_{j+k}\,.$$

Why might you have expected this result?

3 When expressed in terms of Fourier modes, the Lorentz Noether charges of the open string with free ends are,

$$\mathcal{J}^{mn} = L^{mn} + S^{mn}\,, \qquad L^{mn} = 2x^{[m}p^{n]}\,, \quad S^{mn} = \sum_{k\neq 0} \frac{i}{k} \alpha_k^{[m} \alpha_{-k}^{n]}\,.$$

Write down the components of L^{mn} in light-cone coordinates $\{x^+, x^-, x^I\}$ and use the canonical commutation relations to show that $[L^{-I}, L^{-J}] = 0$. Hence deduce that Lorentz invariance requires $[S^{-I}, S^{-J}] = 0$. Why is the factor of i in S^{mn} needed? Does this expression suffer from an operator ordering ambiguity in the quantum theory?

4 A Nambu-Goto string has one end fixed at the origin. The other end is free. Find the Fourier series expansions of the phase space D-vector variables (X, P) for these boundary conditions.

5 A 2-form potential B with 3-form field-strength $H = dB$, in a D-dimensional Minkowski spacetime, has Lagrangian density

$$\mathcal{L} = -\frac{1}{12} H_{mnp} H^{mnp} \,, \qquad \left(H_{mnp} = \partial_m B_{np} + \partial_p B_{mn} + \partial_n B_{pm} \equiv 3\partial_{[m} B_{np]}\right) \,.$$

Verify invariance under the gauge transformation $B \to B + da$, where a is a 1-form parameter, i.e. $B_{mn} \to B_{mn} + 2\partial_{[m} a_{n]}$.

Write the Minkowski metric in light-cone coordinates (X^+, X^-, X^I) $(I = 1, \ldots, D-2)$. Given invertibility of ∂_-, verify that (i) it is possible to choose a gauge in which

$$B_{-m} = 0 \,.$$

Show that it is possible, in this gauge, to eliminate the components B_{+I} and that the resulting Lagrangian density can be written (omitting total derivatives) as

$$\mathcal{L} = \frac{1}{4} B_{IJ} \Box B_{IJ} \,.$$

Hence deduce that the the physical polarizations of B transform as a 2nd-rank antisymmetric tensor of the transverse rotation group $SO(D-2)$. How would you expect this result to generalize to a $(p+1)$-form potential, with $(p+2)$-form field strength, for $p > 1$?

6 The Fourier-space action for an open string with free ends is

$$I = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum_n \lambda_{-n} L_n \right\} \,,$$

where

$$L_n = \frac{1}{2} \sum_k \alpha_k \cdot \alpha_{n-k} \,, \qquad \alpha_0 = \frac{p}{\sqrt{\pi T}} \,.$$

Verify that $\{\alpha_k, L_n\}_{PB} = -ik\,\alpha_{k+n}$, and hence that the non-zero gauge transformations generated by the constraint functions are

$$\delta x = \frac{1}{\sqrt{\pi T}} \sum_n \xi_{-n}\, \alpha_n \,, \qquad \delta\alpha_k = -ik \sum_n \xi_{-n}\, \alpha_{k+n} \,,$$

where $\xi_n(t)$ are the parameters. Now verify that $\{L_m, L_n\}_{PB} = -i(m-n)L_{m+n}$, and hence that the action is invariant provided that

$$\delta\lambda_n = \dot{\xi}_n + i \sum_k (2k - n)\, \xi_k\, \lambda_{n-k} \,.$$

7 In D spacetime dimensions, a symmetric traceless tensor of rank J describes particles of "spin" J. Let $J_{max}(N)$ be the maximum spin at level N of the open Nambu-Goto string, with free ends. Show that $J_{max}(N) = \alpha' M^2$, where M^2 is the invariant mass-squared, and α' is the Regge slope parameter: $\alpha' = 1/(2\pi T)$. Now show that $J_{max}(N) = (\alpha'/2)M^2$ for the closed Nambu-Goto string.

8 For an open string with free-end boundary conditions, in D spacetime dimensions, and quantized in light-cone gauge, write down the states at level 2. What is the mass of these states in terms of the string tension T? What are their $SO(D-2)$ representations? Show that they can be assembled into a unique irreducible non-trivial representation of $SO(D-1)$. How can you conclude from your result that there is no Lorentz scalar particle in the spectrum at level 2?

Now write down the states at level 3. How can these be assembled into a unique, but reducible, representation of $SO(D-1)$?