3P7a **Supersymmetry: Example Sheet 1** Lent 2012

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk.

1. Prove the following identities:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} = \left(\mathbb{1}_2,\ -\sigma^1,\ -\sigma^2,\ -\sigma^3 \right)$$

$$(\sigma^\mu)_{\alpha\dot{\beta}}\,(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} \ = \ 2\,\delta^\delta_\alpha\,\delta^{\dot{\gamma}}_{\dot{\beta}}$$

$$(\sigma^\mu\bar{\sigma}^\nu \ + \ \sigma^\nu\bar{\sigma}^\mu)^\beta_\alpha \ = \ 2\,\eta^{\mu\nu}\delta^\beta_\alpha$$

$$\mathrm{Tr}\left[\,\sigma^\mu\bar{\sigma}^\nu\,\right] \ = \ 2\,\eta^{\mu\nu}$$

Now use these identities to show that the definition

$$V_{\alpha\dot{\alpha}} \ \equiv \ \sigma^\mu_{\alpha\dot{\alpha}}V_\mu$$

can be inverted to give

$$V^\mu \ = \ \frac{1}{2}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,V_{\alpha\dot{\alpha}}\ .$$

Show also that the tensor product of left- and right-handed Weyl spinors can be written:

$$\psi_\alpha\bar{\chi}_{\dot{\alpha}} \ = \ \frac{1}{2}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,(\psi\sigma_\mu\bar{\chi})\ ,$$

which makes explicit the decomposition $(1/2, 0) \otimes (0, 1/2) \ = \ (1/2, 1/2)$.

2. Show that $\sigma^{\mu\nu} \ = \ \frac{i}{4}\,(\sigma^\mu\bar{\sigma}^\nu \ - \ \sigma^\nu\bar{\sigma}^\mu)$ satisfy the Lorentz algebra in the same manner as the Lorentz generators $M^{\mu\nu}$

$$\left[M^{\rho\sigma}, M^{\tau\nu}\right] = i\left(\eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}\right)$$

and can therefore be used as generators of the (fundamental) spinor representation of the Lorentz group.

3. Prove that given $N \in \mathrm{SL}(2, C)$ the expression

$$\Lambda^\mu{}_\nu(N) \ = \ \frac{1}{2}\,\mathrm{Tr}\left[\bar{\sigma}^\mu\ N\ \sigma_\nu\ N^\dagger\right]$$

provides an explicit map from $\mathrm{SL}(2, C)$ to $\mathrm{SO}(3, 1)$.

4. Using the definition $\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$, and the $\mathrm{SL}(2, C)$ spinor transformation property

$$\psi'_\alpha = N_\alpha{}^\beta\psi_\beta \tag{1}$$

work through the following:

(a) Prove that under SL(2,C), ψ^α transforms as $\psi'^\alpha \ = \ \psi^\beta(N^{-1})_\beta{}^\alpha$.

(b) Using the $\mathrm{SL}(2, C)$ invariance of $\epsilon_{\alpha\beta}$, find an expression for N^{-1} in terms of N.

(c) Prove that:

$$\psi\chi \;\equiv\; \psi^\alpha \chi_\alpha \;=\; -\psi_\alpha \chi^\alpha \equiv \chi\psi$$

and check that $\psi\chi$ as well as $\bar\psi\bar\chi \;=\; \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$ are SL$(2,C)$ scalars.

5. Using that

$$(\sigma^{\mu\nu})_\alpha^{\ \beta}\,(\sigma_{\mu\nu})_\gamma^{\ \delta} \;=\; \epsilon_{\alpha\gamma}\,\epsilon^{\beta\delta} \;+\; \delta_\alpha^\delta\,\delta_\gamma^\beta$$

Prove that the tensor product of two left-handed Weyl spinors can be decomposed as

$$(1/2,0)\otimes(1/2,0) = (0,0)\oplus(1,0)$$

or, more explicitly:

$$\psi_\alpha\chi_\beta \;=\; \frac{1}{2}\epsilon_{\alpha\beta}\,(\psi\chi) \;+\; \frac{1}{2}(\sigma^{\mu\nu}\,\epsilon^T)_{\alpha\beta}\,(\psi\,\sigma_{\mu\nu}\,\chi).$$

6. Prove the following Fierz identities:

$$(\theta\psi)\,(\bar\chi\bar\eta) \;=-\;\frac{1}{2}(\theta\sigma^\mu\bar\eta)\,(\bar\chi\bar\sigma_\mu\psi)$$

$$(\theta\sigma^\mu\bar\theta)\,(\theta\sigma^\nu\bar\theta) \;=\; \frac{1}{2}\eta^{\mu\nu}\,(\theta\theta)\,(\bar\theta\bar\theta)$$

7. The Pauli-Ljubánski vector $W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ satisfies

$$[W_\mu, P_\nu] \;=\; 0 \qquad [W_\mu, W_\nu] \;=\; i\epsilon_{\mu\nu\sigma\rho}P^\sigma W^\rho.$$

Prove that it also satisfies:

$$[W^\mu, Q_\alpha] \;=\; -i(\sigma^{\mu\nu})_\alpha^{\ \beta}\,Q_\beta P_\nu.$$

Thus show that $W^\mu W_\mu$ is a Casimir operator of the Poincaré algebra but not of the super Poincaré algebra. This tells us that irreducible representations of the super Poincaré algebra ("supermultiplets") contain particles of different spin.

8. Consider the superspin operator $\tilde C_2$:

$$\tilde C_2 \equiv C_{\mu\nu}C^{\mu\nu} \text{ where } C_{\mu\nu} \equiv B_\mu P_\nu - B_\nu P_\mu \text{ with } B_\mu \equiv W_\mu - \frac{1}{4}\,\bar Q_{\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\alpha}Q_\alpha\ .$$

What does the index structure of $\tilde C_2$ tell you about its Lorentz transformation properties? Check explicitly that $\tilde C_2$ commutes with P_μ and Q_α. Deduce that $\tilde C_2$ is a Casimir of the super Poincaré algebra.

9. Write down the $N=1$ supersymmetry algebra in the four-component (Dirac) spinor formalism. Assume the supersymmetry generators are Majorana spinors.

10. Write explicitly all the components of the $N=1$ supersymmetry multiplets corresponding to:

(i) massive particles of maximum spins 1/2 and 1.

(ii) massless particles with maximum helicity 2 and 1.

Verify in each case that the number of bosons equals the number of fermions. The Higgs mechanism converts a massless complex scalar and a massless vector into a massive vector and a real scalar. Is the Higgs mechanism compatible with supersymmetry in terms of the number of bosonic and fermionic degrees of freedom?