Mathematical Tripos Part III
String Theory 2018: Example Sheet 4

1 Let $|\pm\rangle$ be two states such that $b|+\rangle = |-\rangle$ and $c|-\rangle = |+\rangle$. Given that $b^2 = c^2 = 0$, show that $b|-\rangle = c|+\rangle = 0$, and deduce that $\{b, c\} = 1$. Show that both b and c are hermitian with respect to the inner product defined (over the reals) by

$$\langle +|+\rangle = \langle -|-\rangle = 0 \,, \qquad \langle +|-\rangle = \langle -|+\rangle = 1 \,.$$

Show also that the norm-squared implied by this inner product is

$$||(\alpha|-\rangle + \beta|+\rangle)||^2 = 2\alpha\beta \,, \qquad (*)$$

where α and β are arbitrary real numbers.

In the realization of the anti-commutation relation $\{b, c\} = 1$ for which $b = \partial/\partial c$, show that the state $\alpha|-\rangle + \beta|+\rangle$ is realized by the wavefunction $\psi(c) = \alpha + c\beta$. Show that the norm defined by $(*)$ is equivalent to

$$||\psi(c)||^2 = \frac{\partial}{\partial c}\left[\psi^2\right] \,.$$

2 Let n_{-1} be the (complex) dimension of the space of globally defined analytic vector fields $\xi(z)\partial_z$ on a given Riemann surface (without boundary) of genus g (number of "holes"), and let n_2 be the dimension of the space of analytic quadratic differentials $h(z)dz^2$. For the Riemann sphere ($g = 0$) show that $n_{-1} = 3$ and that $n_2 = 0$. By choosing a suitable basis for the analytic vector fields, verify that their commutation relations are those of $sl(2; C)$. Now find n_{-1} and n_2 for the flat torus ($g = 1$) defined by the identifications $z \sim z + n + im$ for any integers (n, m).

For both the sphere and the torus, verify that $n_2 - n_{-1} = 3(g - 1)$. This is true for any Riemann surface of genus g as a consequence of the Riemann-Roch theorem. Given that there are no conformal Killing vector fields on a Riemann surface with $g \geq 2$, how many independent analytic quadratic differentials are there? [*These give gauge-invariant deformations of a given conformally flat metric, so that the general metric depends on n_2 complex "moduli" over which we must integrate in the path-integral representation of amplitudes.*]

3 The Virasoro amplitude is

$$A(s,t) = \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(3+s+t)}{\Gamma(t+2)\Gamma(s+2)\Gamma(-2-s-t)}$$

Stirling's approximation to $\Gamma(z+1)$, valid for large $|z|$ away from the negative real axis, is $\Gamma(z+1) = \exp[z\ln z - z - \frac{1}{2}\ln z + \mathcal{O}(1)]$. Assuming that corrections to this formula from the poles of $\Gamma(z+1)$ on the negative real axis can be ignored (they produce oscillations that do not contribute "on average"), show that

$$A(s,t) \approx f(t)s^{2(t+1)} \qquad \text{as} \;\; s \to \infty \;\; \text{for fixed } t \,,$$

for some function $f(t)$. This is the *Regge limit* of the amplitude. Show that for large s,

$$2t \approx -s(1 - \cos\theta_s) \qquad (*)$$

where θ_s is the scattering angle. Hence deduce that $\theta_s \to 0$ in the Regge limit.

4 The Veneziano amplitude, for the scattering of two open string tachyons, with $p^2 = 2\pi T$, is

$$A(s,t) = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)} \,,$$

where

$$s = -\frac{(p_1+p_2)^2}{2\pi T} \,, \qquad t = -\frac{(p_1+p_3)^2}{2\pi T} \,.$$

The incoming tachyons have D-momenta p_1 and p_2 and the outgoing ones have D-momenta $-p_3$ and $-p_4$. Find the positions of the poles of $A(s,t)$ as a function of s for fixed t. What is their interpretation? Find the residues at the poles. What does the result tell you about the spectrum of the open string? Does it agree with what is found by light-cone gauge quantization of the open NG string with free ends?

The *hard scattering* limit is the large s limit in which θ_s is kept fixed. Using $(*)$ and Stirling's approximation to $\Gamma(z+1)$, show that the Veneziano amplitude in the hard scattering limit is given by the asymptotic formula

$$A \sim [F(\theta_s)]^{-s} \,, \qquad F = \left[\sin^2(\theta_s/2)\right]^{-\sin^2(\theta_s/2)} \left[\cos^2(\theta_s/2)\right]^{-\cos^2(\theta_s/2)} \,.$$

Conclude that the hard scattering amplitude falls exponentially fast to zero with increasing s. [*This disagrees with experimental results for hadron scattering, which are explained by QCD. This disagreement ended the hope that perturbative string theory might be a theory of the strong interactions.*]

5 The Fourier space action for the NS sector of the open spinning string is

$$S = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k} \cdot \dot{\alpha}_k + i \sum_{r=1/2}^{\infty} b_{-r} \cdot \dot{b}_r - \sum_n \lambda_{-n} L_n - \sum_s \chi_{-s} G_s \right\} ,$$

where

$$L_n = \frac{1}{2} \sum_k \alpha_k \cdot \alpha_{n-k} + \frac{1}{2} \sum_r r b_{n-r} \cdot b_r \,, \qquad G_r = \sum_k \alpha_k \cdot b_{r-k} \,, \qquad \left(\alpha_0 = p/\sqrt{\pi T}\right) .$$

Write down the non-zero Poisson bracket relations for the canonical variables and verify that

$$\{L_m, L_n\} = -i(m-n)L_{m+n} \qquad \{L_m, G_r\}_{PB} = -i(\frac{m}{2} - r)G_{r+m} \,, \qquad \{G_r, G_s\}_{PB} = -2iL_{r+s} \,.$$

Now show that the gauge transformations of the canonical variables generated by the linear combination of constraint functions $\sum_n \xi_{-n} L_n + i \sum_r \epsilon_{-r} G_r$, where the parameters ϵ_r are anticommuting, is

$$\delta\alpha_k = -ik \sum_n \xi_n \alpha_{k-n} + k \sum_r \epsilon_r b_{k-r} \,, \qquad \delta b_r = -i \sum_m \left(r - \frac{m}{2}\right) \xi_m b_{r-m} - \sum_s \alpha_{r-s} \epsilon_s \,.$$

Given that $p_- \neq 0$, verify that all gauge transformations are fixed, excepting the transformation with parameter ξ_0, by the gauge-fixing conditions $\alpha_n^+ = 0$,for $n \neq 0$ and b_r^+ . Verify that this choice allows the constraints to be solved for α_n^- for $n \neq 0$ and b_r^-. Write down the gauge-fixed action, and show that the remaining mass-shell constraint is $p^2 + \mathcal{M}^2 = 0$ with $\mathcal{M}^2 = 2\pi T\,(N_{\rm bose} + N_{\rm fermi} - a)$, where $N_{\rm bose}$ and $N_{\rm fermi}$ are, respectively, the level numbers for the Bose and Fermi oscillators, and a is a constant to allow for operator ordering ambiguities. Why does Lorentz invariance of the quantum theory require $a = 1/2$? Show that the sum of zero point energies of the Bose and Fermi oscillators is, formally, $\zeta(-1,0) - \zeta(-1,1/2)$, where $\zeta(s,q) = \sum_{n=0}^{\infty}(n+q)^{-s}$. Using the fact that $\zeta(-1,q) = -(6q^2 - 6q + 1)/12$, and assuming $-a$ to be the sum of zero point energies, "deduce" that $D = 10$.