# String Theory: Example Sheet 4

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**1.** The low-energy effective action in string frame is given by

$$S = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\,\left(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\,\partial^\mu\Phi\right) \tag{1}$$

Show that the equations of motions for $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ are equivalent to the vanishing of the beta functions

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu{}^{\lambda\kappa} \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi\,H_{\lambda\mu\nu} \\
\beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\,\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}
\end{aligned}$$

**2.** Consider the string frame action (1) in $D$ spacetime dimensions. Show that, when written in terms of the Einstein frame metric

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}\,G_{\mu\nu}(X)$$

the low-energy effective action becomes

$$S = \frac{1}{2\kappa^2}\int d^DX\,\sqrt{-\tilde{G}}\left(\tilde{R} - \frac{1}{12}e^{-8\tilde{\Phi}/D-2}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\right)$$

where $\kappa^2 = \kappa_0^2\,e^{2\Phi_0}$ and $\Phi = \Phi_0 + \tilde{\Phi}$.

**3a.** The string frame metric produced by $N$ infinite static strings lying in the $(X^0, X^1) \equiv (t, x)$ direction is

$$ds^2 = f(r)^{-1}(-dt^2 + dx^2) + d\vec{X}\cdot d\vec{X}$$

where $\vec{X} = (X_2, \ldots, X_{25})$ labels the space transverse to the string and

$$f(r) = 1 + \frac{g_s^2 N l_s^{22}}{r^{22}}$$

with $r^2 = \vec{X}\cdot\vec{X}$. Consider one further infinite probe string in this background, lying parallel to the others. Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = R\tau$ and $x = R\sigma$, the low-energy excitations of the string are governed by the effective action,

$$L \approx T \int dtdx \left[ -f(r)^{-1} + \frac{1}{2}\left( \frac{d\vec{X}}{dt} \cdot \frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx} \cdot \frac{d\vec{X}}{dx} \right) + \ldots \right]$$

Interpret this result.

**3b.** Now include the coupling of the probe string to background $B$-field, which is given by

$$B_{01} = f(r)^{-1} - 1$$

Show that the probe string, suitably oriented and lying parallel to the initial strings, feels no static force.

**4.** Consider an open string whose ends are constrained to lie on a D$p$-brane with a background field strength $F_{ab}$ turned on. Show that the Neumann boundary conditions for the string must be replaced by

$$\partial_\sigma X^a - 2\pi\alpha' \, F^{ab} \, \partial_\tau X_b = 0$$

**5a.** Show that the Born-Infeld Lagrangian can be written in the form,

$$\mathcal{L}_{BI} \equiv \sqrt{\det(1 + F)} = \exp\left( \frac{1}{4} \mathrm{tr} \, \ln(1 - F^2) \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, 1 means the unit matrix, and we have set $2\pi\alpha' = 1$.

**5b.** Show that the equations of motion arising from the Born-Infeld action are equivalent to the beta function condition for the open string,

$$\beta_\sigma(F) = \left( \frac{1}{1 - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma} = 0$$

**Note:** To do this, it will prove very useful if you can first show the following results:

$$\partial_\mu \left[ \mathrm{tr} \, \ln(1 - F^2) \right] = - \left( \frac{F}{1 - F^2} \right)^{\mu\nu} \partial_\rho F_{\mu\sigma} \left( \frac{F}{1 - F^2} \right)^{\sigma\rho}$$

which requires use of the Bianchi identity for $F_{\mu\nu}$ and

$$\partial_\mu \left( \frac{F}{1 - F^2} \right)^{\mu\nu} = \left( \frac{F}{1 - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left( \frac{F}{1 - F^2} \right)^{\sigma\nu} + \left( \frac{1}{1 - F^2} \right)^{\mu\rho} \partial_\mu F_{\rho\sigma} \left( \frac{1}{1 - F^2} \right)^{\sigma\nu}$$