3P7b **Supersymmetry: Example Sheet 2** Lent 2012

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1. Show that

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ \mathcal{Q}_\alpha &= -i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar{\mathcal{Q}}_{\dot\alpha} &= i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\gamma(\sigma^\mu)_{\gamma\dot\alpha}\partial_\mu \end{aligned}$$

satisfy the $N=1$ supersymmetry algebra.

2. Given the general scalar superfield:

$$S(x,\theta,\bar\theta) = \varphi(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta M(x) + \bar\theta\bar\theta N(x)$$

$$+(\theta\sigma^\mu\bar\theta)V_\mu(x) + (\theta\theta)\bar\theta\bar\lambda(x) + (\bar\theta\bar\theta)\theta\rho(x) + \theta\theta\bar\theta\bar\theta D(x)$$

Show that under a supersymmetry transformation

$$\delta S = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})S$$

the components of the superfield transform as:

$$\begin{aligned} \delta\varphi &= \epsilon\psi + \bar\epsilon\bar\chi, & \delta\psi &= 2\epsilon M + (\sigma^\mu\bar\epsilon)(i\partial_\mu\varphi + V_\mu) \\ \delta\bar\chi &= 2\bar\epsilon N - (\epsilon\sigma^\mu)(i\partial_\mu\varphi - V_\mu) & \delta M &= \bar\epsilon\bar\lambda - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar\epsilon \\ V_\mu &= \epsilon\sigma_\mu\bar\lambda + \rho\sigma_\mu\bar\epsilon + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar\sigma_\nu\epsilon - \bar\epsilon\bar\sigma_\nu\sigma_\mu\partial^\nu\bar\chi\right) & \delta\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar\chi \\ \delta\bar\lambda &= 2\bar\epsilon D + \frac{i}{2}\left(\bar\sigma^\nu\sigma^\mu\bar\epsilon\right)\partial_\mu V_\nu + i(\bar\sigma^\mu\epsilon)\partial_\mu M & \delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar\lambda - \rho\sigma^\mu\bar\epsilon) \\ \delta\rho &= 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\bar\sigma^\mu\epsilon\right)\partial_\mu V_\nu + i(\sigma^\mu\bar\epsilon)\partial_\mu N & & \end{aligned}$$

Notice in particular that the transformation of the field D is a total derivative.

3. Verify that the superfield:

$$\Phi(x,\theta,\bar\theta) = \varphi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x)$$

$$+i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar\theta$$

is a chiral superfield ($\bar{\mathcal{D}}\Phi = 0$). Find how each of the components φ, ψ, F transform under supersymmetry.

4. Find the D-term (i.e. the coefficient of $(\theta\theta)(\bar\theta\bar\theta)$) of $\Phi^\dagger\Phi$ where Φ is a chiral superfield.

5. Find the F-term (i.e. the coefficient of $\theta\theta$) of the function $\frac{1}{2}m\Phi^2 + \frac{1}{3!}g\Phi^3$ where Φ is a chiral superfield and m, g are constants.

6. In the Wess-Zumino model where the superpotential is

$$W = \alpha + \kappa\Phi + \frac{m^2}{2}\Phi^2 + \frac{g}{3!}\Phi^3,$$

find the minima of the scalar potential. By expanding around one of the minima find the relation between the mass of the scalar field and the mass of the fermion field. Find also the relation between the quartic self-coupling of the scalar field and the Yukawa coupling.

7. For an abelian vector superfield V, show that the field strength superfield

$$W_\alpha(x,\theta,\bar\theta) \;=\; -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}})\mathcal{D}_\alpha V(x,\theta,\bar\theta)$$

is chiral ($\bar{\mathcal{D}}W_\alpha = 0$) and gauge invariant ($V \to V + i(\Lambda - \Lambda^\dagger), W_\alpha \to W_\alpha$). Find the F-component of $\frac{1}{4}W^\alpha W_\alpha$ in terms of $F_{\mu\nu}$, λ and D.

8. Prove that the Lagrangian:

$$\mathcal{L} \;=\; (\partial_\mu\varphi\;\partial^\mu\varphi^*) - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + FF^* + m[\varphi F - \frac{1}{2}\psi\psi + h.c.]$$

is invariant, up to a total derivative, under the supersymmetry transformations:

$$\begin{aligned}\delta\varphi &= \sqrt{2}\epsilon\;\psi\\ \delta\psi &= i\sqrt{2}\;\sigma^\mu\bar\epsilon\;\partial_\mu\varphi \;+\; \sqrt{2}\;\epsilon F\\ \delta F &= \sqrt{2}\;i\;\bar\epsilon\bar\sigma^\mu\partial_\mu\psi\end{aligned} \qquad (1)$$

where ϵ is an a Grassmann-valued parameter and m a constant (mass). Here $h.c.$ stands for complex or hermitian conjugate.

9. Many remarkable properties of supersymmetric theories follow from the observation that the superpotential must be a *holomorphic* function of chiral superfields. One famous example is the perturbative non-renormalisation of the superpotential. To illustrate this we follow an example given by Seiberg[1] and consider a simple Wess-Zumino model describing the behaviour of a single chiral superfield Φ. The theory is defined at high energies by the superpotential:

$$W_{\rm tree} = \frac{1}{2}\,m\,\Phi^2 \;+\; \frac{1}{3}\,g\,\Phi^3\;. \qquad (2)$$

We would like to find the form of the effective superpotential that governs the behaviour of Φ at low energies — in general we would expect that integrating out the high energy fluctuations of Φ would lead to the appearance of new operators in the superpotential, or at the very least would force us to adjust couplings m and g, causing them to run. We will now argue that superpotential in Eq. 2 does not, in fact, get modified at all by quantum effects. The first trick is to think of the couplings m and g as actually being the lowest components of SPURION superfields.

(a) The superpotential Eq. 2 has two $U(1)$ symmetries. Fill in the following table to indicate the charge of the spurion superfields under each symmetry (the charge

	$U(1)$	$U(1)_R$
Φ	1	1
m		
g		

of Φ has been normalised to 1 in each case):
Hint: Recall the R-charge of the superpotential. When the spurions acquire VEVs, these symmetries are spontaneously broken, but they can still be used to constrain the structure of the low energy effective action: this must be a *holomorphic* function of the superfields Φ, m and g.

(b) Calculate the charge of

$$\frac{g\,\Phi}{m}$$

under the action of each symmetry. Deduce that the most general low energy effective action must take the form:

$$W_{\rm eff} \;=\; \frac{1}{2}\,A\,f\left(\frac{g\,\Phi}{m}\right)\;, \tag{3}$$

for some function $f(z) = \sum_{-\infty}^{\infty} f_n z^n$ — you should determine the combination of parameters A in such a way that $W_{\rm eff}$ has the correct charge under each symmetry.

Now see what we can learn by taking a few carefully chosen limits:

- When $g \to 0$ for fixed m, the theory is free, so we deduce that there can be no negative powers of z in $f(z)$.
- Requiring a smooth massless limit $m \to 0$ shows quadratic and higher powers of z must also be absent.

(c) Find the form of $f(z)$ required to match with the microscopic superpotential $W_{\rm tree}$ at weak coupling, and thus conclude that

$$W_{\rm eff} \;=\; W_{\rm tree}\;. \tag{4}$$

In other words, the superpotential is not renormalised at all. Notice that this argument only requires the existence of a perturbation expansion in a specific limit, so the form of the effective superpotential we have determined also takes account of possible non-perturbative effects. In more general theories, the symmetries used to constrain the superpotential can be anomalous, in which case the above argument holds to all orders in perturbation theory, but there is some scope for the superpotential to receive non-perturbative corrections. In many cases these are well understood, and so the effective superpotential can be determined exactly; *that* is the power of holomorphy.

[1] "The Power of Holomorphy: Exact Results in 4-D SUSY Field Theories," arXiv:hep-th/9408013.