1. Prove the following identities:

\[ (\bar{\sigma}^\mu)^\alpha_\alpha \equiv \epsilon^{\alpha\beta} \epsilon_{\beta\bar{\beta}}^\mu (1_2, -\sigma^1, -\sigma^2, -\sigma^3) \]

\[ (\sigma^\mu)_{\alpha\beta} (\bar{\sigma}^\nu)^\beta_\delta = 2 \delta^\alpha_\alpha \delta^\delta_\beta \]

\[ (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_\alpha = 2 \eta^{\mu\nu} \delta^\beta_\alpha \]

\[ \text{Tr} [\sigma^\mu \bar{\sigma}^\nu] = 2 \eta^{\mu\nu} \]

Now use these identities to show that the definition

\[ V_{\alpha\bar{\alpha}} \equiv \sigma^\mu_{\alpha\bar{\alpha}} V_\mu \]

can be inverted to give

\[ V^\mu = \frac{1}{2} (\bar{\sigma}^\mu)^\alpha_\alpha V_{\alpha\bar{\alpha}} . \]

Show also that the tensor product of left- and right-handed Weyl spinors can be written:

\[ \psi_\alpha \bar{\chi}_{\bar{\alpha}} = \frac{1}{2} (\sigma^\mu)_{\alpha\bar{\alpha}} (\psi_\sigma \bar{\chi}_\tau) , \]

which makes the decomposition \((1/2, 0) \otimes (0, 1/2) = (1/2, 1/2)\) explicit.

2. Show that \(\sigma^{\mu\nu} = \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)\) satisfy the Lorentz algebra in the same manner as the Lorentz generators \(M^{\mu\nu}\)

\[ [M^{\rho\sigma}, M^{\tau\nu}] = i \left( \eta^{\sigma\tau} M^{\mu\nu} - \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\rho\sigma} M^{\nu\tau} + \eta^{\rho\tau} M^{\sigma\nu} \right) \]

and can therefore be used as generators of the (fundamental) spinor representation of the Lorentz group.

3. Prove that given \(N \in \text{SL}(2, \mathbb{C})\) the expression

\[ \Lambda^\mu_\nu(N) = \frac{1}{2} \text{Tr} \left[ \bar{\sigma}^\mu N \sigma_\nu N^\dagger \right] \]

provides an explicit map from \(\text{SL}(2, \mathbb{C})\) to \(\text{SO}(3, 1)\).

4. Using the definition \(\psi^\alpha \equiv \epsilon^{\alpha\beta} \psi_\beta\), and the \(\text{SL}(2, \mathbb{C})\) spinor transformation property

\[ \psi'_{\alpha} = N^\beta_{\alpha} \psi_{\beta} \quad (1) \]

work through the following:

(a) Prove that under \(\text{SL}(2, \mathbb{C})\), \(\psi^\alpha\) transforms as \(\psi'^\alpha = \psi^\beta (N^{-1})^\alpha_\beta\).

(b) Using the \(\text{SL}(2, \mathbb{C})\) invariance of \(\epsilon_{\alpha\beta}\), find an expression for \(N^{-1}\) in terms of \(N\).
(c) Prove that:
\[
\psi \chi \equiv \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha \equiv \chi \psi
\]
and check that \(\psi \chi\) as well as \(\bar{\psi} \bar{\chi}\) = \(\bar{\psi} \bar{\chi}\) are \(\text{SL}(2, C)\) scalars.

5. Using that
\[
(\sigma^{\mu \nu})^\beta_\alpha (\sigma_{\mu \nu})^\delta_\gamma = \epsilon_{\alpha \gamma} \epsilon^{\beta \delta} + \delta^\beta_\alpha \delta^\gamma_\delta
\]
Prove that the tensor product of two left-handed Weyl spinors can be decomposed as
\[
(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0)
\]
or, more explicitly:
\[
\psi_\alpha \chi_\beta = \frac{1}{2} \epsilon_{\alpha \beta} (\psi \chi) + \frac{1}{2} (\sigma^{\mu \nu} \epsilon^T)_{\alpha \beta} (\psi \sigma_{\mu \nu} \chi).
\]

6. Prove the following Fierz identities:
\[
(\theta \psi) (\bar{\chi} \bar{\eta}) = -\frac{1}{2} (\theta \sigma^{\mu \nu}) (\bar{\chi} \sigma_\mu \psi)
\]
\[
(\theta \sigma^{\mu} \bar{\theta}) (\theta \sigma^\nu \bar{\theta}) = \frac{1}{2} \eta^{\mu \nu} (\theta \theta) (\bar{\theta} \bar{\theta})
\]

7. The Pauli-Ljubánski vector \(W_\mu \equiv \frac{1}{2} \epsilon_{\mu \rho \sigma} P^\rho M^\sigma\) satisfies
\[
[W_\mu, P_\nu] = 0 \quad [W_\mu, W_\nu] = i \epsilon_{\mu \nu \sigma \rho} P^\sigma W^\rho.
\]
Prove that it also satisfies:
\[
[W_\mu, Q_\alpha] = -i (\sigma^{\mu \nu})_\alpha^\beta Q_\beta P_\nu.
\]
Thus show that \(W_\mu W_\mu\) is a Casimir operator of the Poincaré algebra but not of the super Poincaré algebra. This tells us that irreducible representations of the super Poincaré algebra ("supermultiplets") contain particles of different spin.

8. Consider the superspin operator \(\tilde{C}_2\):
\[
\tilde{C}_2 \equiv C_{\mu \nu} C^{\mu \nu} \quad \text{where} \quad C_{\mu \nu} \equiv B_\mu P_\nu - B_\nu P_\mu \quad \text{with} \quad B_\mu \equiv W_\mu - \frac{1}{4} \bar{Q}_\alpha (\bar{\sigma}_\mu)^{\dot{\alpha}}_\alpha Q_\alpha.
\]
What does the index structure of \(\tilde{C}_2\) tell you about its Lorentz transformation properties and therefore it’s commutator with \(M^{\mu \nu}\)? Assuming that \([\tilde{C}_2, M^{\mu \nu}]\) takes such a form, check that \(\tilde{C}_2\) commutes with \(P_\mu\) and \(Q_\alpha\) explicitly. Deduce that \(\tilde{C}_2\) is a Casimir of the super Poincaré algebra.

9. By applying a parity transformation to \(\{Q_\alpha, \bar{Q}_\beta\}\), where \(\bar{P} Q_\alpha \bar{P}^{-1} = \eta_P (\sigma^0)^\alpha_\beta \bar{Q}^\beta\) and \(\bar{P} \bar{Q}^\alpha \bar{P}^{-1} = -\eta_P (\sigma^0)^{\dot{\alpha}}_\dot{\beta} Q_\beta\), show that \(\bar{P} P_\mu \bar{P}^{-1} = P^\mu\) (note that \(|\eta_P| = 1\)).

10. Consider states in massive \(N = 1\) supermultiplets, using the commutator \([J_i, \bar{Q}_a] = \frac{1}{2} (\sigma_i)^{\dot{\beta}}_\beta \bar{Q}_\beta\), calculate \([J^2, \bar{Q}_a]\) and then \([J^2, \bar{Q}_1 \bar{Q}_2]\). What does the operator \((\bar{Q}_1 \bar{Q}_2)\) do to the total spin of some state?