Mathematical Tripos Part III
String Theory 2018: Example Sheet 3

1 Given $L_n = \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{n-k}$, use the canonical commutation relations to show that $[L_n, \alpha_k] = -k\alpha_{k+n}$, and hence verify (without changing the operator ordering) that

$$[L_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}} \left[k\alpha_{n+k}\cdot\alpha_{m-k} + (m-k)\,\alpha_k\cdot\alpha_{m+n-k}\right] .$$

Show that the substitution $k \to k+m$ in the first term leads (apparently) to the result that $[L_m, L_n] = (m-n)L_{m+n}$. To analyse the $n+m=0$ case, first split the sum over k as $\sum_{k\in\mathbb{Z}} = \sum_{k<0} + \sum_{k=0}^{m} + \sum_{k>m}$. Next, assuming (without loss of generality) that $m>0$, show that the sum over $k<0$ does not contribute to $\langle 0|[L_m, L_{-m}]|0\rangle$, where $|0\rangle$ is the oscillator vacuum. Then use the identity

$$k\,\alpha_{k-m}\cdot\alpha_{-k+m} + (m-k)\,\alpha_k\cdot\alpha_{-k} \equiv k\,\alpha_{-k+m}\cdot\alpha_{k-m} + (m-k)\,\alpha_{-k}\cdot\alpha_k$$

to show that the sum over $k>m$ does not contribute either. Hence show that

$$\langle 0|\,[L_m, L_{-m}]\,|0\rangle = m\alpha_0^2 + \frac{D}{2}\sum_{k=1}^{m-1} k(m-k)\,.$$

Use the identities $\sum_{k=1}^{m-1} k = \frac{1}{2}m(m-1)$ and $\sum_{k=1}^{m-1} k^2 = \frac{1}{6}m(m-1)(2m-1)$ to put this result in the form

$$\langle 0|\,[L_m, L_{-m}]\,|0\rangle = m\alpha_0^2 + \frac{D}{12}m(m^2-1)\,.$$

Deduce that the L_n span a Virasoro algebra with central charge $c = D$.

2 In the "old covariant" approach to quantization of the open string with free ends, the general Lorentz-scalar state of momentum p at level 2 is $|p\rangle \otimes |\phi\rangle$, where the oscillator Fock space state $|\phi\rangle$ takes the form

$$|\phi\rangle = [A\,\alpha_{-1}^2 + B\,\alpha_0\cdot\alpha_{-2} + C\,(\alpha_0\cdot\alpha_{-1})^2]|0\rangle\,,$$

for constants A, B, C, where $|0\rangle$ is the Fock space vacuum. What is the value of α_0^2 for this state? Show that $|\phi\rangle$ satisfies the Virasoro conditions if $5B = (D-1)A$ and $10C = (D+4)A$, where D is the spacetime dimension. Now show, for $A=1$, that

$$\langle\phi|\phi\rangle = -(2/25)\,(D-1)\,(D-26)\ .$$

What can you conclude from this result?

3 Let $A = \sum_n e^{in\sigma}A_n(t)$ be a function on the phase space of the closed NG string. Given that $\delta_\xi A = -\sum_m \xi_{-m}\{L_m, A\}_{PB}$ for parameter $\xi = \sum_n e^{in\sigma}\xi_n(t)$, and

$$\{L_m, A_n\}_{PB} = -i\,[m(h-1)-n]\,A_{n+m}\,,$$

show that $-\delta_\xi A = \xi A' + h\xi' A$. Hence deduce that if $A = A_-$ and $\xi = \xi^-$, both depending *only* on σ^-, then

$$\delta_{\xi^-}A_- = \xi^-\partial_- A_- + h(\partial_-\xi^-)A_-\,.$$

What is special about the $h=1$ case?

4 The FP ghost action for the open string (with free ends) is

$$I_{FP} = 2i \int dt \int_0^\pi d\sigma \left\{ b\,\partial_+ c + \tilde{b}\partial_- \tilde{c} \right\} .$$

Show that the boundary conditions on (b, c) must be such that $b\delta c = \tilde{b}\delta\tilde{c}$ at the endpoints. Verify that this condition is satisfied by the boundary conditions

$$(\tilde{b} - b)|_{\text{ends}} = 0 , \qquad (\tilde{c} - c)|_{\text{ends}} = 0 ,$$

and verify that these boundary conditions are satisfied by the Fourier series expansions

$$\tilde{c}(-\sigma) = c(\sigma) = \sum_{n\in\mathbb{Z}} e^{in\sigma} c_n , \qquad \tilde{b}(-\sigma) = b(\sigma) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{in\sigma} b_n .$$

Hence show that the open string "quantum" action is

$$I_{\text{qu}} = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} + \sum_{n\in\mathbb{Z}} i b_{-n} \dot{c}_n - H_{\text{qu}} \right\}$$

where $H_{\text{qu}} = L_0 + N_{\text{gh}}$ with

$$N_{\text{gh}} = \sum_{k=1}^{\infty} k \left(b_{-k} c_k + c_{-k} b_k \right)$$

5 Let (b_k, c_k) be anticommuting variables with Poisson brackets $\{b_{-k}, c_k\}_{PB} = -i$, and let

$$L_m = \sum_{k\in\mathbb{Z}} [(J-1)m - k]\, b_{m+k} c_{-k}$$

for some number J. Show that

$$\{L_m, c_n\}_{PB} = i\,(Jm + n)\, c_{n+m}, \qquad \{L_m, b_n\}_{PB} = -i\,[(J-1)m - n]\, b_{n+m} .$$

and hence that

$$\{L_m, L_n\}_{PB} = -i(m-n) L_{m+n} .$$

Now quantise, and show that

$$[L_m, L_{-m}] = \left(\sum_{k<0} + \sum_{k=0}^{m} + \sum_{k>m} \right) (k - Jm)[k + (J-1)m] \left(b_k c_{-k} - b_{-m+k} c_{m-k} \right) .$$

Using the identity $b_k c_{-k} - b_{-m+k} c_{m-k} \equiv c_{m-k} b_{-m+k} - c_{-k} b_k$, and assuming that $m > 0$, show that

$${}_{\text{gh}}\langle 0 | [L_m, L_n] | 0 \rangle_{\text{gh}} = -\frac{1}{6} m \left[\left(6J^2 - 6J + 1 \right) m^2 - 1 \right] ,$$

where $|0\rangle_{\text{gh}}$ is annihilated by both b_k and c_k for $k > 0$. Hence deduce that

$$[L_m, L_{-m}] = 2m(L_0 - a) + \frac{c}{12} m(m^2 - 1) , \qquad a = \frac{1}{2} J(J-1) , \qquad c = -2\left(6J^2 - 6J + 1 \right) .$$

Comment on the $J = 2$ case. Why are a and c unchanged if $J \to 1 - J$? Comment on the $J = 1/2$ case. What is c when $J = 1/2$?