Part III Maths **Lent Term 2019**
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Supersymmetry: Example Sheet 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk $(*)$ *may be more difficult.*

1. In lectures we showed that

$$\int \mathrm{e}^{\frac{1}{2}A_{ab}\psi^a\psi^b}\,\mathrm{d}^{2k}\psi = \mathrm{Pfaff}(A)$$

where A is an real, invertible antisymmetric matrix and ψ^a are $2k$ Grassmann variables. By writing $\psi^a = N^a_{\ b}\psi'^b$ for some $N \in \mathrm{GL}(2k,\mathbb{R})$, show that $\mathrm{Pfaff}(N^{\mathrm{T}}AN) = \det(N)\,\mathrm{Pfaff}(A)$. Show that N may be chosen so as to put A into the form

$$N^{\mathrm{T}}AN = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix},$$

and hence show that $\mathrm{Pfaff}(A) = \pm\sqrt{\det A}$.

2. Suppose $V = V_0 \oplus V_1$ is a super vector space and $M : V \to V$ a linear map. Decomposing

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A : V_0 \to V_0$, $B : V_1 \to V_0$, *etc.*, we define the supertrace of M by $\mathrm{str}(M) = \mathrm{tr}A - \mathrm{tr}D$. The superdeterminant of an invertible matrix M is then defined by

$$\delta(\ln \mathrm{sdet}(M)) = \mathrm{str}(M^{-1}\delta M) \qquad \text{and} \qquad \mathrm{sdet}(I) = 1\,,$$

where δM is an arbitrary variation of M and I is the identity map.

i) Show that $\mathrm{sdet}(MN) = \mathrm{sdet}(M)\,\mathrm{sdet}(N)$.

ii) In the case $B = C = 0$, show that $\mathrm{sdet}(M) = \det A/\det D$ provided D is invertible.

iii) Find $\mathrm{sdet}(M)$ in the case $A = D = I$, $C = 0$ but $B \neq 0$.

iv) By writing

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix},$$

find a general expression for $\mathrm{sdet}(M)$ that holds whenever D is invertible.

3. Consider the change of variables $(x, \psi, \bar{\psi}) \mapsto (y, \chi, \bar{\chi})$ given by

$$y = x - \frac{\bar{\psi}\psi}{h(x)}, \qquad \chi = \sqrt{h(x)}\,\psi, \qquad \bar{\chi} = \bar{\psi}$$

where $h(x)$ is a real polynomial. Using your results to the previous question, show that $\mathrm{d}x\,\mathrm{d}^2\psi = \sqrt{h(y)}\,\mathrm{d}y\,\mathrm{d}^2\chi$.

4. Write down the form of the most general real superfield on $\mathbb{R}^{0|2}$. Find the transformations of the component fields under the transformation $\delta X = (\bar{\epsilon}Q + \epsilon\bar{Q})X$, where $Q = \partial/\partial\theta$ and $\bar{Q} = \partial/\partial\bar{\theta}$. Check that these component transformations obey $\{Q, \bar{Q}\} = 0$, $\{Q, Q\} = 0$ and $\{\bar{Q}, \bar{Q}\} = 0$. Use your superfield to write the action

$$S(x, \psi, \bar{\psi}) = \frac{1}{2}(\partial W)^2 - \bar{\psi}\psi\,\partial^2 W$$

in a manifestly supersymmetric invariant way.

5. Consider the action

$$S(z^a, \psi_1^a, \psi_2^a) = \sum_{a=1}^{n} \left|\frac{\partial W}{\partial z^a}\right|^2 - \psi_1^a \psi_2^b \frac{\partial^2 W}{\partial z^a \partial z^b} - \bar{\psi}_1^a \bar{\psi}_2^b \frac{\partial^2 \overline{W}}{\partial \bar{z}^a \partial \bar{z}^b}$$

where $W(z)$ is a polynomial in the n complex variables z^a that obeys

$$W(\lambda^{q_1} z_1, \lambda^{q_2} z_2, \ldots, \lambda^{q_n} z_n) = \lambda W(z_1, z_2, \ldots, z_n)$$

for some $q_i \in \mathbb{Q}$. What is the chiral ring $\mathcal{R}$ in this model? [*It may help to think of the q_is as 'electric charges'*]. Show that

$$\sum_{X_\alpha \in \mathcal{R}} t^{Q_\alpha} = \prod_i \frac{1 - t^{1-q_i}}{1 - t^{q_i}},$$

where Q_α is the homogeneity of X_α.

6*. Let (M, ω) be a compact, symplectic manifold of dimension $2m$, and let $H : M \to \mathbb{R}$ generate a $U(1)$ action on M with associated Hamiltonian vector field V. Show that the action

$$S(x, \psi) = -\mathrm{i}\alpha\left(H(x) + \omega_{ab}(x)\psi^a\psi^b\right)$$

is invariant under transformations generated by

$$Q = \psi^a \frac{\partial}{\partial x^a} + V^a(x) \frac{\partial}{\partial \psi^a}\,.$$

Compute $\{Q, Q\}$ and give the resulting operator a geometric meaning. Now let g be a $U(1)$-invariant metric on M, and let $\Psi = g(\psi, V)$. Show that $Q^2\Psi = 0$. By considering $S' = S + \lambda Q(\Psi)$, explain why

$$\frac{1}{(2\pi)^m} \int \mathrm{e}^{-S(x,\psi)}\, \mathrm{d}^{2m}x\, \mathrm{d}^{2m}\psi \;=\; \sum_{x_*\,:\,V(x_*)=0} \mathrm{e}^{\mathrm{i}\alpha H(x_*)} \left.\frac{\mathrm{Pfaff}(\partial_{[a}V_{b]})}{\sqrt{\det(\partial_a\partial_b g(V,V))}}\right|_{x_*}\,,$$

where $V_b = g_{bc}V^c$. Near each critical point we can choose g to be the flat metric $\mathrm{d}s^2 = \sum_{i=1}^m \mathrm{d}x_i^2 + \mathrm{d}y_i^2$, and V to act as $V = \sum_i k_i(y_i\partial/\partial x_i - x_i\partial/\partial y_i)$ for some $k_i \in \mathbb{Z}$. Use this to show

$$\frac{1}{m!} \int_M \mathrm{e}^{\mathrm{i}\alpha H}\, \omega^m = \left(\frac{2\pi}{\mathrm{i}\alpha}\right)^m \sum_{x_*} \frac{\mathrm{e}^{\mathrm{i}\alpha H(x_*)}}{\prod_{i=1}^m k_i(x_*)}\,.$$

(This is the Duistermaat-Heckman formula – the original localization formula in mathematics.)

7. Let ψ and $\bar{\psi}$ be fermionic operators obeying the anticommutation relations $\{\psi, \bar{\psi}\} = 1$ and let the vacuum $|0\rangle$ be defined by $\psi|0\rangle = 0$. For a complex fermionic parameter η, define the fermionic coherent state $|\eta\rangle \equiv \mathrm{e}^{\bar{\psi}\eta}|0\rangle$ and its adjoint $\langle\bar{\eta}| = \langle 0|\mathrm{e}^{\bar{\eta}\psi}$. Show that

   i) $\psi|\eta\rangle = \eta|\eta\rangle$,
   ii) $\langle\bar{\eta}|\bar{\psi} = \langle\eta|\bar{\eta}$,
   iii) $\langle\bar{\eta}|\eta\rangle = \mathrm{e}^{\bar{\eta}'\eta}$,
   iv) $\int \mathrm{e}^{-\bar{\eta}\eta}\, |\eta\rangle\langle\bar{\eta}|\, \mathrm{d}^2\eta = 1_{\mathcal{H}}$,
   v) $\mathrm{Tr}(A) = \int \mathrm{e}^{-\bar{\eta}\eta}\, \langle -\bar{\eta}|A|\eta\rangle\, \mathrm{d}^2\eta$,
   vi) $\mathrm{Str}(A) = \mathrm{Tr}((-1)^F A) = \int \mathrm{e}^{-\bar{\eta}\eta}\, \langle\bar{\eta}|\hat{A}|\eta\rangle\, \mathrm{d}^2\eta$,

   where $1_{\mathcal{H}}$ is the identity operator, F the fermion number operator and A an arbitrary (trace-class) operator.

8. Consider SQM for a single real boson and complex fermion, with potential function $h(x) = \frac{1}{2}\omega x^2$ where $\omega \in \mathbb{R}$ is a constant frequency. Using canonical methods, compute the partition function, Witten index and ground state for this model. Compute the complete spectrum of the Hamiltonian. By expanding $x(\tau)$ and $\psi(\tau)$ as Fourier series and taking the path integral to be an integral over all the modes, recover your results for $\mathcal{Z}(\beta)$ and $\mathrm{Tr}((-1)^F \mathrm{e}^{-\beta H})$ from a path integral.