3P7a **Supersymmetry: Example Sheet 1** Lent 2016

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1. Prove the following identities:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} = \left(\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3 \right)$$

$$(\sigma^\mu)_{\alpha\dot{\beta}}\,(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} \;=\; 2\,\delta^\delta_\alpha\,\delta^{\dot{\gamma}}_{\dot{\beta}}$$

$$(\sigma^\mu\bar{\sigma}^\nu \;+\; \sigma^\nu\bar{\sigma}^\mu)^\beta_\alpha \;=\; 2\eta^{\mu\nu}\delta^\beta_\alpha$$

$$\mathrm{Tr}\left[\,\sigma^\mu\bar{\sigma}^\nu\,\right] \;=\; 2\,\eta^{\mu\nu}$$

   Now use these identities to show that the definition

$$V_{\alpha\dot{\alpha}} \;\equiv\; \sigma^\mu_{\alpha\dot{\alpha}}V_\mu$$

   can be inverted to give

$$V^\mu \;=\; \frac{1}{2}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,V_{\alpha\dot{\alpha}}\;.$$

   Show also that the tensor product of left- and right-handed Weyl spinors can be written:

$$\psi_\alpha\bar{\chi}_{\dot{\alpha}} \;=\; \frac{1}{2}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,(\psi\sigma_\mu\bar{\chi})\;,$$

   which makes the decomposition $(1/2,0)\otimes(0,1/2) \;=\; (1/2,1/2)$ explicit.

2. Show that $\sigma^{\mu\nu} \;=\; \frac{i}{4}\,(\sigma^\mu\bar{\sigma}^\nu \;-\; \sigma^\nu\bar{\sigma}^\mu)$ satisfy the Lorentz algebra in the same manner as the Lorentz generators $M^{\mu\nu}$

$$\left[M^{\rho\sigma}, M^{\tau\nu}\right] = i\big(\eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}\big)$$

   and can therefore be used as generators of the (fundamental) spinor representation of the Lorentz group.

3. Prove that given $N \in \mathrm{SL}(2,C)$ the expression

$$\Lambda^\mu{}_\nu(N) \;=\; \frac{1}{2}\,\mathrm{Tr}\left[\bar{\sigma}^\mu\,N\,\sigma_\nu\,N^\dagger\right]$$

   provides an explicit map from $\mathrm{SL}(2,C)$ to $\mathrm{SO}(3,1)$.

4. Using the definition $\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$, and the $\mathrm{SL}(2,C)$ spinor transformation property

$$\psi'_\alpha = N_\alpha{}^\beta\psi_\beta \tag{1}$$

   work through the following:

   (a) Prove that under SL(2,C), ψ^α transforms as $\psi'^\alpha \;=\; \psi^\beta(N^{-1})_\beta{}^\alpha$.

   (b) Using the $\mathrm{SL}(2,C)$ invariance of $\epsilon_{\alpha\beta}$, find an expression for N^{-1} in terms of N.

(c) Prove that:
$$\psi\chi \;\equiv\; \psi^\alpha\chi_\alpha \;=\; -\psi_\alpha\chi^\alpha \equiv \chi\psi$$
and check that $\psi\chi$ as well as $\bar\psi\bar\chi \;=\; \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$ are SL$(2,C)$ scalars.

5. Using that
$$(\sigma^{\mu\nu})_\alpha^{\;\beta} \; (\sigma_{\mu\nu})_\gamma^{\;\delta} \;=\; \epsilon_{\alpha\gamma}\,\epsilon^{\beta\delta} \;+\; \delta_\alpha^\delta\,\delta_\gamma^\beta$$
Prove that the tensor product of two left-handed Weyl spinors can be decomposed as
$$(1/2,0) \otimes (1/2,0) = (0,0) \oplus (1,0)$$
or, more explicitly:
$$\psi_\alpha\chi_\beta \;=\; \frac{1}{2}\epsilon_{\alpha\beta}\,(\psi\chi) \;+\; \frac{1}{2}(\sigma^{\mu\nu}\,\epsilon^T)_{\alpha\beta}\,(\psi\,\sigma_{\mu\nu}\,\chi).$$

6. Prove the following Fierz identities:
$$(\theta\psi)\,(\bar\chi\bar\eta) \;=\; -\,\frac{1}{2}(\theta\sigma^\mu\bar\eta)\,(\bar\chi\bar\sigma_\mu\psi)$$

$$(\theta\sigma^\mu\bar\theta)\,(\theta\sigma^\nu\bar\theta) \;=\; \frac{1}{2}\eta^{\mu\nu}\,(\theta\theta)\,(\bar\theta\bar\theta)$$

7. The Pauli-Ljubánski vector $W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ satisfies
$$[W_\mu, P_\nu] \;=\; 0 \qquad [W_\mu, W_\nu] \;=\; i\epsilon_{\mu\nu\sigma\rho}P^\sigma W^\rho.$$
Prove that it also satisfies:
$$[W^\mu, Q_\alpha] \;=\; -i(\sigma^{\mu\nu})_\alpha^{\;\beta}\,Q_\beta P_\nu.$$
Thus show that $W^\mu W_\mu$ is a Casimir operator of the Poincaré algebra but not of the super Poincaré algebra. This tells us that irreducible representations of the super Poincaré algebra ("supermultiplets") contain particles of different spin.

8. Consider the superspin operator $\tilde C_2$:
$$\tilde C_2 \equiv C_{\mu\nu}C^{\mu\nu} \text{ where } C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu \text{ with } B_\mu \equiv W_\mu - \frac{1}{4}\,\bar Q_{\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\alpha}Q_\alpha\;.$$
What does the index structure of $\tilde C_2$ tell you about its Lorentz transformation properties and therefore it's commutator with $M^{\mu\nu}$? Assuming that $[\tilde C_2,\, M^{\mu\nu}]$ takes such a form, check that $\tilde C_2$ commutes with P_μ and Q_α explicitly. Deduce that $\tilde C_2$ is a Casimir of the super Poincaré algebra.

9. By applying a parity transformation to $\{Q_\alpha, \bar Q_{\dot\beta}\}$, where $\hat P Q_\alpha \hat P^{-1} = \eta_P(\sigma^0)_{\alpha\dot\beta}\bar Q^{\dot\beta}$ and $\hat P\bar Q^{\dot\alpha}\hat P^{-1} = -\eta_P^*\,(\bar\sigma^0)^{\dot\alpha\beta}\,Q_\beta$, show that $\hat P P_\mu\hat P^{-1} = P^\mu$ (note that $|\eta_P| = 1$).

with supersymmetry in terms of the number of

10. Consider states in massive $N = 1$ supermultiplets, using the commutator $[J_i,\,\bar Q_{\dot\alpha}] = \frac{1}{2}(\sigma_i)_{\dot\alpha}^{\;\dot\beta}\bar Q_{\dot\beta}$, calculate $[J^2,\,\bar Q_{\dot\alpha}]$ and then $[J^2,\,\bar Q_{\dot1}\bar Q_{\dot2}]$. What does the operator $(\bar Q_{\dot1}\bar Q_{\dot2})$ do to the total spin of some state?