String Theory: Example Sheet 3

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1. The "Polyakov-style" action for a massive relativistic point particle, involving the Minkowski space coordinate $X^\mu(\tau)$ and the einbein $e(\tau)$, is given by

$$S = \frac{1}{2}\int d\tau \;\left(e^{-1}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - em^2\right)$$

a. Show that this action has the reparameterization invariance $\tau \to \tilde{\tau} - \eta(\tau)$, to linear order in η, with

$$\delta e \;=\; \frac{d}{d\tau}(\eta(\tau)e) \quad , \quad \delta X^\mu = \frac{dX^\mu}{d\tau}\eta(\tau)$$

b. Consider a path starting at $X^\mu(\tau_1) = X_1^\mu$ and finishing at $X^\mu(\tau_2) = X_2^\mu$. Show that reparameterization invariance allows the choice of gauge

$$e(\tau) = \frac{l}{\tau_2 - \tau_1}$$

where l is the invariant length of the worldline.

c. The Feynman propagator for a massive scalar particle defined by the path integral,

$$G(X_1 - X_2) = \mathcal{N}\int De\, DX\; e^{iS[X,e]}$$

with $\mathcal{N}$ the usual normalization constant. By transforming to the gauge $e = l/(\tau_2 - \tau_1)$, together with an appropriate redefinition of the parameter τ, show that

$$G(X_1 - X_2) = \mathcal{N}' \int_0^\infty dl\, DX\; \exp\left(\frac{i}{2}\int_0^l d\tau\; (\dot{X}^2 - m^2)\right)$$

Why does the integral over the length of path l remain? (You may find the discussion in Polchinski chapter 5.1 useful to answer this).

d. Compute the functional integral over X to deduce that the propagator in momentum space is given by

$$\tilde{G}(p) = \frac{1}{p^2 + m^2}$$

2. The scattering amplitude for m closed string tachyons is given by,

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))}\ \delta^{26}(\sum_i p_i)\ \int \prod_{i=1}^{m} d^2 z_i\ \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l}$$

a. Show that the integral is invariant under the $SL(2;\mathbf{C})$ transformation

$$z_i \to \frac{a z_i + b}{c z_i + d}$$

only when the momenta are on-shell, i.e. $p_i^2 = 4/\alpha'$.

b. Explain why this means that the 4-point amplitude can be reduced to the integral

$$\mathcal{A}^{(4)} \sim g_s^2\ \delta^{26}(\sum_i p_i)\ \int d^2 z\ |z|^{\alpha' p_2 \cdot p_3} |1 - z|^{\alpha' p_3 \cdot p_4}$$

c. Evaluate this integral in terms of gamma functions. Show that, when written in Mandelstam variables, it is given by the Virasoro-Shapiro amplitude

$$\mathcal{A}^{(4)} \sim g_s^2\ \frac{\Gamma(-1 - \alpha' s/4)\Gamma(-1 - \alpha' t/4)\Gamma(-1 - \alpha' u/4)}{\Gamma(2 + \alpha' s/4)\Gamma(2 + \alpha' t/4)\Gamma(2 + \alpha' t/4)}$$

3. Explain why the limit $s \to \infty$, with t fixed corresponds to small angle scattering at high energy. Show that in this limit the Virasoro-Shapiro amplitude exhibits so-called Regge behaviour,

$$\mathcal{A}^{(4)} \to\ g_s^2\ \delta^{26}(\sum_i p_i)\ \frac{\Gamma(-1 - \alpha' t/4)}{\Gamma(2 + \alpha' t/4)}\ s^{2 + \alpha' t/2}$$

4a. Write down the vertex operator for a massless closed string state with polarization $\zeta_{\mu\nu}$ and momentum p^μ. What are the restrictions on p^μ and $\zeta_{\mu\nu}$?

b. Consider the scattering of a massless closed string mode with momentum p_1 and two tachyons with momentum p_2 and p_3. Show that $p_1 \cdot p_2 = p_1 \cdot p_3 = 0$ and $p_2 \cdot p_3 = -4/\alpha'$.

c. Show that the 3-point scattering amplitude for these particles is given by

$$\mathcal{A}^{(3)} \sim \frac{g_s}{\mathrm{Vol}(SL(2;\mathbf{C}))}\,\delta^{26}(\sum_i p_i)\,\int \prod_{i=1}^{3} d^2 z_i\,\frac{1}{|z_{23}|^4}\,\zeta_{\mu\nu}\left(\frac{p_2^\mu}{z_{12}}+\frac{p_3^\mu}{z_{13}}\right)\left(\frac{p_2^\nu}{\bar{z}_{12}}+\frac{p_3^\nu}{\bar{z}_{13}}\right)$$

where $z_{ij} = z_i - z_j$.

d. Explain why the $SL(2;\mathbf{C})$ gauge symmetry allows us to simplify this to

$$\mathcal{A}^{(3)} \sim g_s \delta^{(26)}(\sum_i p_i)\ \zeta_{\mu\nu}(p_2^\mu - p_3^\mu)(p_2^\nu - p_3^\nu)$$

5a. After using $SL(2;\mathbf{C})$ to fix the positions of 3 vertex operators, the tree-level m-point amplitude for tachyon scattering reduces to an integral over the positions of the remaining $m-3$ vertex operator insertions,

$$\mathcal{A}^{(4)} \sim g_s^{m-2}\,\delta^{26}(\sum_i p_i)\,\int \prod_{i=4}^{m} d^2 z_i\,\prod_{j\neq l} |z_{jk}|^{\alpha' p_j\cdot p_l}$$

where $z_{jl} = z_j - z_l$. The variables describing the exchange of momentum are $s_{ij} = -(p_i + p_j)^2$. The hard scattering limit is defined by $s_{ij} \to \infty$. Explain why the integral can be evaluated using a saddle-point approximation in this limit.

b. For the 4-point amplitude, use the saddle point approximation to show that

$$\mathcal{A}^{(4)} \sim g_s^2\,\delta^{26}(\sum_i p_i)\ \exp\left(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right)$$

6. The low-energy effective action in string frame is given by

$$S = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\,\partial^\mu\Phi\right) \qquad (1)$$

Show that the equations of motions for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are equivalent to the vanishing of the beta functions

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu{}^{\lambda\kappa} \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi\,H_{\lambda\mu\nu} \\
\beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\,\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}
\end{aligned}$$

7. Consider the string frame action (1) in D spacetime dimensions. Show that, when written in terms of the Einstein frame metric

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}\, G_{\mu\nu}(X)$$

the low-energy effective action becomes

$$S = \frac{1}{2\kappa^2}\int d^DX\,\sqrt{-\tilde{G}}\left(\tilde{R} - \frac{1}{12}e^{-8\tilde{\Phi}/D-2}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\right)$$

where $\kappa^2 = \kappa_0^2\, e^{2\Phi_0}$ and $\Phi = \Phi_0 + \tilde{\Phi}$.

8a. The string frame metric produced by N infinite static strings lying in the $(X^0, X^1) \equiv (t, x)$ direction is

$$ds^2 = f(r)^{-1}(-dt^2 + dx^2) + d\vec{X}\cdot d\vec{X}$$

where $\vec{X} = (X_2, \ldots, X_{25})$ labels the space transverse to the string and

$$f(r) = 1 + \frac{g_s^2 N l_s^{22}}{r^{22}}$$

with $r^2 = \vec{X}\cdot\vec{X}$. Consider one further infinite probe string in this background, lying parallel to the others. Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = R\tau$ and $x = R\sigma$, the low-energy excitations of the string are governed by the effective action,

$$L \approx T\int dtdx\,\left[-f(r)^{-1} + \frac{1}{2}\left(\frac{d\vec{X}}{dt}\cdot\frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx}\cdot\frac{d\vec{X}}{dx}\right) + \ldots\right]$$

Interpret this result.

8b. Now include the coupling of the probe string to background B-field, which is given by

$$B_{01} = f(r)^{-1} - 1$$

Show that the probe string, suitably oriented and lying parallel to the initial strings, feels no static force.