String Theory Answers to Ex. Sheet 4

1. $b|+\rangle = |-\rangle$ $\qquad b|-\rangle = b^2|+\rangle = 0$ if $b^2 = 0$

$c|-\rangle = |+\rangle$ $\qquad c|+\rangle = c^2|-\rangle = 0$ if $c^2 = 0$

$$\left.\begin{array}{l} \left.\begin{array}{l} cb|+\rangle = c|-\rangle = |+\rangle \\ bc|+\rangle = 0 \end{array}\right\} \therefore \{c,b\}|+\rangle = |+\rangle \\ \left.\begin{array}{l} bc|-\rangle = b|+\rangle = |-\rangle \\ cb|-\rangle = 0 \end{array}\right\} \therefore \{b,c\}|-\rangle = |-\rangle \end{array}\right\} \therefore \{c,b\} = 1 \text{ (on all states)}$$

$b^\dagger$ is defined by $\langle i|b^\dagger|j\rangle = \langle j|b|i\rangle^*$ for any states $|i\rangle, |j\rangle$

$\therefore$ b is hermitian if $\boxed{\langle j|b|i\rangle^* = \langle i|b|j\rangle}$ (+)

Check this for $\{|i\rangle\} = \{|+\rangle, |-\rangle\}$

$$\boxed{\begin{array}{l} \langle +|b|+\rangle^* = \langle +|-\rangle^* = 1 \\ \langle +|b|+\rangle = \langle +|-\rangle = 1 \end{array}} \checkmark \qquad \boxed{\begin{array}{l} \langle -|b|-\rangle^* = 0 \\ \langle -|b|-\rangle = 0 \end{array}} \checkmark$$

$$\boxed{\begin{array}{l} \langle +|b|-\rangle^* = 0 \\ \langle -|b|+\rangle = \langle -|-\rangle = 0 \end{array}} \checkmark \qquad \boxed{\begin{array}{l} \langle -|b|+\rangle^* = \langle -|-\rangle^* = 0 \\ \langle +|b|-\rangle = 0 \end{array}} \checkmark$$

$\therefore$ b is hermitian for this inner product

Similar argument applies to c

$$\| \alpha|-\rangle + \beta|+\rangle \|^2 = \left(\alpha^*\langle -| + \beta^*\langle +|\right)\left(\alpha|-\rangle + \beta|+\rangle\right)$$

$$= \alpha^*\beta\langle -|+\rangle + \alpha\beta^*\langle +|-\rangle \qquad \text{since } \langle \pm|\pm\rangle = 0$$

$$= (\alpha^*\beta + \alpha\beta^*) \qquad \text{since } \langle +|-\rangle = \langle -|+\rangle = 1$$

$$= 2\alpha\beta \quad \text{for real } \alpha\ \beta$$

1 cont'd

Realize $\{b, c\} = 1$ by $b = \frac{\partial}{\partial c}$ on wavefunctions $\psi(c) = \alpha + c\beta$

$$c\,\psi(c) = c\alpha = 0 \text{ iff } \alpha = 0 \quad \therefore \; c\beta \propto \beta|+\rangle$$

$$b\,\psi(c) = \frac{\partial}{\partial c}\psi(c) = \beta \quad (= 0 \text{ iff } \beta = 0 \quad \text{so} \quad \alpha \propto \alpha|-\rangle$$

Assume normalization is such that $\left\{ \begin{array}{l} c \leftrightarrow |+\rangle \\ 1 \leftrightarrow |-\rangle \end{array} \right\} \Leftrightarrow \boxed{\psi(c) \leftrightarrow \alpha|-\rangle + \beta|+\rangle}$

(without loss of generality).

Then $||\psi(c)||^2$ should equal $||\alpha|-\rangle + \beta|+\rangle||^2 = 2\alpha\beta$ (real α, β)

But $$\frac{\partial}{\partial c}\psi^2 = \frac{\partial}{\partial c}\left[(\alpha + c\beta)(\alpha + c\beta)\right] = \frac{\partial}{\partial c}\left(\alpha^2 + c\,2\alpha\beta\right) = 2\alpha\beta$$

$$\therefore \quad ||\psi(c)||^2 = \frac{\partial}{\partial c}\psi^2 = \int dc\, \psi^2(c)$$

Alternatively $\quad \alpha|-\rangle + \beta|+\rangle \longleftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \langle +|+\rangle & \langle +|-\rangle \\ \langle -|+\rangle & \langle -|-\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv g$

Inner product of column vectors u, v is $\langle u|v\rangle = u^T g v$

$$\therefore \quad b \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad c \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

b is hermitian iff $\langle u|b|v\rangle = \langle v|b|u\rangle^*$ or $u^T g b v = (v^T g b u)^*$

For real u, v, this is equivalent to

$$u^T g b v = v^T g b u \equiv u^T b^T g^T v = u^T b^T g v \quad (\text{since } g^T = g)$$

True $\forall u, v \in \mathbb{R}^2$ iff $\boxed{gb = b^T g}$

$$\text{LHS} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{RHS} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ LHS = RHS $\Rightarrow$ b is hermitian (//ly for c)

2. $\underline{g=0}$ (Riemann sphere) Need two patches $\lfloor z$, $\lfloor w$ $z = 1/w$

$$dz = -\frac{1}{w^2}\,dw \Rightarrow \frac{dw}{dz} = -w^2 \Rightarrow \partial_z = -w^2\partial_w$$

On $\lfloor z$ $\quad \xi(z)\partial_z = \underbrace{\left[\xi_0 + \xi_1 z + \frac{1}{2}\xi_2 z^2 + \frac{1}{6}\xi_3 z^3 + \ldots\right]}_{\text{No negative powers of } z \text{, to avoid singularity at } z=0}\partial_z$

analytic vector field

$$= \left[\xi_0 + \frac{\xi_1}{w} + \frac{\xi_2}{2w^2} + \frac{\xi_3}{6w^3} + \ldots\right](-w^2\partial_w)$$

$$= \Big[\underbrace{-w^2\xi_0 - w\xi_1 - \frac{1}{2}\xi_2}_{\text{non-sing.}} \underbrace{- \frac{\xi_3}{6w} - \ldots}_{\text{singular at } w=0}\Big]\partial_w$$

$\therefore$ ξ_0, ξ_1, ξ_2 are three complex parameters of general non-singular analytic vector field on RS. A basis is

$$\left\{\partial_z,\ z\partial_z,\ \frac{1}{2}z^2\partial_z\right\} \Rightarrow \boxed{n_{-1} = 3}$$

Analytic quadratic differentials $\quad \xi(z)\,dz^2 = \underbrace{\left[\xi_0 + \xi_1 z + \ldots\right]}_{\text{non-sing. at } z=0}dz^2$

$$= \left[\xi_0 + \frac{\xi_1}{w} + \ldots\right]\left(\frac{dz}{dw}\right)^2 dw^2$$

$$= \underbrace{\left[\frac{\xi_0}{w^4} + \frac{\xi_1}{w^5} + \ldots\right]}_{\text{all sing. at } w=0}dw^2$$

$\therefore$ $\underline{n_2 = 0}$

2 cont'd. $g = 1$ (torus) $z \sim z + n + im$ (square torus)

One coord. patch suffices but $\xi(z)$ must be doubly periodic for both $\xi(z)\partial_z$ and $\xi(z)dz^2$ $\Rightarrow$ $\xi(z) = \xi_0$ in both cases

$\therefore$ $n_{-1} = n_2 = 1$

$g=0$ $\quad n_2 - n_1 = 0 - 3 = -3$

$g=1$ $\quad n_2 - n_1 = 0$

agrees with formula $n_2 - n_1 = 3(g-1)$

N.B. Algebra of analytic vector fields on Riemann sphere

$$\left[\partial_z, \tfrac{1}{2}z^2\partial_z\right] = z\partial_z$$

$$\left[z\partial_z, \partial_z\right] = -\partial_z$$

$$\left[z\partial_z, \tfrac{1}{2}z^2\partial_z\right] = z^2\partial_z - \tfrac{1}{2}z^2\partial_z = \tfrac{1}{2}z^2\partial_z$$

cf. $[J_-, J_+] = J_0$

$[J_0, J_\pm] = \pm J_0$

realized by $J_\pm = \sigma_1 \pm i\sigma_2$

$J_0 = \sigma_3$

Pauli matrices.

$\{\sigma_1, \sigma_2, \sigma_3\}$ with complex coeffs span Lie algebra of group $SL(2;\mathbb{C})$ (= 2×2 complex matrices with unit determinant)

$$3 \quad A(s,t) = \frac{\Gamma(-1-t)\,\Gamma(-1-s)\,\Gamma(3+s+t)}{\Gamma(t+2)\,\Gamma(s+2)\,\Gamma(-2-s-t)}$$

$s \to \infty$
fixed t

$$= \underbrace{\left[\frac{\Gamma(-1-t)}{\Gamma(t+1)}\right]}_{\text{fn of } t \text{ only}} \cdot \frac{\Gamma(-(s+2)+1)}{\Gamma((s+1)+1)} \cdot \frac{\Gamma((s+t+2)+1)}{\Gamma(-(s+t+3)+1)}$$

Stirling approx $\quad \Gamma(z+1) \sim \exp\{z\ln z - z - \frac{1}{2}\ln z + O(1)\}$

$$\therefore \Gamma(-u+1) \sim \exp\{-u(\ln u + i\pi) + u - \tfrac{1}{2}(\ln u + i\pi) + O(1)\}$$

$$\sim -ie^{-i\pi u}\exp\{-u\ln u + u - \tfrac{1}{2}\ln u + O(1)\}$$

u real
(+ small imag. part)

$$\frac{\Gamma(-(s+2)+1)}{\Gamma((s+1)+1)} \sim \text{phase} \times \exp\left\{ \begin{array}{l} -(s+2)\ln(s+2) + s - \frac{1}{2}\ln(s+2) \\ -(s+1)\ln(s+1) + s + \frac{1}{2}\ln(s+1) + O(1) \end{array}\right\}$$

ignore

but $\ln(s+2) = \ln s + O(1/s)$ etc, so

$$\frac{\Gamma(-(s+2)+1)}{\Gamma((s+1)+1)} \sim \exp\{-2s\ln s + 2s - 3\ln s + O(1)\}$$

$$\text{Ily} \quad \frac{\Gamma((s+t+2)+1)}{\Gamma(-(s+t+3)+1)} \sim \exp\left\{ \begin{array}{l} (s+t+2)\ln s - s - \frac{1}{2}\ln s \\ +(s+t+3)\ln s - s + \frac{1}{2}\ln s + O(1) \end{array}\right\}$$

$$\sim \exp\{2s\ln s - 2s + (2t+5)\ln s + O(1)\}$$

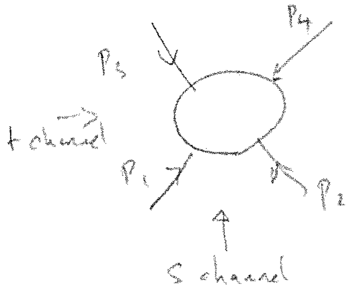
$$\therefore A(s,t) \sim f(t)\exp\{-3\ln s + (2t+5)\ln s + O(1)\} = f(t)\exp\{2(t+1)\ln s\}$$

$$\sim f(t)\, s^{2(t+1)}$$

From notes $\quad t = -2\left(1+\frac{s}{4}\right)(1-\cos\theta_s) \approx -\frac{s}{2}(1-\cos\theta_s) \quad$ as $s \to \infty$

$\therefore$ For fixed t, $\quad s \to \infty \Rightarrow 1-\cos\theta_s \to 0$, i.e. $\theta_s \to 0$

4 $\quad A(s,t) = \dfrac{\Gamma(-1-s)\,\Gamma(-1-t)}{\Gamma(-2-s-t)}$ $\qquad \left.\begin{array}{l} s = -(p_1+p_2)^2 \\ t = -(p_1+p_3)^2 \\ p^2 = 1 \end{array}\right\}$ in units for which $2\pi T = 1$.



$$t = -2\left(1+\frac{s}{4}\right)(1-\cos\theta_s)$$

For fixed t, poles come from single poles of $\Gamma(-1-s)$ at $\underbrace{s = -1, 0, 1, 2 \dots}_{\text{agrees with energy levels of open string}}$

Residue of pole at $s = n$ is

$$\lim_{s\to n}\Big[(s-n)\Gamma(-1-s)\Big]\frac{\Gamma(-1-t)}{\Gamma(-2-n-t)} \equiv R_n$$

$$\boxed{\Gamma(z+1) = z\Gamma(z) \implies \Gamma(z+n+1) = (z+n)(z+n-1)\dots z\,\overbrace{(z-1)\,\Gamma(z-1)}^{\Gamma(z)}} \quad (\dagger)$$

$$\Rightarrow \Gamma(-s+n+1) = (-1)^{n+1}(s-n)(s-n+1)\dots s(s+1)\Gamma(-s-1)$$

$$\Rightarrow (s-n)\Gamma(-1-s) = \frac{(-1)^{n+1}\Gamma(-s+n+1)}{(s-n+1)\dots s(s+1)} \xrightarrow{s\to n} \frac{(-1)^{n+1}}{(n+1)!}$$

$$\therefore R_n = \frac{(-1)^{n+1}}{(n+1)!}\frac{\Gamma(-1-t)}{\Gamma(-2-n-t)} \qquad \Rightarrow \underline{R_{-1} = 1}$$

For $n \geq 0$ use $(\dagger)$ again with $z = -t-n-2$ to get

$$\Gamma(-1-t) = (-1)^{n+1}(t+2)(t+3)\dots(t+n+2)\,\Gamma(-t-n-2)$$

$$\Rightarrow \frac{\Gamma(-1-t)}{\Gamma(-t-n-2)} = (-1)^{n+1}(t+n+2)\dots(t+3)(t+2)$$

$$\therefore \underset{(n\geq 0)}{R_n} = \frac{1}{(n+1)!}\underbrace{(t+n+2)\dots(t+3)(t+2)}_{(n+1)\text{th order polynomial}} \qquad \Rightarrow \boxed{\text{spin } n + \text{lower spins for } s = n}$$

e.g. $n = 0 \longrightarrow$ spin 1 $\quad$ (agrees with open string spectrum)

4 cont'd Hard scattering $s \to \infty$ at fixed θ_s

$2t \simeq -s(1-\cos\theta_s)$ as $s \to \infty$

fixed θ_s $\Rightarrow$ $t \to \infty$ s.t. $\boxed{t = -\gamma s}$ $\gamma = \sin^2 \theta_s/2$ $\Rightarrow 1-\gamma = \cos^2\frac{\theta_s}{2}$

Stirling : $\Gamma(-u+1) \sim \underset{\text{ignore}}{\text{phase}} \times \exp\{-u\ln u + u + \underset{\substack{\text{gives powers of } s \\ \text{slowly varying, so ignore}}}{O(\ln u)}\}$

$u = s+2$

$\Gamma(-1-s) \sim \exp\{-s\ln s + s + O(\ln s)\}$

$u = t+2 = -\gamma s + 2$ gives phase factor, ignore

$\Gamma(-1-t) \sim \exp\{\gamma s(\ln s + \ln\gamma + i\pi) - \gamma s + O(\ln s)\}$

$\therefore \boxed{\Gamma(-1-s)\Gamma(-1-t) \sim \exp\{-(1-\gamma)s\ln s + (1-\gamma)s + (\gamma\ln\gamma)s\}}$

$u = s+t+3 = (1-\gamma)s + 3$

$\Gamma(-2-s-t) \sim \exp\{-(1-\gamma)s[\ln s + \ln(1-\gamma)] + (1-\gamma)s + O(\ln s)\}$

$\sim \exp\{-(1-\gamma)s\ln s + (1-\gamma)s - [(1-\gamma)\ln(1-\gamma)]s\}$

$\therefore A(s,t) \sim \exp\{-f(\gamma)s\}$ $f(\gamma) = \underbrace{-\gamma\ln\gamma - (1-\gamma)\ln(1-\gamma)}_{\text{positive}}$ $\Rightarrow e^f > 1$

$\sim [e^f]^{-s} \to 0$ at $s \to \infty$

$e^f = \gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)} = [\sin^2\theta_s/2]^{-\sin^2\frac{\theta_s}{2}} [\cos^2\frac{\theta_s}{2}]^{-\cos^2\frac{\theta_s}{2}}$

5. Open NS string

Lemma 1 $\{L_m, \alpha_k\}_{PB} = ik\alpha_{k+m}$ ← in lectures

$$\boxed{\{L_m, b_r\}_{PB} = i\left(r+\frac{m}{2}\right)b_{m+r}.} \quad (*)$$

Proof of (*) $\{L_m, b_r\}_{PB} = \frac{1}{2}\sum_s s\{b_{m-s}\cdot b_s, b_r\}_{PB}$ (N.B. Lorentz indices are omitted)

$$= \frac{1}{2}\sum_s\left(s\, b_{m-s}\cdot\{b_s, b_r\}_{PB} - s\{b_r, b_{m-s}\}_{PB}\cdot b_s\right)$$

↑ anticommutativity of b and symmetry of {b, b} PB

$$= \frac{i}{2}rb_{m+r} + \frac{i}{2}(r+m)b_{m+r}$$

$$= \underline{i\left(r+\frac{m}{2}\right)b_{m+r}}$$

Lemma 2

$$\boxed{\begin{aligned}\{G_r, \alpha_k\}_{PB} &= ik\, b_{r+k}\\ \{G_r, b_s\}_{PB} &= -i\alpha_{r+s}\end{aligned}}$$

← follows directly from can. PB rels

$\{\alpha_k^m, \alpha_{-k}^n\}_{PB} = -ik\eta^{mn}$

$\{b_{-r}^m, b_r^n\}_{PB} = -i\eta^{mn}.$

Lemma 3 $\boxed{\sum_r b_{n-r}\cdot b_r \equiv 0}$

Proof LHS $\underset{r\to r+n}{=} \sum_r b_{-r}\cdot b_{r+n} \underset{r\to -r}{=} \sum_r b_r\cdot b_{n-r} \overset{\text{anticommutativity}}{=} -\sum_r b_{n-r}\cdot b_r$

$= -$LHS $\quad\therefore$ $\underline{\text{LHS} = 0}$

$$\underline{L_m = L_m^{(\alpha)} + L_m^{(b)}}$$

$$\{L_m^{(b)}, L_n^{(b)}\}_{PB} = \frac{1}{2}\sum_r\{L_m, b_{n-r}\cdot b_r\}_{PB} = \frac{1}{2}\sum_r r\{L_m, b_{n-r}\}\cdot b_r + \frac{1}{2}\sum_r r\, b_{n-r}\{L_m, b_r\}_{PB}$$

$$= \frac{i}{2}\sum_r r\left(n-r+\frac{m}{2}\right)b_{m+n-r}\cdot b_r + \underbrace{\frac{i}{2}\sum_r r\left(r+\frac{m}{2}\right)b_{n-r}\cdot b_{m+r}}_{\frac{i}{2}\sum_r (r-m)\left(r-\frac{m}{2}\right)b_{m+n-r}\cdot b_r}$$

5 cont'd $\quad \{L_m^{(b)}, L_n^{(b)}\}_{PB} = \frac{i}{2}\sum_r \left[r(n-m) + \frac{m^2}{2}\right] b_{n+m-r}\cdot b_r$

$$= -i(m-n)\, L_{n+m}^{(b)} + \underbrace{\frac{im^2}{4}\sum_r b_{n+m-r}\cdot b_r}_{0 \text{ by Lemma 3.}}$$

$\{L_m^{(\alpha)}, L_n^{(\alpha)}\}_{PB} = -i(m-n)\, L_{m+n}^{(\alpha)}$ done in lectures.

$\therefore \{L_m, L_n\}_{PB} = -i(m-n)\, L_{m+n}$.

$/\!/$ly $\{L_m, G_r\}_{PB} = -i\left(\frac{m}{2} - r\right) G_{r+m}$

Tricky case in $\{G_r, G_s\}_{PB}$:

$$\{G_r, G_s\}_{PB} = \sum_k \{G_r, \alpha_k\}_{PB}\cdot b_{s-k} + \sum_k \alpha_k \cdot \{G_r, b_{s-k}\}_{PB}$$

$$= \underbrace{\boxed{i\sum_k k\, b_{r+k}\cdot b_{s-k}}}_{?} \underbrace{- i\sum_k \alpha_k\cdot\alpha_{r+s-k}}_{-2i\, L_{r+s}^{(\alpha)}}$$

set $k = s+s'$. s is fixed so $\sum_k$ is equiv. to $\sum_{s'}$, so

$$i\sum_k k\, b_{r+k}\cdot b_{s-k} = i\sum_{s'} (s+s')\, b_{r+s+s'}\cdot b_{-s'}$$

$$= \underbrace{i\sum_{s'} s'\, b_{r+s+s'}\cdot b_{-s'}}_{\underbrace{-i\sum_{s'} s'\, b_{(r+s)-s'}\cdot b_{s'}}_{-2i\, L_{r+s}^{(b)}}} + \underbrace{is\sum_{s'} b_{(r+s)+s'}\cdot b_{-s'}}_{0 \text{ by Lemma 3}}$$

$\therefore \{G_r, G_s\}_{PB} = -2i\, L_{r+s}$

5 Cont'd $\quad \Phi = \sum_n \xi_{-n} L_n + i \sum_r \epsilon_{-r} G_r$

ϵ_{-r} ← anticommuting params.

$$\delta \alpha_k = \{\alpha_k, \Phi\}_{PB} = -\sum_n \xi_{-n} \{L_n, \alpha_k\}_{PB} - i \sum_r \epsilon_{-r} \{G_r, \alpha_k\}_{PB}$$

$$\underline{= -ik \sum_n \xi_{-n} \alpha_{n+k} + k \sum_r \epsilon_{-r} b_{r+k}}$$

$$\delta b_r = \{b_r, \Phi\}_{PB} = -\sum_n \xi_{-n} \{L_n, b_r\}_{PB} - i \sum_s \epsilon_{-s} \underbrace{\{b_r, G_s\}_{PB}}_{\text{symmetric PB}}$$

$$= -i \sum_n \left(r + \frac{n}{2}\right) \xi_{-n} b_{n+r} - \sum_s \epsilon_{-s} \alpha_{r+s}$$

$$\underline{= -i \sum_m \left(r - \frac{m}{2}\right) \xi_m b_{r-m} - \sum_s \epsilon_s \alpha_{r-s}}$$

$$\underbrace{\text{set} \left\{ \begin{array}{ll} \alpha_n^+ = 0 & n \neq 0 \\ b_r^+ = 0 & \forall r \end{array} \right\}}_{\text{lcg}} \qquad \delta \alpha_k^+ = -ik \xi_k \alpha_0^+ = \frac{-ik p_-}{\sqrt{\pi T}} \xi_k \qquad \left(\alpha_0^+ = \frac{p_-}{\sqrt{\pi T}}\right)$$

$\therefore \alpha_k^+ = 0, k \neq 0$ is preserved by gauge trans. iff

$0 = p_- \xi_k \Rightarrow \xi_k = 0$ (no residual gauge inv) $\underline{\text{if } p_- \neq 0}$

Illy $\quad \underline{\delta b_r^+ = -\alpha_0^+ \epsilon_r \rightarrow \epsilon_r = 0 \text{ if } p_- \neq 0}$

$$\left. \begin{array}{l} L_n\Big|_{\text{lcg}} = \alpha_0^+ \alpha_n^- + \frac{1}{2} \sum_k \underline{\alpha}_k \cdot \underline{\alpha}_{n-k} + \frac{1}{2} \sum_r r\, \underline{b}_{n-r} \cdot \underline{b}_r \\ G_r\Big|_{\text{lcg}} = \alpha_0^+ b_r^- + \sum_k \underline{\alpha}_k \cdot \underline{b}_{r-k} \end{array} \right\}$$

$\therefore$ if $p_- \neq 0$
$L_n = 0 \quad (n \neq 0)$
$G_r = 0 \quad (\forall r)$
can be solved for
$\underline{\alpha_n^- \ (n \neq 0) \ \& \ b_r^-}$

Only $L_0 = 0$ constraint survives this l.c.g.

$$L_0\Big|_{\text{lcg}} = \underbrace{\alpha_0^+ \alpha_0^- + \frac{1}{2} |\underline{\alpha}_0|^2}_{p^2/2\pi T} + \underbrace{\sum_{n=1}^{\infty} \underline{\alpha}_{-n} \cdot \underline{\alpha}_n}_{N_b} + \underbrace{\sum_{r=1/2}^{\infty} r\, \underline{b}_{-r} \cdot \underline{b}_r}_{N_f}$$

S cont'd $\quad L_0 = \frac{1}{2\pi T}\left[p^2 + \underbrace{2\pi T(N_b+N_f)}_{M^2}\right]$

l.c.g. action is

$$S = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k}\alpha_{-k}\cdot\dot{\alpha}_k + i\sum_{r=1/2}^{\infty} b_{-r}\cdot\dot{b}_r - \left(\frac{\lambda_0}{2\pi T}\right)(p^2+M^2)\right\}$$

Quantize $\quad \begin{cases} \alpha_k^I|0\rangle = 0, & k>0 \\ b_r^I|0\rangle = 0 & r>0 \end{cases} \qquad \begin{rcases} N_b|0\rangle = 0 \\ N_f|0\rangle = 0 \end{rcases} \Rightarrow \begin{cases} N_b = 0,1,2,\dots \\ N_f = 0,\frac{1}{2},1,\frac{3}{2}\dots \end{cases}$

$\alpha' M^2 = N_b + N_f - a \qquad \left(\alpha' = \frac{1}{2\pi T}\right) \qquad (I = 1,2,\dots D-2)$

zero-pt. contribution

g.s. $(N_b = N_f = 0)$, $\alpha' M^2 = -a$

1st-excited states $\underbrace{b_{-1/2}^I|0\rangle}_{(D-2)\text{-vector}}$ $\quad (N_b = 0, N_f = \frac{1}{2}) \quad M^2 = \frac{1}{2} - a$

Lorentz inv. $\Rightarrow$ (D-2)-vector is massless $\Rightarrow a = \frac{1}{2}$

For Weyl ordering $\quad \alpha' M^2 = \frac{1}{2}\sum_{k=1}^{\infty}\{\alpha_{-k}^I, \alpha_k^I\} + \frac{1}{2}\sum_{r=1/2}^{\infty} r\left[b_{-r}^I, b_r^I\right]$

$$= N_f + N_b + \frac{1}{2}\sum_{k=1}^{\infty}\underbrace{\left[\alpha_k^I, \alpha_{-k}^I\right]}_{(D-2)k} - \frac{1}{2}\sum_{r=1/2}^{\infty}\underbrace{\{b_r^I, b_{-r}^I\}}_{D-2}$$

$$= N_f + N_b \underbrace{+ \frac{(D-2)}{2}\left(\sum_{k=1}^{\infty} k - \sum_{r=1/2}^{\infty} r\right)}_{-a}$$

$$\therefore -a = \frac{(D-2)}{2}\Big[\underbrace{\zeta(-1,0)}_{-1/12} - \underbrace{\zeta(-1,\tfrac{1}{2})}_{1/24}\Big] = -\frac{(D-2)}{16} \quad \left(\begin{array}{c}\text{using given formula} \\ \text{for } \zeta(-1,q)\end{array}\right)$$

$\therefore \; a = \frac{1}{2} \Rightarrow D = 10$