Answers to String Theory Ex. Sheet 3

1. $L_n = \frac{1}{2}\sum_k \alpha_k \cdot \alpha_{n-k}$ $\qquad [\alpha_k^\mu, \alpha_{-k}^\nu] = k\eta^{\mu\nu}$

$$[L_n, \alpha_k] = \frac{1}{2}\sum_j \alpha_j \cdot [\alpha_{n-j}\, \alpha_k] - \frac{1}{2}\sum_j [\alpha_k, \alpha_j]\cdot \alpha_{n-j}$$

$$= -\frac{1}{2} k\, \alpha_{n+k} - \frac{1}{2} k\, \alpha_{n+k} = \underline{-k\,\alpha_{n+k}}$$

$$[L_m, L_n] = \frac{1}{2}\sum_k \left(\alpha_k \cdot [\alpha_{m-k}, L_n] + [\alpha_k, L_n]\cdot \alpha_{m-k}\right)$$

$$= \frac{1}{2}\sum_k \left[(m-k)\,\alpha_k \cdot \alpha_{m+n-k} + k\,\alpha_{n+k}\cdot\alpha_{m-k}\right]$$

$$= \frac{1}{2}\sum_k (m-k)\,\alpha_k \cdot \alpha_{m+n-k} + \underbrace{\frac{1}{2}\sum_{k-n}(k-n)\,\alpha_k \cdot \alpha_{m+n-k}}_{(?)\ \frac{1}{2}\sum_k (k-n)\,\alpha_k\cdot\alpha_{m+n-k}}$$

$$\Rightarrow \frac{1}{2}(m-n)\sum_k \alpha_k \cdot \alpha_{m+n-k} = (m-n)\, L_{m+n-k}$$

$\underline{m>0}$ $\quad [L_m, L_{-m}] = \frac{1}{2}\left(\sum_{k<0} + \sum_{k=0}^{m} + \sum_{k>m}\right)\left[k\,\alpha_{k-m}\cdot\alpha_{m-k} + (m-k)\,\alpha_k\cdot\alpha_{-k}\right]$

① ② ③

$$\langle 0|①|0\rangle = \frac{1}{2}\sum_{k<0}\left[k\,\cancelto{0}{\langle 0|\alpha_{k-m}\cdot\alpha_{m-k}|0\rangle} + (m-k)\cancelto{0}{\langle 0|\alpha_k\cdot\alpha_{-k}|0\rangle}\right] = 0$$

using given identity

$$\langle 0|③|0\rangle = \frac{1}{2}\sum_{k>m}\left[k\,\cancelto{?}{\langle 0|\alpha_{m-k}\cdot\alpha_{k-m}|0\rangle} + (m-k)\cancelto{0}{\langle 0|\alpha_{-k}\cdot\alpha_k|0\rangle}\right] = 0$$

$$\langle 0|②|0\rangle = \underbrace{\frac{1}{2}\sum_{k=1}^{m} k\,\langle 0|\alpha_{k-m}\cdot\alpha_{m-k}|0\rangle}_{\frac{1}{2}m\,\alpha_0^2} + \underbrace{\frac{1}{2}\sum_{k=0}^{m-1}(m-k)\langle 0|\alpha_k\cdot\alpha_{-k}|0\rangle}_{\frac{1}{2}m\,\alpha_0^2 + \frac{1}{2}\sum_{k=1}^{m-1}(m-k)\langle 0|\alpha_k\cdot\alpha_{-k}|0\rangle}$$

$$= m\,\alpha_0^2 + \frac{D}{2}\sum_{k=1}^{m-1} k(m-k)$$

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but
$$\sum_{k=1}^{m-1} k(m-k) = m\sum_{k=1}^{m-1} k - \sum_{k=1}^{m-1} k^2$$

$$= \frac{m^2}{2}(m-1) - \frac{1}{6}m(m-1)(2m-1)$$

$$= \frac{m(m-1)}{6}\left(3m - 2m+1\right) = \frac{m(m-1)(m+1)}{6}$$

$$\therefore \langle 0|[L_m, L_{-m}]|0\rangle = m\alpha_0^2 + \frac{D}{12}m(m^2-1) \qquad (*)$$

From classical result we know that

$$[L_m, L_n] = (m-n)L_{m+n} + A_m\delta_{m+n} \qquad (+)$$

for some quantum ordering of L_0, e.g.

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k \quad \Rightarrow \quad \langle 0|L_0|0\rangle = \frac{1}{2}\alpha_0^2$$

Then $\langle 0|[L_m, L_{-m}]|0\rangle = m\alpha_0^2 + A_m$.

Comparison with $(*)$ shows that $A_m = \frac{D}{12}m(m^2-1)$

$\Rightarrow$ (+) is Virasoro algebra with $c = D$

2. $\frac{1}{2}\alpha_0^2 + (N-1)$ at level N

$\therefore \boxed{\alpha_0^2 = -2}$ for $N=2$

$$\phi = \left[A\,\alpha_{-1}^2 + B\alpha_0\cdot\alpha_{-2} + C(\alpha_0\cdot\alpha_{-1})^2\right]|0\rangle$$

level 2 state

$$0 = L_1 \phi = \left(A\left[L_1, \alpha_{-1}^2\right] + B\,\alpha_0 \cdot \left[L_1 \cdot \alpha_{-2}\right] + C\left[L_1, (\alpha_0 \cdot \alpha_{-1})^2\right]\right)|0\rangle$$

$$= \left(A\left(\alpha_0 \cdot \alpha_{-1} + \alpha_{-1} \cdot \alpha_0\right) + 2B\,\alpha_0 \cdot \alpha_{-1} + 2C\alpha_0^2\,\alpha_0 \cdot \alpha_{-1}\right)|0\rangle$$

$$= 2\left(A + B + C\alpha_0^2\right)\alpha_0 \cdot \alpha_{-1}|0\rangle$$

$$\Rightarrow \boxed{A + B = 2C} \qquad (\text{using } \alpha_0^2 = -2)$$

$$0 = L_2 \phi = \left(A\left[L_2, \alpha_{-1}^2\right] + B\,\alpha_0 \cdot \left[L_2, \alpha_{-2}\right] + C\left[L_2, (\alpha_0 \cdot \alpha_{-1})^2\right]\right)|0\rangle$$

$$= \left(A\,\alpha_1 \cdot \alpha_{-1} + 2B\,\alpha_0^2 + C\,\alpha_0 \cdot \alpha_1\,\alpha_0 \cdot \alpha_{-1}\right)|0\rangle$$

$$= \left(AD + 2B\alpha_0^2 + C\alpha_0^2\right)|0\rangle$$

$$\Rightarrow AD = 2(2B + C) \qquad (\text{using } \alpha_0^2 = -2)$$

$$= 4B + (A + B)$$

$$\therefore \boxed{A(D-1) = 5B} \qquad \therefore 10C = 5A + A(D-1)$$

$$\text{or } \boxed{10C = A(D+4)}$$

For $A = 1$

$$\phi = \left[\alpha_{-1}^2 + B\,\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2\right]|0\rangle$$

where $B = \frac{D-1}{5}$, $C = \frac{D+4}{10}$

$$\|\phi\|^2 = \langle 0|\left[\alpha_1^2 + \bar{B}\,\alpha_0 \cdot \alpha_2 + C(\alpha_0 \cdot \alpha_1)^2\right]\left[\alpha_{-1}^2 + B\,\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2\right]|0\rangle$$

$$= \underbrace{\langle 0|\left[\alpha_1^2 + C(\alpha_0\alpha_1)^2, \alpha_{-1}^2 + C(\alpha_0 \cdot \alpha_{-1})^2\right]|0\rangle}_{2D + 2C^2(\alpha_0^2)^2 + 4C\alpha_0^2} + B^2 \underbrace{\langle 0|\left[\alpha_0 \cdot \alpha_2, \alpha_0 \cdot \alpha_{-2}\right]|0\rangle}_{2\alpha_0^2}$$

$$\therefore \|\phi\|^2 = 2\left[D + (B^2 + 2C)\alpha_0^2 + C^2(\alpha_0^2)^2_1\right]$$

$$= 2\left[D - 2(B^2 + 2C) + 4C^2\right]$$

$$= 2\left[D - 2\left(\frac{(D-1)^2}{25} + \frac{D+4}{5}\right) + \frac{(D+4)^2}{25}\right]$$

$$= \frac{2}{25}\left[25D - 2(D-1)^2 + (D+4)(D-6)\right]$$

$$= -\frac{2}{25}\left(D^2 - 27D + 26\right) = \underline{-\frac{2}{25}(D-1)(D-26)}$$

$\therefore$ ϕ is a null state for $D = 26$.

N.B. This is consistent with fact that in the light-cone gauge there is <u>no scalar at level 2</u>.

3. $\delta_\xi A = \{A, \sum_m \xi_{-m} L_m\}_{PB}$ is gauge trans. of A.

$$= \sum_n e^{in\sigma} \sum_m \xi_{-m} \{A_n, L_m\}_{PB}$$

$$\sum_m \xi_{-m} \sum_n e^{in\sigma}\, i[m(h-1) - n]\, A_{n+m}$$

$$= i\sum_n \xi_{-m} e^{-im\sigma} \underbrace{\sum_n e^{i(n+m)\sigma}[m(h-1) - n]\, A_{n+m}}_{\underbrace{\sum_n e^{in\sigma}(mh - n) A_n}_{mhA + iA'}}$$

$$= h\left(i\sum_m m\,\xi_{-m} e^{-im\sigma}\right)A - \xi A'$$

$$\delta_\xi A = -h\underbrace{\left(i\sum_m m\,\xi_m e^{im\sigma}\right)}_{\xi'}A - \xi A'$$

$$= -h\xi' A - \xi A'$$

$$\therefore \underline{-\delta_\xi A = \xi A' + h\xi' A}$$

If $A(t,\sigma) = A(\sigma^-) \equiv A_-$ $\quad (\sigma^- = t-\sigma)$
$\quad \xi(t,\sigma) = \xi(\sigma^-) \equiv \xi^-$

then $\delta_{\xi^-} A = \xi^- \partial_- A_- + h(\partial_- \xi^-)A_-$

For $h=1$ $\quad \underline{\delta_{\xi^-} A = \partial_-(\xi^- A_-)}$

4 $\quad I_{FP} = 2i\int dt \int_0^\pi d\sigma \{ b\,\partial_+ c + \tilde{b}\,\partial_- \tilde{c} \}$ $\qquad \partial_\pm = \frac{1}{2}(\partial_t \pm \partial_\sigma)$

$$\delta I_{FP} = 2i\int dt \int_0^\pi d\sigma \{ \delta b\,\partial_+ c + \delta c\,\partial_+ b + \delta\tilde{b}\,\partial_- \tilde{c} + \delta\tilde{c}\,\partial_- \tilde{b} \}$$

$$+\ i\int dt \left\{ \left[b\,\delta c - \tilde{b}\,\delta\tilde{c} \right]_0^\pi + \partial_t(\;) \right\}$$

$\therefore$ b.c.s must be such that $\left[b\,\delta c - \tilde{b}\,\delta\tilde{c} \right]_0^\pi = 0$

If $\begin{cases} \tilde{b} = b \\ \tilde{c} = c \end{cases}$ at $\sigma = 0, \pi$ then $\left.(b\,\delta c - \tilde{b}\,\delta\tilde{c})\right|_{\text{ends}} = 0$ ✓

$\tilde{c}(-\sigma) = c(\sigma) = \sum_n e^{in\sigma} c_n \Rightarrow$ $\tilde{c}(0) = c(0)$ ✓
$\qquad \tilde{c}(\pi) = \sum_n (-1)^n c_n = c(\pi)$ ✓

Illy for $b, \tilde{b}$

$$2i\int_0^{\pi} d\sigma\left(b\partial_+ c + \tilde{b}\partial_-\tilde{c}\right) = \underbrace{i\int_0^{\pi} d\sigma\left(b\dot{c} + \tilde{b}\dot{\tilde{c}}\right)}_{①} + \underbrace{i\int_0^{\pi} d\sigma\left(bc' - \tilde{b}\tilde{c}'\right)}_{②}$$

$$① = \frac{i}{4\pi}\sum_n\sum_m \underbrace{\left[\int_0^{\pi} d\sigma\left[e^{i(n+m)\sigma} + e^{-i(n+m)\sigma}\right]\right]}_{4\pi\delta_{n+m}} b_n\dot{c}_m$$

(N.B. factor of $\frac{1}{4\pi}$ in Fourier series expansion of $b, \tilde{b}$)

$$= i\sum_n b_{-n}\dot{c}_n$$

$$② = -\frac{1}{4\pi}\sum_n\sum_m n\underbrace{\int_0^{\pi} d\sigma\left[e^{i(n+m)\sigma} + e^{-i(n+m)\sigma}\right]}_{4\pi\delta_{n+m}} b_m c_n$$

$$= -2\sum_n n\, b_{-n}c_n = -2\sum_{n>0} n\, b_{-n}c_n + 2\sum_{n<0} n\, c_n b_{-n}$$

$$= -2\sum_{n=1}^{\infty} n\left(b_{-n}c_n + c_{-n}b_n\right)$$

Hence result

5. $$L_m = \sum_{k\in\mathbb{Z}}\left[(J-1)m - k\right] b_{m+k}c_{-k} \qquad (\text{N.B } J = h)$$

and $\{b_{-k}, c_k\}_{PB} = -i$

$$\{L_m, c_n\}_{PB} = -\sum_k\left[(J-1)m-k\right]\overbrace{\{b_{m+k}, c_n\}_{PB}}^{-i\delta_{k+m+n}} c_{-k}$$

$$= i\left[(J-1)m + (m+n)\right]c_{m+n} = i(Jm+n)c_{m+n}$$

$$\Big(\Rightarrow \underset{\text{conf. dim}}{[c]} = 1 - J\Big)$$

$$\{L_m, b_n\}_{PB} = \sum_k \left[(J-1)m-k\right] b_{m+k} \overbrace{\{C_{-k}, b_n\}_{PB}}^{-i\delta_{n-k}}$$

$$= -i\left[(J-1)m-n\right] b_{m+n} \qquad (\Rightarrow [b]=J)$$

$$\{L_m, L_n\}_{PB} = \sum_k \left[(J-1)n-k\right]\left[\{L_m, b_{n+k}\}_{PB} C_{-k} + b_{n+k}\{L_m, C_{-k}\}_{PB}\right]$$

$$= -i\sum_k \left[(J-1)n-k\right]\left\{\left[(J-1)m-n-k\right] b_{n+m+k} C_{-k} - \left[Jm-k\right] b_{n+k} C_{m-k}\right\}$$

$$= -i\sum_k \left[(J-1)n-k\right]\left[(J-1)m-n-k\right] b_{n+m+k} C_{-k}$$

$$+ i\sum_k \left[(J-1)n-m-k\right]\left[(J-1)m-k\right] b_{n+m+k} C_{-k}$$

$$= -i(J-1)(m^2-n^2)\sum_k b_{n+m+k} C_{-k} + i(m-n)\sum_k k\, b_{n+m+k} C_{-k}$$

$$= -i(m-n)\underbrace{\sum_k \left[(J-1)(m+n)-k\right] b_{n+m+k} C_{-k}}_{L_{m+n}}$$

$$= -i(m-n)L_{m+n}$$

In quantum theory:

$$[L_m, C_n] = -(Jm+n)C_{m+n}$$

$$[L_m, b_n] = \left[(J-1)m-n\right] b_{m+n}$$

$$[L_m, L_{-m}] = -\sum_k \left[(J-1)m+k\right]\left([L_m, b_{-m+k}] C_{-k} + b_{-m+k}[L_m, C_{-k}]\right)$$

$$= -\sum_k \left[(J-1)m+k\right]\left[(Jm-k)\, b_k C_{-k} - (Jm-k)\, b_{-m+k} C_{m-k}\right]$$

$$= \sum_k (k-Jm)\left[k+(J-1)m\right]\left(b_k C_{-k} - b_{-m+k} C_{m-k}\right)$$

$$\therefore [L_m, L_{-m}] = \left(\sum_{k<0} + \sum_{k=0}^{m} + \sum_{k>m}\right)(k-Jm)\left[k+(J-1)m\right]\left(b_k c_{-k} - b_{-m+k} c_{m-k}\right)$$

Take $m>0$

$$\sum_{k<0} (k-Jm)\left[k+(J-1)m\right]\langle 0|\left(\cancel{b_k c_{-k}} - \cancel{b_{-m+k} c_{m-k}}\right)|0\rangle = 0$$

using given identity $\rightarrow$ $$\sum_{k>m} (k-Jm)\left[k+(J-1)m\right]\langle 0|\left(\cancel{c_{m-k} b_{k-m}} - \cancel{c_{-k} b_k}\right)|0\rangle = 0$$

$$\sum_{k=0}^{m} (k-Jm)\left[k+(J-1)m\right]\langle 0|\left(b_k c_{-k} - b_{-m+k} c_{m-k}\right)|0\rangle$$

$$= \underbrace{-m^2 J(J-1)\langle 0|\left(b_0 c_0 - \cancel{b_{-m} c_m}\right)|0\rangle}_{k=0 \text{ term}} \underbrace{- m^2 J(J-1)\langle 0|\left(b_m c_{-m} - b_0 c_0\right)|0\rangle}_{k=m \text{ term}}$$

$$+ \sum_{k=1}^{m-1}(k-Jm)\left[k+(J-1)m\right]\langle 0|\left(\underbrace{\{b_k, c_{-k}\}}_{1} - \cancel{b_{-m+k} c_{m-k}}\right)|0\rangle$$

$$= -m^2 J(J-1)\underbrace{\langle 0|\{b_m, c_{-m}\}|0\rangle}_{1} + \sum_{k=1}^{m-1}(k-Jm)\left[k+(J-1)m\right]$$

$$= -m^2 J(J-1) - m^2 J(J-1)\underbrace{\sum_{k=1}^{m-1}}_{m-1} - m\sum_{k=1}^{m-1} k + \sum_{k=1}^{m-1} k^2$$

$$= -m^3 J(J-1) - m\frac{1}{2}m(m-1) + \frac{1}{6}m(m-1)(2m-1)$$

$$= -m^3 J(J-1) - \frac{(m^3-m)}{6} = -(m^3-m)\left[J(J-1)+\frac{1}{6}\right] - mJ(J-1)$$

$$= -mJ(J-1) - \frac{(m^3-m)}{6}\left(6J^2-6J+1\right)$$

$$\therefore \quad \langle 0 | [L_m, L_{-m}] | 0 \rangle = -mJ(J-1) - \frac{(m^3-m)}{6}(6J^2-6J+1) \qquad (*)$$

Compare with

$$[L_m, L_{-m}] = 2m(L_0 - a) + \frac{c}{12}(m^3-m)$$

for operator ordering in L_0 s.t. $L_0|0\rangle = 0$

then

$$\langle 0 | [L_m, L_{-m}] | 0 \rangle = -2ma + \frac{c}{12}(m^3-m)$$

Comparing with $(*)$ we find that

$$\boxed{a = \frac{1}{2}J(J-1) \qquad c = -2(6J^2-6J+1)}$$

e.g. $J=2 \quad a=1, \quad c=-26$ (result for FP ghosts of NG string)

$[ib\partial_+ c] = (1,1)$ if $[b] = (J,0)$ and $[c] = (1-J, 0)$

conf. dims

but $ib\partial_+ c = -i\partial_+ c\, b$ (anticommutativity)

$$= ic\partial_+ b + \partial_+(\;)$$

so roles of b, c are reversed and $J \leftrightarrow 1-J$.
$\Rightarrow$ expression for a, c should be invariant under $J \to 1-J$

$J = \frac{1}{2} \Rightarrow [b] = [c]$ and $c = 1$
If we take $b = c$ then $c = \frac{1}{2}$ (applies to R×NS worldsheet fermions).