2015 Part III Lectures on Extra Dimensions

Fernando Quevedo

(Second part of the course Supersymmetry and Extra Dimensions with Ben Allanach)

Contents

Chapter 1

Extra dimensions

1.1 Basics of Higher Dimensional Theories

1.1.1 Why Consider $D \geq 4$ Dimensions?

- *One argument against* $D \geq 4$. The gravitational force between two bodies $F \propto 1/R^2$ (potential $\Phi \propto 1/R$) for distance between bodies R in 4-dimensions and in general dimension D, $F \propto 1/R^{D-2}$ ($\Phi \propto 1/R^{D-3}$). Only potentials $\Phi \propto 1/R$ have stable orbits. So only in 4D there can be stable planetary systems. Exercise: Prove it.

- *However*. Experimentally we observe $D = 4$. But we know this only puts a bound on the size of the extra dimensions. Extra time dimensions are complicated (usually imply closed time-like curves affecting causaliy) but there may be as many extra space-like dimensions as long as they are small enough not to have been observed ($r \lesssim 10^{-18}$m for nongravitational physics and $r \lesssim 10^{-6}$m for gravity).

  Gauss' law implies for the electric field $\vec{E}$ and its potential Φ of a point charge Q:

$$\oint_{S^2} \vec{E} \cdot d\vec{S} = Q \implies \|\vec{E}\| \propto \frac{1}{R^2}, \quad \Phi \propto \frac{1}{R} \qquad \text{4 dimensions}$$

$$\oint_{S^3} \vec{E} \cdot d\vec{S} = Q \implies \|\vec{E}\| \propto \frac{1}{R^3}, \quad \Phi \propto \frac{1}{R^2} \qquad \text{5 dimensions}$$

  So in D spacetime dimensions

$$\|\vec{E}\| \propto \frac{1}{R^{D-2}}, \quad \Phi \propto \frac{1}{R^{D-3}}.$$

  If one dimension is compactified (radius r) like in $\mathbb{M}_4 \times S^1$, then

$$\|\vec{E}\| \propto \begin{cases} \frac{1}{R^3} & : R < r \\ \frac{1}{R^2} & : R \gg r \end{cases}.$$

  Analogues arguments hold for gravitational fields and their potentials.

- *Another argument against* $D \geq 4$. Only in 4 dimensions gauge couplings are dimensionless $S = -1/g^2 \int d^D x F_{MN} F^{MN} + \cdots$. Since $[A_M] = 1, [F_{MN}] = 2$, so $[g] = (4-D)/2$. So properly defined quantum field theories of gauge fields exist only in 4 dimensions (gauge field theories in $D \geq 4$ are non-renormalisable). However knowing that gravity is non-renormalisable already in four dimensions, a theory including both gravity and gauge interactions is already non renormalisable.

- *Another curiosity.* Gravity is 'non-trivial' for $D \geq 4$ (in the sense that the graviton field has no propagating degrees of freedom in lower dimensions).
- *Enhancing spacetime symmetries.* It is important to look for alternative ways to address the problems that supersymmetry solves and also to address other trouble spots of the Standard Model. We mentioned in the first lecture that supersymmetry and extra dimensions are the natural extensions of spacetime symmetries that may play an important role in our understanding of nature and addressing the problems of the Standard Model.
- *Technical simplifications.* Often supersymmetric theories in 4D are easier to define starting from $D \geq 4$.
- *Potential part of fundamental theories?.* String/M-theory consistent in $D = 10, 11$.
- *Maybe best argument.* Why not? After all the best way to address the question of why we observe 4-dimensions is to study physics in arbitrary number of dimensions.

Here we will start the discussion of physics in extra dimensions.

1.1.2 Brief History

- $\sim$ 150 AD Ptolemy "On dimensionality"
- 19th century Cayley, Möbius, Riemann N-dimensional geometry, ...
- In 1914 Nordstrom and 1919 - 1921 Kaluza independently tried to unify gravity and electromagnetism. Nordstrom was attempting an unsuccessful theory of gravity in terms of scalar fields, prior to Einstein. Kaluza used general relativity extended to five dimensions. His concepts were based on Weyl's ideas.
- 1926 Klein: cylindric universe with 5th dimension of small radius R
- After 1926, several people developed the KK ideas (Einstein, Jordan, Pauli, Ehrenfest,...)
- 1960's: de Witt obtaining 4 dimensional Yang Mills theories in 4d from $D > 5$. Also strings with $D = 26$.
- In 1970's and 1980's: Superstrings required $D = 10$. Developments in supergravity required extra dimensions and possible maximum numbers of dimensions for SUSY were discussed: $D = 11$ turned out to be the maximum number of dimensions (Nahm). Witten examined the coset

$$\begin{aligned} G/H &= \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \\ \dim(G/H) &= (8 + 3 + 1) - (3 + 1 + 1) = 7 \end{aligned}$$

which also implied $D = 11$ to be the minimum. 11 dimensions, however, do not admit chirality since in odd dimensions, there is no analogue of the γ_5 matrix in four dimensions.

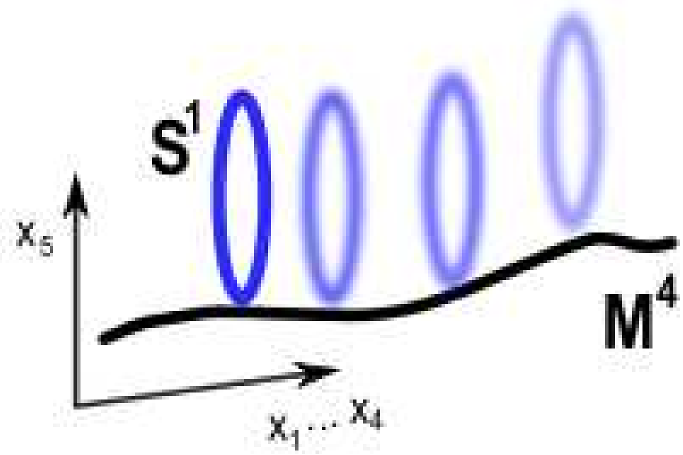

Figure 1.1: Example of a five dimensional spacetime $M^4 \times S^1$ where S^1 is a circular extra dimension in addition to four dimensional M^4.

- 1990's-2000's: Superstrings revived $D = 11$ (M theory). Brane world scenario (large extra dimensions). AdS_D/CFT_{D-1} dualities...

Exercise: *Consider the Schrödinger equation for a particle moving in two dimensions x and y. The second dimension is a circle or radius r. The potential corresponds to a square well ($V(x) = 0$ for $x \in (0, a)$ and $V = \infty$ otherwise). Derive the energy levels for the two-dimensional Schrödinger equation and compare the result with the standard one-dimensional situation in the limit $r \ll a$.*

1.2 Bosonic Field theories in extra dimensions

1.2.1 Scalar field in 5 dimensions

Before discussing higher dimensional gravity, we will start with the simpler cases of scalar fields in extra dimensions, followed by vector fields and other bosonic fields of helicity $\lambda \leq 1$. This will illustrate the effects of having extra dimensions in simple terms. We will be building up on the level of complexity to reach gravitational theories in five and higher dimensions. In the next chapter we extend the discussion to include fermionic fields.

Consider a massless 5D scalar field $\varphi(x^M)$, $M = 0, 1, ..., 4$ with action

$$\mathcal{S}_{5\text{D}} = \int \mathrm{d}^5x \, \partial^M\varphi \, \partial_M\varphi \, .$$

Set the extra dimension $x^4 = y$ defining a circle of radius r with $y \equiv y + 2\pi r$.

Our spacetime is now $\mathbb{M}_4 \times S^1$. Periodicity in y direction implies discrete Fourier expansion

$$\varphi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \varphi_n(x^\mu) \exp\left(\frac{iny}{r}\right) .$$

Notice that the Fourier coefficients are functions of the standard 4D coordinates and therefore are (an infinite number of) 4D scalar fields. The equations of motion for the Fourier modes are (in general massive) wave equations

$$\partial^M\partial_M\varphi = 0 \implies \sum_{n=-\infty}^{\infty} \left(\partial^\mu\partial_\mu - \frac{n^2}{r^2}\right) \varphi_n(x^\mu) \exp\left(\frac{iny}{r}\right) = 0$$

$$\Longrightarrow \quad \boxed{\partial^\mu \partial_\mu \varphi_n(x^\mu) \; - \; \frac{n^2}{r^2}\, \varphi_n(x^\mu) \;\; = \;\; 0 \; .}$$

These are then an infinite number of Klein Gordon equations for massive 4D fields. This means that each Fourier mode φ_n is a 4D particle with mass $m_n^2 = \frac{n^2}{r^2}$. Only the zero mode ($n = 0$) is massless. One can visualize the states as an infinite tower of massive states (with increasing mass proportional to n). This is called *Kaluza Klein tower* and the massive states ($n \neq 0$) are called *Kaluza Klein-* or *momentum states*, since they come from the momentum in the extra dimension:

Figure 1.2: The Kaluza Klein tower of massive states due to an extra S^1 dimension. Masses $m_n = |n|/r$ grow linearly with the fifth dimension's wave number $n \in \mathbb{Z}$.

In order to obtain the effective action in 4D for all these particles, let us plug the mode expansion of φ into the original 5D action,

$$\begin{aligned} \mathcal{S}_{5\mathrm{D}} \;\; &= \;\; \int \mathrm{d}^4x \int \mathrm{d}y \; \partial^M \varphi \partial_M \varphi = \int \mathrm{d}^4 x \sum_{n=-\infty}^{\infty} \Big(\partial^\mu \varphi_n(x^\mu) \, \partial_\mu \varphi_n(x^\mu)^* \; - \; \frac{n^2}{r^2} |\varphi_n|^2 \Big) \\ &= \;\; 2\,\pi\, r \int \mathrm{d}^4 x \; \big(\partial^\mu \varphi_0(x^\mu) \, \partial_\mu \varphi_0(x^\mu)^* \; + \; ... \big) \;\; = \;\; \mathcal{S}_{4\mathrm{D}} \; + \; ... \; . \end{aligned}$$

This means that the 5D action reduces to one 4D action for a massless scalar field plus an infinite sum of massive scalar actions in 4D. If we are only interested in energies smaller than the $\frac{1}{r}$ scale, we may concentrate only on the 0 mode action. If we restrict our attention to the zero mode (like Kaluza did), then $\varphi(x^M) = \varphi(x^\mu)$. This would be equivalent to just truncating all the massive fields. In this case speak of *dimensional reduction*. More generally, if we keep all the massive modes we talk about *compactification*, meaning that the extra dimension is compact and its existence is taken into account as long as the Fourier modes are included.

1.2.2 Vector fields in 5 dimensions and higher

Let us now move to the next simpler case of an abelian vector field in 5D, similar to an electromagnetic field in 4D. We can split a massless vector field $A_M(x^M)$ into

$$A_M \;\; = \;\; \begin{cases} A_\mu & \text{(vector in 4 dimensions)} \\ A_4 =: \rho & \text{(scalar in 4 dimensions)} \end{cases} \; .$$

Each component has a discrete Fourier expansion

$$A_\mu \;\; = \;\; \sum_{n=-\infty}^{\infty} A_\mu^n \, \exp\left(\frac{iny}{r}\right) \; , \qquad \rho \;\; = \;\; \sum_{n=-\infty}^{\infty} \rho_n \, \exp\left(\frac{iny}{r}\right) \; .$$

Consider the action

$$\mathcal{S}_{\rm 5D} = \int {\rm d}^5x \, \frac{1}{g_{\rm 5D}^2} \, F_{MN} \, F^{MN}$$

with field strength

$$F_{MN} := \partial_M A_N - \partial_N A_M$$

implying

$$\partial^M \partial_M A_N - \partial^M \partial_N A_M = 0 \ .$$

Choose a gauge, e.g. Lorenz

$$\partial^M A_M = 0 \implies \partial^M \partial_M A_N = 0 \ ,$$

then this obviously becomes equivalent to the scalar field case (for each component A_M) indicating an infinite tower of massive states for each massless state in 5D. In order to find the 4D effective action we once again plug this into the 5D action:

$$\mathcal{S}_{\rm 5D} \mapsto \mathcal{S}_{\rm 4D} = \int {\rm d}^4x \, \left(\frac{2\pi r}{g_{\rm 5D}^2} \, F_{(0)}{}^{\mu\nu} \, F_{(0)\mu\nu} + \frac{2\pi r}{g_{\rm 5D}^2} \, \partial_\mu \rho_0 \, \partial^\mu \rho_0 + \ldots \right) \ ,$$

Therefore we end up with a 4D theory of a gauge particle (massless), a massless scalar and infinite towers of massive vector and scalar fields. Notice that the gauge couplings of 4- and 5 dimensional actions (coefficients of $F_{MN}F^{MN}$ and $F_{\mu\nu}F^{\mu\nu}$) are related by

$$\frac{1}{g_4^2} = \frac{2\pi r}{g_5^2} \ .$$

In D spacetime dimensions, this generalises to

$$\frac{1}{g_4^2} = \frac{V_{D-4}}{g_D^2}$$

where V_n is the volume of the n dimensional compact space (e.g. an n sphere of radius r). Higher dimensional electromagnetic fields have further interesting issues that we pass to discuss.

Comments on spin and degree of freedom counting

We know that a gauge particle in 4 dimensions has spin one and carries two degrees of freedom. We may ask about the generalization of these results to a higher dimensional gauge field.
Recall Lorentz algebra in 4 dimension

$$\Big[M^{\mu\nu} \, , \, M^{\rho\sigma} \Big] = i \big(\eta^{\mu\sigma} \, M^{\nu\rho} + \eta^{\nu\rho} \, M^{\mu\sigma} - \eta^{\nu\sigma} \, M^{\mu\rho} - \eta^{\mu\rho} \, M^{\nu\sigma} \big)$$

$$J_i = \epsilon_{ijk} \, M_{jk} \ , \qquad J \propto M_{23} \ .$$

For finite dimensional massless representations in D dimensions, $O(D-2)$ is little group:

$$P^\mu = (E, \, E \, , \underbrace{0 \, , \, \ldots \, , \, 0}_{O(D-2)})$$

The Lorentz algebra is just like in 4 dimensions, replace μ, ν, ... by M, N, ..., so M_{23} commutes with M_{45} and M_{67} for example. Define the spin to be the maximum eigenvalue of any $M^{i(i+1)}$. The number of degrees of freedom in 4 dimensions is 2 ($A_\mu \mapsto A_i$ with $i = 2,3$) corresponding to the 2 photon polarizations and $(D-2)$ in D dimension, $A_M \mapsto A_i$ where $i = 1, 2, ..., D-2$.

1.2.3 Antisymmetric tensor fields, Duality and p-branes

So far we considered scalar- and vector fields:

	scalar	vector	index - range
$D=4$	$\varphi(x^\mu)$	$A_\mu(x^\mu)$	$\mu=0,1,2,3$
$D>4$	$\varphi(x^M)$	$A_M(x^M)$	$M=0,1,...,D-1$

We will see now that in extra dimensions there are further fields corresponding to bosonic particles of helicity $\lambda \leq 1$. These are antisymmetric tensor fields, which in 4D are just equivalent to scalars or vector fields by a symmetry known as *duality.* But in extra dimensions these will be new types of particles (that play an important role in string theory for instance).
Consider for example

- Massless antisymmetric tensor $B_{\mu\nu}$ in $D=4$ with field strength $H_{\mu\nu\rho}\partial_{[\mu}B_{\nu\rho]}$ with action $S = 1/g^2\int d^4x H^{\mu\nu\rho}H_{\mu\nu\rho}$

$$H_{\mu\nu\rho} \;=\; \partial_{[\mu}B_{\nu\rho]} \;\Longrightarrow\; \tilde{F}_\sigma \;=\; \epsilon_{\sigma\mu\nu\rho}\,H^{\mu\nu\rho} \;=\; \partial_\sigma a$$

  The action S can be shown to be equivalent to $S \propto g^2\int d^4x\,\partial^\mu a\partial_\mu a$ (see example sheet). Therefore a two-index massless antisymmetric tensor $B_{\mu\nu}$ is said to be dual to a massless scalar a.

- In 4 dimensions, define a dual field strength to the Faraday tensor $F^{\mu\nu}$ via

$$\tilde{F}^{\mu\nu} \;:=\; \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma}\ ,$$

  then Maxwell's equations in vacuo read:

$$\partial^\mu F_{\mu\nu} \;=\; 0 \qquad \text{(field equations)}$$
$$\partial^\mu \tilde{F}_{\mu\nu} \;=\; 0 \qquad \text{(Bianchi identities)}$$

  The exchange $F \leftrightarrow \tilde{F}$ (the *electromagnetic duality*) corresponding to $\vec{E} \leftrightarrow \vec{B}$ swaps field equations and Bianchi identities.

  Similarly in 5 dimensions, one could define in analogy

$$\tilde{F}^{MNP} \;=\; \epsilon^{MNPQR}\,F_{QR}\ .$$

- $D=6$

  So far in $D=4,5$ antisymmetric tensors of higher rank have been shown to be equivalent (dual) to known objects such as scalars and electromagnetic fields. However in $D=6$ and higher they can be seen to be a new kind of physical fields.

$$F_{MNP} \;=\; \partial_{[M}B_{NP]} \;\Longrightarrow\; \tilde{F}_{QRS} \;=\; \epsilon_{MNPQRS}\,F^{MNP} \;=\; \partial_{[Q}\tilde{B}_{RS]}$$

  Here the potentials $B_{NP} \leftrightarrow \tilde{B}_{RS}$ are of the same type. Contrary to the $D=4,5$ cases these are NEW objects that are not dual to scalaras or vectors.

One can generally start with an antisymmetric $(p+1)$ - tensor $B_{M_1\dots M_{p+1}}$ and derive a field strength

$$H_{M_1\dots M_{p+2}} \;=\; \partial_{[M_1}A_{M_2\dots M_{p+2}]}$$

and its dual (with $D-(p+2)$ indices)

$$\tilde{H}_{M_1\dots M_{D-p-2}} = \epsilon_{M_1\dots M_D} H^{M_{D-p-1}\dots M_D} .$$

Note that under duality, couplings g are mapped to (multiples of) their inverses,

$$\mathcal{L} = \frac{1}{g^2}(\partial_{[M_1}B_{M_2\dots M_{p+2}]})^2 \leftrightarrow g^2(\partial_{[M_1}\tilde{B}_{M_2\dots M_{D-(p+2)}]})^2 .$$

In these simple cases the g^2 factors can in principle be absorbed in the redefinition of the fields but for more general cases, such as supersymmetric Yang-Mills theories (or discrete cases like the Ising model) the duality actually maps strong coupling to weak couplings [3].

Exercise: *Consider the following Lagrangian*

$$\mathcal{S} = \int d^4x \left(\frac{1}{g^2} H_{\mu\nu\rho}H^{\mu\nu\rho} + a\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma}\right) .$$

Solve the equation of motion for the Lagrange multiplier a to obtain an action for a propagating massless Kalb-Ramond field $B_{\mu\nu}$. Alternatively, solve the equation of motion for the field $H_{\nu\rho\sigma}$, to obtain an action for the propagating axion field a. What happens to the coupling g under this transformation?

The procedure mentioned in the previous exercise can be generalised to any system that has a global symmetry. That is a sufficient condition for the existence of a dual theory is the existence of a global symmetry. The steps to dualisation are then: 1. Gauge the global symmetry by introducing a gauge field, 2. Introduce a Lagrange multplier constrain that sets the corresponding field strength to zero, 3. Change the order of integration to obtain the dual theory after fixing an appropriate gauge. For instance let us take the simplest case: a tensor of rank 0 (scalar) in 2-dimensions. The action $S = R^2\int d^2\sigma\partial^\mu X\partial_\mu X$, the global symmetry is $X\to X+c$. Gauging it means we change $\partial^\mu X\to D^\mu X = \partial^\mu X + A^\mu$. Then we set the field strength to zero by adding a Lagrange multiplier constraint to the action: $S=\int d^2\sigma\,(D^\mu X D_\mu X+\epsilon^{\mu\nu}\Lambda F_{\mu\nu})$ integrating over Λ sets $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu = 0$ then the gauge field is a pure gauge and can set the gauge choice $A^\mu=0$ and then get back the original action. But instead of integrating over Λ we integrate by parts the Lagrange multiplier term and solve for A^μ and fix the gauge $X=0$ leads to the dual action $\tilde{S}=(2R)^{-2}\int d^2\sigma\partial^\mu\Lambda\partial_\mu\Lambda$. This is the case for the worldsheet of string theory with one of the coordinates X living on a circle of radius R. This duality maps large radius to small radius $R\to 1/2R$ and is called $T-$*duality*. For the general antisymmetric tensor $B_{M_1\cdots M_q}$ the global symmetry is $B_{M_1\cdots M_q}\to B_{M_1\cdots M_q}+c_{M_1\cdots M_q}$ and the gauge field would be a $q+1$ rank tensor. All the steps are the same as above. Notice that the path integrals are gaussian and so solving the equations of motion are the same as integrating out.

Antisymmetric tensors carry spin 1 or less, *e.g.* in 6 dimensions:

$$B_{MN} = \begin{cases} B_{\mu\nu} & : \text{ rank two tensor in 4 dimensions} \\ B_{\mu 5}\ ,\ B_{\mu 6} & : \text{ 2 vectors in 4 dimensions} \\ B_{56} & : \text{ scalar in 4 dimensions} \end{cases}$$

To see the number of degrees of freedom, consider the little group and count the number of components in the transverse dimensions

$$B_{M_1\dots M_{p+1}} \mapsto B_{i_1\dots i_{p+1}} , \qquad i_k = 1,\dots,(D-2) .$$

These are $\binom{D-2}{p+1}$ independent components.

p branes

Antisymmetric tensors of different ranks introduce new kinds of extended objects known as p branes which are extended objects with p spatial dimensions spanning a $p+1$ dimensional worldvolume while they move in $D > p+1$ dimensions.

Let us recall the situation with point particles in $D = 4$. Electromagnetic fields couple to the worldline of particles via

$$\mathcal{S} \sim \int A_\mu \, \mathrm{d}x^\mu \ ,$$

This can be seen as follows: the electromagnetic field couples to a conserved current in 4 dimensions as $\int \mathrm{d}^4x A_\mu J^\mu$ (with Dirac current $J^\mu = \bar{\psi}\gamma^\mu\psi$ for an electron field for instance). For a particle of charge q, the current can be written as an integral over the world line of the particle $J^\mu = q \int \mathrm{d}\xi^\mu \delta^4(x-\xi)$ such that $\int J^0 \mathrm{d}^3x = q$ and so the coupling becomes $\int \mathrm{d}^4x J^\mu A_\mu = q \int \mathrm{d}\xi^\mu A_\mu$.
We can extend this idea for higher dimensional theories with higher rank tensors. For a potential $B_{[MN]}$ with two indices, the analogue is

$$\int B_{MN} \, \mathrm{d}x^M \wedge \mathrm{d}x^N \ ,$$

i.e. need a string (or 1-brane) with 2 dimensional worldsheet to couple. Further generalisations are

$$\int B_{MNP} \, \mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \mathrm{d}x^P \qquad \text{(membrane or 2-brane)}$$

$$\int B_{M_1 \ldots M_{p+1}} \, \mathrm{d}x^{M_1} \wedge \ldots \wedge \mathrm{d}x^{M_{p+1}} \qquad (p \text{ brane})$$

Therefore we can see that antisymmetric tensors of higher rank couple naturally to extended objects. This leads to the concept of a p brane as a generalisation of a particle that couples to antisymmetric tensors of rank $p+1$.

A particle carries charge under a vector field, such as electromagnetism $Q = \int d^3x J^0$ with J^μ the conserved current. In the same sense, p branes carry a **new** kind of charge with respect to a higher rank antisymmetric tensor $Z^{i_1 \ldots i_p} = \int d^{D-1}x \, J^{0 i_1 \ldots i_p}$. In the same way that in $D = 4$ there are dual objects corresponding to point-like magnetic monopoles, in arbitrary dimensions D the dual objects are (magnetic) $D-p-4$ branes that couple to the dual fields $\tilde{B}_{M_1 \ldots M_{D-p-3}}$.

1.2.4 Gravitation in extra dimensions: Kaluza Klein theory

After discussing scalar-, vector- and antisymmetric tensor fields

	spin	deg. of freedom
scalar φ	0	$1+1$
vector A_M	0 , 1	$D-2$
antisymmetric tensor $A_{M_1 \ldots M_{p+1}}$	0 , 1	$\binom{D-2}{p+1}$

we are now ready to consider the graviton G_{MN} of Kaluza Klein theory. Let us start again with $D = 5$ dimensions

$$G_{MN} = \begin{cases} G_{\mu\nu} & \text{graviton} \\ G_{\mu 4} & \text{vectors} \\ G_{44} & \text{scalar} \end{cases}$$

where $\mu, \nu = 0, 1, 2, 3$.

The background metric appears in the 5 dimensional *Einstein Hilbert action* and field equations

$$\mathcal{S} \;=\; -M_*^3 \int \mathrm{d}^5x \, \sqrt{|G|}\, {}^{(5)}R \,, \qquad {}^{(5)}R_{MN} \;=\; 0 \,.$$

Here M_* is the fundamental mass scale of the high dimensional theory (not to be confused with the four-dimensional Planck mass!). One possible solution is the 5 dimensional Minkowski metric $G_{MN} = \eta_{MN} = (+,-,-,-,-)$, another one is that of 4 dimensional Minkowski spacetime M_4 times a circle S^1, i.e. the metric is of the $\mathbb{M}_4 \times S^1$ type

$$\mathrm{d}s^2 \;=\; W(y)\, \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \;-\; \mathrm{d}y^2$$

where $\mathbb{M}_3 \times S^1 \times S^1$ is equally valid. In this setting, $W(y)$ is a warp factor that is allowed by the symmetries of the background and y is restricted to the interval $[0, 2\pi r]$. For simplicity we will set the warp factor to a constant but will consider it later where it will play an important role.
Consider the physical excitations to the background metric

$$G_{MN} \;=\; \phi^{-\frac{1}{3}} \begin{pmatrix} \left(g_{\mu\nu} \;-\; \kappa^2\, \phi\, A_\mu\, A_\nu\right) & -\kappa\, \phi\, A_\mu \\ -\kappa\, \phi\, A_\nu & -\phi \end{pmatrix}$$

where κ is a constant to be fixed. Performing the Fourier expansion

$$g_{\mu\nu} = \sum_{-\infty}^{\infty} g_{\mu\nu}^n e^{iny/r} \,, \qquad A_\mu = \sum_{-\infty}^{\infty} A_\mu^n e^{iny/r} \,, \qquad \phi = \sum_{-\infty}^{\infty} \phi^n e^{iny/r}$$

we can write

$$G_{MN} \;=\; \underbrace{\phi^{(0)-\frac{1}{3}} \begin{pmatrix} \left(g_{\mu\nu}^{(0)} \;-\; \kappa^2\, \phi^{(0)}\, A_\mu^{(0)}\, A_\nu^{(0)}\right) & -\kappa\, \phi^{(0)}\, A_\mu^{(0)} \\ -\kappa\, \phi^{(0)}\, A_\nu^{(0)} & -\phi^{(0)} \end{pmatrix}}_{\text{Kaluza Klein ansatz}} + \infty \text{ tower of massive modes}$$

and plug the zero mode part into the Einstein Hilbert action:

$$\mathcal{S}_{4D} \;=\; -\int \mathrm{d}^4x \, \sqrt{|g|} \left\{ M_{\mathrm{pl}}^2\, {}^{(4)}R \;+\; \frac{1}{4}\phi^{(0)}\, F_{\mu\nu}^{(0)}\, F^{(0)\mu\nu} \;+\; \frac{M_{\mathrm{pl}}^2}{6}\, \frac{\partial^\mu \phi^{(0)}\, \partial_\mu \phi^{(0)}}{(\phi^{(0)})^2} \;+\; \ldots \right\}$$

Where in order to absorb the constant in the Maxwell term we have set $\kappa^{-1} = M_{\mathrm{pl}} = \sqrt{hc/G}$ the 4-dimensional Planck scale. Notice we have obtained a unified theory of gravity, electromagnetism and scalar fields!

Exercise: *Show that the last equation follows from a pure gravitational theory in five-dimensions, using* ${}^{(5)}R = {}^{(4)}R - 2e^{-\sigma}\nabla^2 e^\sigma - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}$ *where* $G_{55} = e^{2\sigma}$*. Relate the gauge coupling to the* $U(1)$ *isometry of the compact space.*

Comment

The Planck mass $M_{\mathrm{pl}}^2 = M_*^3 \cdot 2\pi r$ is a derived quantity. We know experimentally that $M_{\mathrm{pl}} \approx 10^{19}$ GeV, therefore we can adjust M_* and r to give the right result. But there is no other constraint to fix M_* and r. So at this level M_* can only be determined after the radius of the circle is fixed.

Symmetries

From the Kaluza-Klein ansatz we can write the line element as

$$\mathrm{d}s^2 = \phi^{(0)-\frac{1}{3}} \left(g^{(0)}_{\mu\nu} dx^\mu dx^\nu - \phi^{(0)} \left(dy + ka A^{(0)}_\mu dx^\mu \right)^2 \right)$$

From here we can explicitly see the symetries of the low energy effective action in $D = 4$.

- general 4 dimensional coordinate transformations

$$x^\mu \quad \mapsto \quad x'^\mu(x^\nu) \ , \qquad g^{(0)}_{\mu\nu} \ \text{(graviton)} \ , \qquad A^{(0)}_\mu \ \text{(vector)}$$

- y transformation

$$y \quad \mapsto \quad y' \quad = \quad F(x^\mu, y)$$

In order to leave $\mathrm{d}s^2$ invariant, need

$$F(x^\mu, y) \quad = \quad y \ + \ f(x^\mu) \quad \Longrightarrow \quad \mathrm{d}y' \quad = \quad \mathrm{d}y \ + \ \frac{\partial f}{\partial x^\mu} \, \mathrm{d}x^\mu \ , \qquad A'^{(0)}_\mu \quad = \quad A^{(0)}_\mu \ - \ \frac{1}{\kappa} \frac{\partial f}{\partial x^\mu}$$

which are gauge transformation for a massless field $A^{(0)}_\mu$! This is the way to understand that standard gauge symmetries can be derived from general coordinate transformations in extra dimensions, explaining the Kaluza Klein programme of unifying all the interactions by means of extra dimensions.

- overall scaling

$$y \quad \mapsto \quad \lambda \, y \ , \qquad A^{(0)}_\mu \quad \mapsto \quad \lambda \, A^{(0)}_\mu \ , \qquad \phi^{(0)} \quad \mapsto \quad \frac{1}{\lambda^2} \, \phi^{(0)} \quad \Longrightarrow \quad \mathrm{d}s^2 \quad \mapsto \quad \lambda^{\frac{2}{3}} \, \mathrm{d}s^2$$

$\phi^{(0)}$ is a massless *modulus field*, a flat direction in the potential, so $\langle \phi^{(0)} \rangle$ and therefore the size of the 5th dimension is arbitrary. $\phi^{(0)}$ is called breathing mode, radion or *dilaton*. This is a major problem for these theories: It looks like **all** the values of the radius (or volume in general) of the extra dimensions are equally good and the theory does not provide a way to fix this size. It is a manifestation of the problem that the theory cannot prefer a flat 5D Minkowski space (infinite radius) over $\mathbb{M}_4 \times S^1$ (or $\mathbb{M}_3 \times S^1 \times S^1$, etc.). This is the *moduli problem* of extra dimensional theories. String theories share this problem. Recent developments in string theory allows to fix the value of the volume and shape of the extra dimension, leading to a large but discrete set of solutions. This is the so-called "landscape" of string solutions (each one describing a different universe and ours is only one among a huge number of them).

Generalization to more dimensions

$$G_{MN} \quad = \quad \begin{pmatrix} \left(g_{\mu\nu} \ + \ \kappa^2 \, \gamma_{mn} K^m_i K^n_j \, A^i_\mu \, A^j_\nu \right) & \kappa \, \gamma_{mn} \, K^n_i \, A^i_\mu \\ \kappa \, \gamma_{mn} \, K^m_i \, A^i_\nu & \gamma_{mn} \end{pmatrix}$$

The K^m_i are Killing vectors of an internal manifold $\mathcal{M}_{D-4}$ with metric γ_{mn}. The theory corresponds to Yang Mills in 4 dimensions with gauge group corresponding to the isometry of the extra dimensional manifold. Note that the Planck mass now behaves like

$$\boxed{M^2_{\text{pl}} \quad = \quad M^{D-2}_* \, V_{D-4} \quad \sim \quad M^{D-2}_* \, r^{D-4} \quad = \quad M^2_* \, (M_* \, r)^{D-4}.}$$

In general we know that the highest energies explored so far require $M_* > 1$ TeV and $r < 10^{-16}$ cm since no signature of extra dimensions has been seen in any experiment. In Kaluza Klein theories there is no reason to expect a large value of the volume and it has been usually assumed that $M_* \approx M_{\rm pl}$.

Finally we can also count the number of degrees of freedom of a graviton in D dimensions using again the transverse dimensions.

$$N_{dof} = \frac{(D-2)(D-1)}{2} - 1 \quad = \quad \frac{D(D-3)}{2}$$

Corresponding to the number of components of a traceless symmetric tensor in the $D-2$ transverse dimensions. Notice that only in $D = 4$ the graviton and the photon have the same number of degrees of freedom ($N_{dof} = 2$).

1.3 The brane world scenario

So far we have been discussion the standard Kaluza Klein theory in which our universe is higher dimensional. We have not seen the extra dimensions because they are very small (smaller than the smallest scale that can be probed experimentally at colliders which is 10^{-16} cm).

We will introduce now a different and more general higher dimensional scenario. The idea here is that our universe is a p brane, or a surface inside a higher dimensional *bulk spacetime*. A typical example of this is as follows: all the Standard Model particles (quarks, leptons but also gauge fields) are trapped on a 3 dimensional spatial surface (the brane) inside a higher dimensional spacetime (the bulk). Gravity on the other hand lives on the full bulk spacetime and therefore only gravity probes the extra dimensions. The total action can be written as:

$$S = S_{bulk} + S_{brane}$$

with

$$S_{bulk} = -M_*^{D-2} \int \mathrm{d}^D x \sqrt{|G|}^{(D)} R$$

and

$$S_{brane} = \int \mathrm{d}^4 x \sqrt{|\gamma|} \left(\mathcal{L}(matter)\right)$$

where $\gamma_{\mu\nu}$ is the induced metric on the brane, which for simplicity we are considering it to be a $p = 3$ brane but in principle it could be any other dimensionality $p \leq D - 1$.

Therefore we have to distinguish the D dimensional bulk space (background spacetime) from the $(p+1)$ world volume coordinates of a p brane. Matter lives in the $d(= 4)$ dimensions of the brane, whereas gravity takes place in the D bulk dimensions. This scenario seems very ad hoc at first sight but it is naturally realized in string theory where matter tends to live on D branes (a particular class of p branes corresponding to surfaces where ends of open strings are attached to). Whereas gravity, coming from closed strings can leave in the full higher dimensional ($D = 10$) spacetime. Then the correspondence is as follows:

$$\text{gravity} \quad \longleftrightarrow \quad \text{closed strings}$$
$$\text{matter} \quad \longleftrightarrow \quad \text{open strings}$$

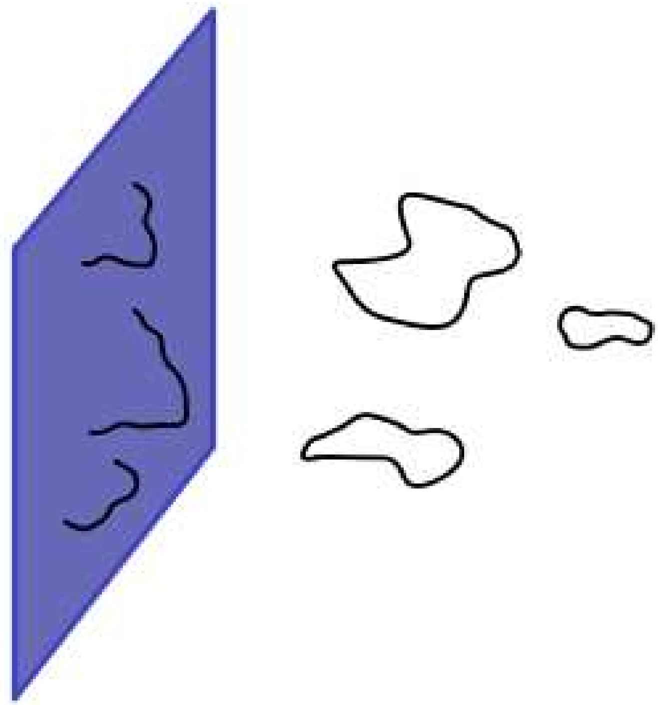

Figure 1.3: Brane world scenario with matter corresponding to open strings which start and end on the brane and gravity incorporated by closed strings probing the full bulk spacetime.

For phenomenological purposes we can distinguish two different classes of brane world scenarios.

1.3.1 Large extra dimensions

Let us first consider an unwarped compactification, that is a constant warp factor $W(y)$. We have remarked that the fundamental higher dimensional scale M_* is limited to be $M_* \geq 1$ TeV in order to not contradict experimental observations which can probe up to that energy. By the same argument we have constrained the size of the extra dimensions r to be $r < 10^{-16}$ cm because this is the length associated to the TeV scale of that accelerators can probe. However, in the brane world scenario, if only gravity feels the extra dimensions, we have to use the constraints for gravity only. Since gravity is so weak, it is difficult to test experimentally and so far the best experiments can only test it to scales larger than ≈ 0.1 mm. This is much larger than the 10^{-16} cm of the Standard Model. Therefore, in the brane world scenario it is possible to have extra dimensions as large as 0.1 mm without contradicting any experiment! This has an important implication also as to the value of M_* (which is usually taken to be of order $M_{\rm pl}$) in Kaluza Klein theories. From the Einstein Hilbert action, the Planck mass $M_{\rm pl}$ is still given by

$$M_{\rm pl}^2 \quad = \quad M_*^{D-2}\, V_{D-4}$$

with $V_{D-4} \sim r^{D-4}$ denoting the volume of the extra dimensions. But now we can have a much smaller fundamental scale M_* if we allow the volume to be large enough. We may even try to have the fundamental scale to be of order $M_* \sim 1$ TeV. In five dimensions, this will require a size of the extra dimension to be of order $r \approx 10^8$ km in order to have a Planck mass of the observed value $M_{\rm pl} \approx 10^{18}$ GeV (where we have used $r = M_{\rm pl}^2/M_*^3$). This is clearly ruled out by experiments. However, starting with a 6 dimensional spacetime we get $r^2 = M_{\rm pl}^2/M_*^4$, which gives $r \approx 0.1$mm for $M_* = 1$ TeV. This is then consistent with all gravitational experiments as well as Standard Model tests. Higher dimensions would give smaller values of r and will also be consistent. The interesting thing about the 6 dimensional case is that it is possible to be tested by the next round of experiments in both, the accelerator experiments probing scales of order TeV and gravity experiments, studying deviations of the squared law at scales smaller than 0.1mm.

Notice that this set up changes the nature of the hierarchy problem because now the small scale (i.e. $M_{\rm ew} \approx M_* \approx 1$ TeV) is fundamental whereas the large Planck scale is a derived quantity. The hierarchy problem now is changed to explain why the size of the extra dimensions is so large to generate the Planck scale of 10^{18} GeV starting from a small scale $M_* \approx 1$ TeV. This changes the nature of the hierarchy problem, because it turns it into a dynamical question of how to fix the size of the extra dimensions. Notice that this will require exponentially large extra dimensions (in units of the inverse fundamental scale M_*). The hierarchy problem then becomes the problem of finding a mechanism that gives rise to exponentially large sizes of the extra dimensions.

Number of extra dimensions	Size of r for $M_* = 1$TeV
1	$10^{11} m$
2	$10^{-4} m$
3	$10^{-9} m$
6	$10^{-12} m$

Exercise: Demonstrate that the volume of a $N-1$ sphere of radius r is

$$V_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}\, r^{N-1} \tag{1.1}$$

Hint: It may help to consider the integral $I_N = \int d^N x e^{-\rho^2}$ with $\rho^2 = \sum_{i=1}^N x_i^2$. Use this result to derive an expression for the electric (and gravitational) potential in D dimensions. Show that the potential due to a point particle in five dimensions reduces to the 4-dimensional potential at distances much larger than the size of the fifth dimension.

1.3.2 Warped compactifications

This is the so-called *Randall Sundrum scenario.* The simplest case is again a 5 dimensional theory but with the following properties. Instead of the extra dimension being a circle S^1, it is now an interval I (which can be defined as an *orbifold* of S^1 by identifying the points $y \equiv -y$, if the original circle had length $2\pi r$, the interval I will have half that size, πr). The surfaces at each end of the interval play a role similar to a brane, being 3 dimensional surfaces inside a 5 dimensional spacetime. The second important ingredient is that the warp factor $W(y)$ is determined by solving Einstein's equations in this background. We then have warped geometries with a y dependent warp factor $\exp\big(W(y)\big)$, in 5 dimensions

$$\mathrm{d}s^2 \;=\; \exp\big(W(y)\big)\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;-\; \mathrm{d}y^2\ .$$

The volume V_{D-4} has a factor

$$V_{D-4} \quad \sim \quad \int_{-\pi}^{+\pi} \mathrm{d}y \;\; \exp\bigl(W(y)\bigr) \;.$$

Consider then the two branes,one at $y = 0$ ("the Planck brane") and one at $y = \pi r$ ("the Standard Model brane"), the total action has contributions from the two branes and the bulk itself:

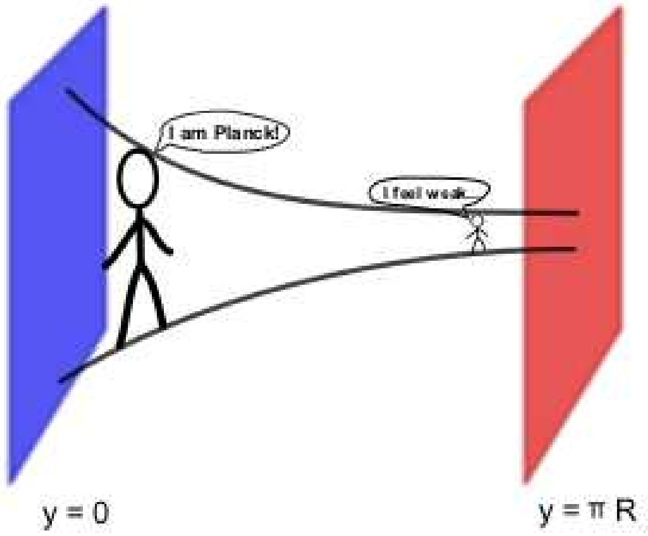

Figure 1.4: Brane configuration in the Randall-Sundrum scenario: The warped geometry in the y direction gives rise to a mass hierarchy between the Planck brane at $y = 0$ and the Standard Model brane at $y = \pi r$. Notice this is only a cartoon. The energy scales are redshifted so that we can have Planck scale on the left brane and the TeV scale on the right brane.

$$\mathcal{S} \quad = \quad \mathcal{S}_{y=0} \; + \; \mathcal{S}_{y=\pi r} \; + \; \mathcal{S}_{\text{bulk}}$$

Einstein's equations imply $W(y) \propto e^{-|ky|}$ with k a constant (see [5] and example sheet 4), so the metric changes from $y = 0$ to $y = \pi r$ via $\eta_{\mu\nu} \longmapsto \exp(-k\pi r)\eta_{\mu\nu}$. This means that all the length and energy scales change with y. If the fundamental scale is $M_* \approx M_{\text{pl}}$, the $y = 0$ brane carries physics at M_{pl}, but as long as we move away from this end of the interval, all the energy scales will be "red shifted" by the factor $e^{-|ky|}$ until we reach the other end of the interval in which $y = \pi r$. This exponential changes of scales is appropriate for the hierarchy problem. If the fundamental scale is the Planck scale, at $y = 0$ the physics will be governed by this scale but at $y = r$ we will have an exponentially smaller scale. In particular we can have the electroweak scale $M_{\text{ew}} \approx M_{\text{pl}} \cdot e^{-\pi kr} \approx 1$ TeV if r is only slightly bigger than the Planck length $r \geq 50\,\ell_{\text{pl}}$. This is a more elegant way to "solve' the hierarchy problem. We only need to find a mechanism to fix the value of r of order $50\,\ell_{\text{pl}}$! Notice that in this scenario 5 dimensions are compatible with experiment (unlike the unwarped case that required a radius many kilometers large).

Exercise: *Consider a five dimensional gravity theory with a negative cosmological constant $\Lambda < 0$, compactified on an interval $(0, \pi)$. Each end of the interval corresponds to a '3-brane' which we choose to have tension $\pm\Lambda/k$ respectively. Here k is a common scale to be determined later in terms of the fundamental scale in 5D M and Λ. Verify that the warped metric*

$$\mathrm{d}s^2 \quad = \quad e^{-2W(\theta)}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;\; - \;\; r^2\,\mathrm{d}\theta^2$$

satisfies Einstein's equations. Here $e^{-2W(\theta)}$ *is the warp factor and* r *is a constant measuring the size of the interval. You can use that Einstein's equations reduce to*

$$\frac{6\,W'^2}{r^2} \;=\; -\frac{\Lambda}{2M^3}\,, \qquad \frac{3\,W''}{r^2} \;=\; \frac{\Lambda}{2\,M^3\,kr}\left[\delta(\theta-\pi)\;-\;\delta(\theta)\right].$$

Solve for $W(\theta)$ *and use the warp factor to show that the effective 4D Planck scale is now*

$$M_{\rm pl}^2 \;=\; M^3\,r\int_{-\pi}^{\pi}{\rm d}\theta\; e^{-2W} \;=\; \frac{M^3}{k}\left(1\;-\;e^{-2kr}\right).$$

Find the value of the constant k*. Consider the Higgs Lagrangian on the brane at* $\theta = \pi$*, bring it into canonical form and show that the mass is proportional to the factor* $e^{-k\pi r}$*. How large can* r *be in order to reproduce the electroweak scale from the Planck scale? Does this solve the hierarchy problem? How does the Planck scale differ from the 5D scale* M*?*

1.3.3 Brane world scenarios and the hierarchy problem

Large and warped extra dimensions are alternatives to supersymmetry to address the hierarchy problem. In the large extra dimensions scenario the hierarchy problem is exchanged to the problem of finding compactifications with very large volumes.

In the warped case the 'solution' is more elegant as we can see on the simple $D = 5$ example mentioned above (Randall-Sundrum).

The Higgs Lagrangian is

$$\begin{aligned}
\mathcal{S}_{\rm vis} \;&\sim\; \int{\rm d}^4x\;\sqrt{|g_{\rm vis}|}\left[g_{\rm vis}^{\mu\nu}\,D_\mu H^\dagger\,D_\nu H\;-\;\lambda\,(|H|^2\;-\;v_0^2)^2\right]\\
&\overset{g_{\rm vis}=e^{-2kr\pi}g_4}{\sim}\; \int{\rm d}^4x\;\sqrt{|g_4|}\,e^{-4kr\pi}\left[g_4^{\mu\nu}\,e^{2kr\pi}\,D_\mu H^\dagger\,D_\nu H\;-\;\lambda\,(|H|^2\;-\;v_0^2)^2\right]\\
&\overset{He^{kr\pi}\to H}{=}\; \int{\rm d}^4x\;\sqrt{|g_4|}\left[g_4^{\mu\nu}\,D_\mu H^\dagger\,D_\nu H\;-\;\lambda\,(|H|^2\;-\;e^{-2kr\pi}v_0^2)^2\right]
\end{aligned}$$

In the last step we canonically normalised the Higgs field such that the kinetic terms are canonical. The Higgs mass then is given by $m_H = e^{-kr\pi}v_0$ and depends on the warp factor. The natural scale for v_0 is the Planck scale. To obtain a Higgs mass at the weak scale we need $\pi kr \sim 50$. The 4-dim Planck scale and the 5-dim scale M are here comparable as e^{-2kr} is tiny.

$$m_{\rm H} \;\sim\; e^{-kr\pi}\,M_{\rm pl}$$

This solves the hierarchy problem through warping as long as a mechanism can be found to stabilise the radius r to the required value. The advantage over the large extra dimensions is that it looks more factible to stabilise r to values $\pi kr \sim 50$ than the hierarchically large values needed for the volume in the unwarped case.

Building concrete models that include the standard model and addressing its other problems on a brane is not straightforward. Notice that in both scenarios, the problem of solving the hierarchy problem has been turned into the problem of fixing the size of the extra dimensions. It is worth remarking that both mechanisms have been found to be realised in string theory (putting them on firmer theoretical grounds since otherwise they are *adhoc* scenarios based on higher dimensional (nonrenormalisable) gravitational theories). Studying mechanisms to fix the moduli that determines the size and shape of extra dimensions is one of the most active areas of research within string theory and higher dimensional theories in general.

Chapter 2

Supersymmetry in higher dimensions

So far we have discussed the possible bosonic fields in extra dimensions (scalars, vectors, antisymmetric tensors and metrics). What about fermionic fields in extra dimensions? Good references for the technical aspects are [6, 8, 9].

2.1 Spinors in higher dimensions

For a theory of fermions in more than four dimensions, need some analogue of the four dimensional Dirac γ matrices, i.e. representations of the Clifford algebra

$$\left\{ \Gamma^M \ , \ \Gamma^N \right\} \ = \ 2\,\eta^{MN} \ , \qquad \Sigma^{MN} \ = \ \frac{i}{4} \left[\Gamma^M \ , \ \Gamma^N \right] \ ,$$

where the Σ^{MN} are generators of $SO(1, D-1)$ subject to the Lorentz algebra

$$\left[\Sigma^{MN} \ , \ \Sigma^{PQ} \right] \ = \ i \left(\Sigma^{MQ}\,\eta^{NP} \ + \ \Sigma^{NP}\,\eta^{MQ} \ - \ \Sigma^{MP}\,\eta^{NQ} \ - \ \Sigma^{NQ}\,\eta^{MP} \right) \ .$$

2.1.1 Spinor representations in even dimensions $D = 2n$

Define n pairs of ladder operators

$$a^0 \ := \ \frac{i}{2} \left(\Gamma^0 \ + \ \Gamma^1 \right) \qquad \Longrightarrow \qquad (a^0)^\dagger \ = \ \frac{i}{2} \left(-\Gamma^0 \ + \ \Gamma^1 \right)$$
$$a^j \ := \ \frac{i}{2} \left(\Gamma^{2j} \ - \ i\Gamma^{2j+1} \right) \qquad \Longrightarrow \qquad (a^j)^\dagger \ = \ \frac{i}{2} \left(\Gamma^{2j} \ + \ i\Gamma^{2j+1} \right) \ , \qquad j = 1, ..., n-1 \ ,$$

whose hermiticity properties are due to $(\Gamma^0)^\dagger = +\Gamma^0$ and $(\Gamma^{M\neq 0})^\dagger = -\Gamma^{M \neq 0}$. From the Clifford algebra in $\eta^{MN} = \mathrm{diag}(+1, \ -1, \ ..., \ -1)$ signature, it follows that the a^j (where $j = 0, 1, ..., n-1$ now) furnish a set of n fermionic oscillators

$$\left\{ a^i \ , \ (a^j)^\dagger \right\} \ = \ \delta^{ij} \ , \qquad \left\{ a^i \ , \ a^j \right\} \ = \ \left\{ (a^i)^\dagger \ , \ (a^j)^\dagger \right\} \ = \ 0 \ .$$

Let $|0\rangle$ denote the vacuum such that $a^i|0\rangle = 0$, then there are states

states	$\|0\rangle$	$(a^i)^\dagger\,\|0\rangle$	$(a^i)^\dagger\,(a^j)^\dagger\,\|0\rangle$	$\cdots$	$(a^n)^\dagger\,(a^{n-1})^\dagger \ldots (a^1)^\dagger\,\|0\rangle$
number	1	n	$\binom{n}{2}$	$\cdots$	1

of total number

$$1 \ + \ n \ + \ \begin{pmatrix} n \\ 2 \end{pmatrix} \ + \ ... \ + \ 1 \ = \ \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \ = \ 2^n \ = \ 2^{\frac{D}{2}} \ .$$

States in the spinor representations are defined by $n = D/2$ quantum numbers $s_i = \pm\frac{1}{2}$

$$|s_0, \dots, s_{n-1}\rangle \;:=\; (a^0)^{\dagger(s_0+\frac{1}{2})} \dots (a^{n-1})^{\dagger(s_{n-1}+\frac{1}{2})} |0\rangle \; .$$

Note that the generators $\Sigma^{(2i)(2i+1)}$ mutually commute. So we diagonalize all of

$$\begin{aligned} (a^0)^\dagger a^0 - \frac{1}{2} &= +\frac{1}{4}\left[\Gamma^0 , \Gamma^1\right] &= -i\Sigma^{01} \\ (a^j)^\dagger a^j - \frac{1}{2} &= \frac{i}{4}\left[\Gamma^{2j} , \Gamma^{2j+1}\right] &= \Sigma^{(2j)(2j+1)} \end{aligned}$$

and find the $|s_0, ..., s_{n-1}\rangle$ defined above to be the simultaneous eigenstates of

$$S^i \;:=\; \begin{cases} (a^0)^\dagger a^0 - \frac{1}{2} = -i\Sigma^{01} & : \; i = 0 \\ (a^i)^\dagger a^i - \frac{1}{2} = \Sigma^{(2i)(2i+1)} & : \; i = 1, ..., n-1 \end{cases}$$

in the sense that

$$S^i \, |s_0, \dots, s_{n-1}\rangle \;=\; s_i \, |s_0, \dots, s_{n-1}\rangle \; .$$

Call those $|s_0, ..., s_{n-1}\rangle$ *Dirac spinors*. In $D = 4$ dimensions with $n = 2$, for instance, the states $|\pm\frac{1}{2}, \pm\frac{1}{2}\rangle$ form a 4 component spinor.

Representations in even dimensions are reducible, since the generalization of γ^5,

$$\Gamma^{2n+1} \;:=\; i^{n-1}\,\Gamma^0\,\Gamma^1 \dots \Gamma^{2n-1} \; ,$$

satisfies

$$\left\{\Gamma^{2n+1} , \Gamma^M\right\} = 0 \; , \qquad \left[\Gamma^{2n+1} , \Sigma^{MN}\right] = 0 \; , \qquad (\Gamma^{2n+1})^2 = \mathbb{1} \; .$$

It follows from

$$\begin{aligned} 2^n \, S^0 \, S^1 \dots S^{n-1} &= 2^n \, \frac{1}{4}\left(+\frac{i}{4}\right)^{n-1} \left[\Gamma^0 , \Gamma^1\right] \dots \left[\Gamma^{2n-2} , \Gamma^{2n-1}\right] \\ &= i^{n-1}\Gamma^0\,\Gamma^1 \dots \Gamma^{2n-1} \;=\; \Gamma^{2n+1} \; . \end{aligned}$$

that all the $|s_0, ..., s_{n-1}\rangle$ are eigenstates to Γ^{2n+1}

$$\Gamma^{2n+1}\,|s_0, \dots, s_{n-1}\rangle \;=\; \pm|s_0, \dots, s_{n-1}\rangle$$

with eigenvalue $+1$ for even numbers of $s_i = -\frac{1}{2}$ and -1 for odd ones. This property is called *chirality*, and spinors of definite chirality are referred to as *Weyl spinors*.

2.1.2 Spinor representations in odd dimensions $D = 2n + 1$

Just add $i\Gamma^{2n+1} = i^n\Gamma^0\Gamma^1 \dots \Gamma^{2n-1}$ to the Γ^M matrices of $D = 2n$ dimensions. From its properties $\{\Gamma^{2n+1}, \Gamma^M\} = 0$ and $(\Gamma^{2n+1})^2 = 1$, it perfectly extends the Clifford algebra in $D = 2n$ dimensions to $D = 2n + 1$ with extended metric $\eta^{\mu\nu} = (+1, -1, ..., -1)$.

Since there is no further Γ matrix with which Γ^{2n+1} could be paired to a further a^i operator, the representation is the same as for $D = 2n$, but now irreducible. The $SO(1, 2n)$ generators in addition to those of $SO(1, 2n-1)$ are given by $\frac{i}{2}\Gamma^M\Gamma^{2n+1}$ with $M = 0, 1, ..., 2n-1$. Since odd dimensions do not have a "γ^5", there is no chirality. The spinor representations' dimension is $2^{\frac{D-1}{2}}$.

In general, define N_D to give the number of spinor components:

$$N_D \;:=\; \begin{cases} 2^n = 2^{\frac{D}{2}} & : \; D = 2n \text{ even} \\ 2^n = 2^{\frac{D-1}{2}} & : \; D = 2n+1 \text{ odd} \end{cases}$$

2.1.3 Majorana spinors (extra material not covered in lectures)

Let us now introduce the notion of *reality* for spinors in Minkowski spacetime. Under infinitesimal Lorentz transformations, spinors ψ transform into $\psi' = \psi + i\omega_{MN}\Sigma^{MN}\psi$. Since the Σ^{MN} are in general complex, it is not guaranteed that relations between ψ and its complex conjugate ψ^* are consistent with Lorentz transformations.

A relation between $\psi \leftrightarrow \psi^*$ is referred to as the *Majorana condition.* It has to be of the form $\psi^* = C\Gamma^0\psi$ where C is the *charge conjugation matrix.* Consistency requires $(C\Gamma^0) * C\Gamma^0 = 1$ which is possible in dimensions $D = 0, 1, 2, 3, 4 \mod 8$. In other words, among the physically sensible dimensions, $D = 5, 6, 7$ do not admit a Majorana condition.

A Majorana condition can be imposed on a Weyl spinor if $D = 0, 1, 2, 3, 4 \mod 8$ and the Weyl representation is conjugate to itself. Weyl spinors exist in even dimensions $D = 2n$, and by analyzing the complex conjugate of the chirality matrix

$$(\Gamma^{2n+1})^* \quad = \quad (-1)^{n+1}\, C^{-1}\, \Gamma_0^{-1}\, \Gamma^{2n+1}\, \Gamma_0\, C \ ,$$

it turns out that charge conjugation only preserves the spinors' chirality if $(-1)^{n+1} = +1$. If n is even, i.e. in $D = 4, 8, 12, ...$ dimensions, the two inequivalent Weyl representations are complex conjugate to each other, and one can either impose the Weyl or Majorana condition, but not both! In dimensions $D = 2 \mod 8$, the Weyl representations are self conjugate and compatible with the Majorana condition, so Majorana Weyl spinors are possible in dimensions $D = 2$ and $D = 10$.

2.2 Supersymmetry algebra

The SUSY algebra in D dimensions consists of generators M_{MN}, P_M, Q^A_α last of which are spinors in D dimensions (with index A counting the number of supersymmetries as in the extended SUSY case in $D = 4$). The algebra has the same structure as in 4 dimensions, with the bosonic generators defining a standard Poincaré algebra in higher dimensions, Q^A_α transforming as spinors imply:

$$[M^{MN}, Q^A_\alpha] = -\left(\Sigma^{MN}\right)_\alpha^{\ \beta} Q^A_\alpha$$

where Σ^{MN} as defined above represent the Lorentz transformation in the spinorial representation. Finally the pure spinorial part:

$$\left\{Q^A_\alpha \ , \ Q^B_\beta\right\} \quad = \quad \left(\Omega^{AB}_{\alpha\beta}\right)^M P_M \ + \ Z^{AB}_{\alpha\beta}$$

where $\left(\Omega^{AB}_{\alpha\beta}\right)^M$ are dimension-dependent constants and the central charges $Z^{AB}_{\alpha\beta}$ now can also include brane charges. This is the $D > 4$ Coleman Mandula- or HLS generalization of the $D = 4$ algebra. The arguments for the proof are identical to those in 4 dimensions and we will skip them here.

A new feature of the Poincaré algebra is that all the generators $M^{(2j)(2j+1)}$ commute with each other and can thus be simultaneously diagonalized as we have seen in the discussion of the higher dimensional spinorial representation. Then we can have several "spins" defined as the eigenvalues of these operators. Of particular relevance is the generator M^{01}. This is used to define a *weight* w of an operator $\mathcal{O}$ by

$$\left[M^{01} \ , \ \mathcal{O}\right] \quad = \quad -iw\,\mathcal{O}$$

where $\mathcal{O}$ and $\mathcal{O}^*$ have the same weight.

2.2.1 Representations of supersymmetry algebra in higher dimensions

Similar to the 4-dimensional case we consider the massless states defined by momenta

$$P^\mu = (E\ ,E\ ,0\ ,...\ ,0)$$

again the little group ($ISO(D-2)$) has infinite dimensional representations. If we limit to only finite dimensional representations we restrict to the smaller little group $O(D-2)$[1]. We define the spin to be the maximum eigenvalue of M_{MN} in the representation. Notice that for the momentum of a massless particle $P^1 - P^0 = 0$ and that

$$\left[M^{01}\ ,\ P^1 \pm P^0\right] \;=\; \mp i\left(P^1 \pm P^0\right) .$$

Therefore the weight of $P^1 \pm P^0$ is $w = \pm 1$. As the "$-$" combination $P^1 - P^0$ is zero in massless representations, the weight $w = -1$ can be excluded and we only need to consider combinations of $\{Q, Q\}$ in which both Q's have weight $w = +\frac{1}{2}$.
So if we start with arbitrary spinors Q_α of the form

$$Q_\alpha \;=\; |\pm\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \cdots,\ \pm\tfrac{1}{2}\rangle\ , \qquad \alpha = 1, ..., N_D$$

with N_D components (recall that $N_D = 2^{\frac{D}{2}}$ for even and $N_D = 2^{\frac{D-1}{2}}$ for odd dimensionality respectively), requiring weight $+\frac{1}{2}$ means that (, as a special case of $[M^{MN}, Q_\alpha] = -\Sigma^{MN}Q_\alpha$,)

$$\left[M^{01}\ ,\ Q_\alpha\right] \;=\; -\Sigma^{01}\,Q_\alpha \;=\; -iS^0\,Q_\alpha \;\stackrel{!}{=}\; -\frac{i}{2}\,Q_\alpha\ ,$$

so Q_α has to be of the form

$$Q_\alpha\Big|_{w=+\frac{1}{2}} \;=\; |\stackrel{\downarrow}{+}\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \cdots,\ \pm\tfrac{1}{2}\rangle\ , \qquad \alpha = 1, ..., \frac{N_D}{2}\ .$$

This leads to half of the number of components of Q_α in the massless case, namely $\frac{N_D}{2}$.
Furthermore, we can separate the Q's into Q^+ and Q^- according to eigenvalues of M_{23} (standard spin in 4d). They furnish an algebra of the form $\{Q^+, Q^+\} = \{Q^-, Q^-\} = 0$ and $\{Q^+, Q^-\} \neq 0$ corresponding to creation- and annihilation operators. To see this, consider the commutator

$$\left[M^{(2j)(2j+1)}\ ,\ Q_{(\alpha}\,Q_{\beta)}\right] \;=\; -Q_{(\alpha}\,S^j\,Q_{\beta)} \;-\; S^j\,Q_{(\alpha}\,Q_{\beta)} \;=\; -(s_j^{(\alpha)} \;+\; s_j^{(\beta)})\,Q_{(\alpha}\,Q_{\beta)}\ .$$

Using the super Poincaré algebra, we can also show this expression to be a linear combination of the $P^2 ... P^{D-1}$ which are all zero in our case $P^\mu = (E\ ,E\ ,0\ ,...\ ,0)$. Consequently, all the combinations $s_j^{(\alpha)} + s_j^{(\beta)}$ have to vanish leaving $\{Q_\alpha^+, Q_{\beta=\alpha}^-\}$ as the only nonzero anticommutators.
This implies that a supersymmetric multiplet can be constructed starting from a "vacuum" state $|\lambda\rangle$ of helicity λ annihilated by the Q^- operators, $Q^-|\lambda\rangle = 0$, and the rest of the states in the multiplet are generated by acting on Q^+. Therefore they will be of the form

$$Q_\alpha^+\Big|_{w=+\frac{1}{2}} \;=\; |+\tfrac{1}{2},\ \stackrel{\downarrow}{+}\tfrac{1}{2},\ \pm\tfrac{1}{2},\ \cdots,\ \pm\tfrac{1}{2}\rangle\ , \qquad \alpha = 1, ..., \frac{N_D}{4}$$

[1]Notice that restricting to only finite dimensional representations is a strong assumption. It is less justified than the 4-dimensional case in which it can be argued that there is no physical evidence for the infinite dimensional massless representations. However in higher dimensions this is a less clear argument since extra dimensions themselves have not been observed. This issue needs better understanding.

and the total number will be $\frac{N_D}{4}$.
Given some state $|\lambda\rangle$ of helicity λ (i.e. $M_{23}|\lambda\rangle = \lambda|\lambda\rangle$), the action of any Q^+_α will lower the M^{23} eigenvalue:

$$\begin{aligned} M^{23}\,Q^+_\alpha\,|\lambda\rangle \;&=\; \Big[M^{23}\;,\;Q^+_\alpha\Big]\,|\lambda\rangle \;+\; Q^+_\alpha\,M^{23}\,|\lambda\rangle \;=\; -\Sigma^{23}\,Q^+_\alpha\,|\lambda\rangle \;+\; \lambda\,Q^+_\alpha\,|\lambda\rangle \\ &=\; \left(\lambda\;-\;\frac{1}{2}\right)\,Q^+_\alpha\,|\lambda\rangle \end{aligned}$$

We therefore obtain the follwing helicities by application of the $Q^+_\alpha\big|_{w=+\frac{1}{2}}$

$$|\lambda\rangle\;, \qquad |\lambda-\tfrac{1}{2}\rangle\;, \qquad \ldots\;,\;|\lambda-\tfrac{1}{2}\cdot\tfrac{N_D}{4}\rangle\;.$$

It follows for the range of occurring λ's that

$$\lambda_{\max}\;-\;\lambda_{\min}\;\;=\;\;\lambda\;-\;\left(\lambda\;-\;\tfrac{N_D}{8}\right)\;\;=\;\;\frac{N_D}{8}\;,$$

imposing $|\lambda| \leq 2$ thus requires $N_D \leq 32$. But remembering that $N_D = 2^{\frac{D}{2}}, 2^{\frac{D-1}{2}}$ for even and odd dimensionality, this implies a maximum number of spacetime dimensions $D = 10, 11$.
Notice the similarity of this argument with the previous proof that the maximum number of supersymmetries in 4 dimensions was $\mathcal{N} = 8$. We will see later that precisely $\mathcal{N} = 8$ supergravity is obtained from the supersymmetric theories in $D = 10$ and $D = 11$.
Let us take a closer look at the spectrum of $D = 11$ and $D = 10$:

- D = 11

  Only $\mathcal{N} = 1$ SUSY is possible. The only multiplet consists of

$$\underbrace{g_{MN}}_{\text{graviton}}\;,\qquad \underbrace{\psi^\alpha_M}_{\text{gravitino}}\;,\qquad \underbrace{A_{MNP}}_{\text{antisymmetric tensor (non-chiral)}}$$

  In order to count the (on shell) degrees of freedom for each field we have to perform the analysis based on the little group $O(D-2)$. The graviton in D dimensions carries $\frac{(D-2)(D-1)}{2}-1$ components, corresponding to a symmetric tensor in $D-2$ dimensions minus the trace, which is $45-1 = 44$ in the $D = 11$ case. An antisymmetric tensor of rank $p+1$ in D dimensions has $\binom{D-2}{p+1}$ degrees of freedom, in the case of A_{MNP} with $p+1 = 3$, this is $\binom{9}{3} = 84$.

  For the gravitino spinor ψ^μ_α, we have $2^{\frac{D-3}{2}}\cdot(D-2)-2^{\frac{D-3}{2}}$ independent components: The first factor is the product of the spinor components times the vector components of the gravitino (since it carries both indices), and the subtraction of the $2^{\frac{D-3}{2}}$ degrees of freedom of a spin $\frac{1}{2}$ particle is similar to the subtraction of the trace for the graviton. In terms of $su(2)$ representations $(1)\otimes\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)\oplus\left(\frac{1}{2}\right)$, one can say that the spin $\frac{1}{2}$ contribution on the right hand side is discarded. More generally, a vector spinor Ψ^α_M only furnishes an irreducible Lorentz representation if contractions with any invariant tensor (such as the metric and the higher dimensional Γ matrices) vanish. If the "gamma trace" $\Psi^\alpha_M\Gamma^M_{\alpha\beta}$ was nonzero, then it would be a lower irreducible representation on its own right. In $D = 11$, we obtain $9\cdot 2^4 - 2^4 = 128$ components for the gravitino which matches the number of bosonic degrees of freedom $84+44$.

- D = 10

  This allows two different $\mathcal{N} = 2$ theories and one $\mathcal{N} = 1$ corresponding to the massless spectrum of type IIA, type IIB string theories ($\mathcal{N} = 2$) and type I or heterotic ($\mathcal{N} = 1$). The spectrum for each of these theories is written in the table.

IIA	g_{MN}	$2\times\psi_M^\alpha$	B_{MN}	ϕ	A_{MNP}	A_M	$2\times\lambda$	
IIB	g_{MN}	$2\times\psi_M^\alpha$	$2\times B_{MN}$	$2\times\phi$	$A^\dagger_{MNPQ}$	$2\times\lambda$		
I	$(g_{MN}$	B_{MN}	ϕ	$\psi_M^\alpha)$	(gravity)	$(A_M$	$\lambda)$	(chiral)

Here the fermions in type IIA case have opposite chirality (so the theory is not chiral) and the fermions in type IIB have the same chirality (so the theory is chiral). Also in the IIB case the field strength of the 4-index field $A^\dagger_{MNPQ}$ is self dual. It is easy to check that the number of bosonic and fermionic degrees of freedom in both IIA and IIB cases adds up to 128. In the type I case there are two different types of $\mathcal{N}=1$ multiplets, the unique gravitational one with $64+64$ degrees of freedom and the chiral or matter multiplets with $8+8$ degrees of freedom each.

2.2.2 Supersymmetry Algebra and p-Branes

Recall that about general antisymmetric tensors $B_{M_1 \dots M_{p+1}}$ of spin 0 or 1, we know:

- B_M couples to a particle $\int B^M \, \mathrm{d}x_M$, where $\mathrm{d}x_M$ refers to the world line
- B_{MN} couples to a string $\int B^{MN} \, \mathrm{d}x_M \wedge \mathrm{d}x_N$ (world sheet)
- B_{MNP} to a membrane ...
- $B_{M_1 \dots M_{p+1}}$ to a p brane

The coupling is dependent of the object's charges $Z^{i_1 \dots i_p} = \int d^{D-1} J^{0 i_1 \dots i_p}$:

object	charge	couples to
particle	q	A_M
string	Z_M	B_{MN}
p brane	$Z_{M_1 \dots M_p}$	$B_{M_1 \dots M_{p+1}}$

Charges of p-branes are new examples of central charges in the SUSY algebra:

$$\left\{ Q \ , \ Q \right\} \ \propto \ a\, P \ + \ b^{M_1 \dots M_p} \, Z_{M_1 \dots M_p}$$

For instance in $D = 11$ the presence of the A_{MNP} tensor implies they couple to 2-branes. The dual tensor corresponds to A_{MNPQRS} (its seven index field strength is dual to the 4-index field strength of A_{MNP}) which couples to the (magnetic) $p = 5$ branes. So the natural extended objects in $D = 11$ supergravity are 2-branes and 5-branes. The corresponding charges are then Z_{MN} and Z_{MNPQR}. Therefore the SUSY algebra can be written as

$$\left\{ Q_\alpha, Q_\beta \right\} = \left(C\Gamma^M \right)_{\alpha\beta} P_M + \left(C\Gamma^{MN} \right)_{\alpha\beta} Z_{MN} + \left(C\Gamma^{MNPQR} \right)_{\alpha\beta} Z_{MNPQR}$$

Where C is the charge conjugation matrix. A test to count the number of independent components of this algebra is that on the LHS there are $32 \times 33/2 = 528$ independent components (since the dimension of the spinors in $D = 11$ is 32) whereas in the LHS, the first term has 11 components from P_M the second $11 \times 10/2 = 55$ from Z_{MN} and the last one $\binom{11}{5} = 462$ so giving a total for the LHS of $11+55+462 = 528$ the same as the number of components of the RHS. So we do not expect more surprises to add to the algebra.

The central charges as in the case of extended supersymmetry in $D = 4$ play an important role defining the BPS states. The p-branes can be the corresponding BPS objects. The generalisation of the BPS condition is setting their charges equal to their tension (like the mass/charge relation in the case of pointlike objects in $D = 4$). Usually the charge that appears in the BPS condition is the projection to the brane of the tensorial charges Z. The sign if this charge defines branes versus antibranes that carry same tension but opposite charges of the corresponding brane. p-branes also appear as solitonic solutions of the supergravity field equations. They appear as black hole kind of solutions known as black-branes of the supergravity with the singularity not being point-like but of higher dimension (p). The BPS condition as usual implies that branes preserve part of the original supersymmetry.

2.3 Dimensional Reduction

Let us review the general procedure of reducing any number of dimensions bigger than 4 to $d = 4$. Recall the example of a scalar in 5 dimensions $M_5 = \mathbb{M}_4 \times S^1$ (the last of which has radius R) where field in 5 dimensions could be replaced by ∞ many fields in $d = 4$. If φ is massless,

$$\partial_M \partial^M \varphi \ = \ 0 \quad \Longrightarrow \quad \partial_\mu \partial^\mu \varphi_n \ - \ \frac{n^2}{R^2}\, \varphi_n \ = \ 0 \ ,$$

then the Fourier mode φ_n with respect to the S^1 dimension has a mass of $\frac{n}{R}$.
For dimensional reduction, only keep the $n = 0$ mode,

$$\begin{aligned}
\varphi(x^M) \ &\mapsto \ \varphi(x^\mu) \\
A_M(x^M) \ &\mapsto \ A_\mu(x^\mu) \ , \quad \underbrace{A_m(x^\mu)}_{\text{scalars}} \ , \qquad m \ = \ 4, ..., D-1 \\
B_{MN} \ &\mapsto \ B_{\mu\nu} \ , \quad \underbrace{B_{\mu n}}_{\text{vectors}} \ , \quad \underbrace{B_{mn}}_{\text{scalars}} \\
\underbrace{\psi}_{2^n} \ &\mapsto \ \underbrace{\psi}_{\frac{1}{4}2^n \ \text{4D}-\text{spinors}} \ .
\end{aligned}$$

Consider e.g. the reduction of $D = 11$ to $d = 4$: The fundamental fields are graviton g_{MN} that carries $\frac{9 \cdot 10}{2} - 1 = 44$ degrees of freedom and the gravitino ψ^α_M with $9 \cdot 2^{\frac{9-1}{2}} - 2^{\frac{9-1}{2}} = 8 \cdot 16 = 128$ components. Again, the subtraction is an extra spinor degree of freedom. The final field is an antisymmetric tensor A_{MNP} that carries $\binom{9}{3} = 84$ degrees of freedom. Note that we have 128 bosonic degrees of freedom and 128 fermionic degrees of freedom. Dimensional reduction to $d = 4$ leads to:

$$\begin{aligned}
g_{MN} \ &\mapsto \ \underbrace{g_{\mu\nu}}_{\text{graviton}} \ , \quad \underbrace{g_{\mu m}}_{7 \ \text{vectors}} \ , \quad \underbrace{g_{mn}}_{\frac{7\cdot 8}{2}=28 \ \text{scalars (symmetry!)}} \\
A_{MNP} \ &\mapsto \ A_{\mu\nu\rho} \ , \quad \underbrace{A_{\mu\nu m}}_{7 \ \text{tensors}} \ , \quad \underbrace{A_{\mu mn}}_{21 \ \text{vectors}} \ , \quad \underbrace{A_{mnp}}_{\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}=35 \ \text{scalars (antisymmetry!)}} \\
\psi^\alpha_M \ &\mapsto \ \underbrace{\psi^\alpha_\mu}_{\frac{32}{4}=8} \ , \quad \underbrace{\psi^\alpha_m}_{7\cdot 8=56 \ \text{fermions}}
\end{aligned}$$

Recall here that a three index antisymmetric tensor $A_{\mu\nu\rho}$ in 4 dimensions carries no degrees of freedom and that two index antisymmetric tensors $A_{\mu\nu m}$ are dual to scalars. The spectrum is the same as the

$\mathcal{N}=8$ supergravity in 4 dimensions:

number	helicity	particle type	on shell degrees of freedom in $d=4$
1	2	graviton	$1\cdot\left(\frac{(4-2)(4-1)}{2}-1\right)=1\cdot 2=2$
8	$\frac{3}{2}$	gravitino	$8\cdot\left(2^{\frac{4-2}{2}}\cdot(4-2)-2^{\frac{4-2}{2}}\right)=8\cdot 2=16$
28	1	vector	$(7+21)\cdot(4-2)=28\cdot 2=56$
56	$\frac{1}{2}$	fermion	$56\cdot 2^{\frac{4-2}{2}}=56\cdot 2=112$
70	0	scalar	$28+7+35=70$

There is a theory of $\mathcal{N}=8$ supergravity based on the g_{MN} and A_{MNP}. Reducing the dimension from 11 to 4 has an effect of $\mathcal{N}=1\mapsto\mathcal{N}=8$. This $\mathcal{N}=8$ model is non-chiral, but other compactifications and p branes in a 10 dimensional string theory can provide chiral $\mathcal{N}=1$ models close to the MSSM. Notice that the statement of why the maximum dimensionality of supersymmetric theories is 11 is identical to the statement that the maximum number of supersymmetries in 4 dimensions is $\mathcal{N}=8$ since both theories are related by dimensional reduction. Actually, the explicit construction of extended supergravity theories was originally done by going to the simpler theory in extra dimensions and dimensionally reduce it.

Other interesting dimensional reductions are: from $D=11$ to $D=10$ it gives precisely the spectrum of IIA supergravity. Also starting from the $\mathcal{N}=1$ matter multiplet in $D=10$ and performing dimensional reduction to $D=4$ gives rise to the spectrum of $\mathcal{N}=4$ vector multplet in $D=4$ and in general dimensional reduction of $D=10$, $\mathcal{N}=1$ supergravity gives rise to $D=4$, $\mathcal{N}=4$ supergravity.

Chapter 3

A brief overview of compactifications

So far we have mostly concentrated on dimensional reduction and the only discussion of the geometry of extra dimensions was through the simplest circle compactifications. The study of compactifications for general geometries is very broad and here we will only touch some of the main points.

The typical starting point is a supergravity theory in high dimensions for which we will search for solutions preserving maximal symmetry in four dimensions times some compact manifold for the extra dimensions. The relevant fields are the bosonic fields in the massless spectrum of the theory that include the metric G_{MN}, antisymmetric tensors of different ranks $B_{M_1 \dots M_q}$ and some scalars ϕ. The effective field theory is determined by the action:

$$S = -\int d^D x \sqrt{|G|} \left(M_*^{D-2D} R + \partial^M \phi \partial_M \phi + f(\phi) H^{M_1 \cdots M_{q+1}} H_{M_1 \dots M_{q+1}} + \ldots \right) \tag{3.1}$$

where the $\cdots$ stand for terms including fermions which are not relevant for our purposes and higher order bosonic terms, such as higher derivatives terms including powers of the curvature, etc.. The antisymmetric tensor term is actually a sum over the different values of q present in the spectrum. The functions $f(\phi)$ have a well defined dependence on the fields ϕ. Each supergravity theory has an spectrum of antisymmetric tensors and very few scalars.

The field equations take the general schematic form:

$$R_{MN} = T_{MN}(G, H, \phi) \tag{3.2}$$

$$D^P \left(f H_{PM_1 \dots M_q} \right) = A(G, H, \phi) \tag{3.3}$$

$$\Box \phi = B(G, H, \phi) \tag{3.4}$$

where T_{MN} is the corresponding stress energy tensor and A, B simple functions of their arguments, explicitly known case by case, that vanish with H and ϕ. The issue is to find explicit solutions of these equations for values of $\langle G_{MN} \rangle$, $\langle H \rangle$ and $\langle \phi \rangle$ such that the geometry is of the type $\mathcal{M}_4 \otimes \mathcal{M}_{D-4}$ where $\mathcal{M}_4$ represents maximally symmetric spacetime in $D = 4$ that can be Minkowski, de Sitter or anti de Sitter depending if the value of the vacuum energy vanishes or is positive or negative respectively. $\mathcal{M}_{D-4}$ is a finite volume, usually compact euclidean manifold that needs to be determined.

Notice first that the simplest solution of these equations is to have $\langle G_{MN} \rangle = \eta_{MN}$ and $\langle H_{MN} \rangle = \langle D^M \phi \rangle = 0$ which is the full D dimensional Minkowski space. This solution usually preserves all supersymmetries and then is stable under quantum corrections. Since it does not clearly describe our world we will have to live with the idea that there will be more than one, probably many, solutions and the geometry of our universe with four, almost flat dimensions and small extra dimensions, is only one of them (if at all).

3.1 Toroidal Compactifications

The second simplest case is the case where $\mathcal{M}_4$ is 4-dimensional Minkowski space and $\mathcal{M}_{D-4} = T_{D-4}$ with $T_{D-4} = (S_1)^{D-4}$ the $D-4$ dimensional torus. This is a valid solution as the one-dimensional circle was in the five dimensional case. It also preserves all supersymmetries and therefore has extended SUSY in four dimensions. Implying non-chirality and therefore unrealistic spectrum to describe our world with chiral weak interactions. The extra-dimensional components of the metric G_{mn} with $m, n = 1, \cdots, D-4$ are massless scalar fields in four dimensions with a flat potential energy. Similarly for antisymmetric tensors, *e.g.* for a rank two tensor the components B_{mn} will be massless scalar fields in four dimensions. These fields play an important role in compactifications and are called *moduli fields* since they measure the size and shape of the extra dimensions.

Let us discuss the simplest case of two extra dimensions. The independent components of the metric and tw0-index tensor in the two extra dimensions are

$$G_{mn} = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} \qquad B_{mn} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

We can collect the four independent components in terms of two complex scalar fields as follows:

$$\begin{aligned} U &\equiv \frac{G_{12}}{G_{11}} + i\frac{\sqrt{G}}{G_{22}} \\ T &\equiv B + i\sqrt{G} \end{aligned} \tag{3.5}$$

Where here $G = G_{11}G_{22} - G_{12}^2$ is the determinant of the two-dimensional metric. The field U is the standard *complex structure* of a two-dimensional torus. Changing the value of U changes the shape of the torus. T is called the *Kähler structure* modulus since the two-dimensional torus is a Kähler manifold. Changing the Kähler modulus T changes the size of the torus (since the volume is determined by $\sqrt{G}$). These are typical fields that also appear in more general compactifications. Each of them parametrise a plane defined by the coset space $SL(2,\mathbb{R})/O(2)$. The full *moduli space* is then $SL(2,\mathbb{R})/O(2) \otimes SL(2,\mathbb{R})/O(2) = O(2,2,\mathbb{R})/O(2)^2$. Since U is the complex structure of the torus it implies the standard $SL(2,\mathbb{Z})$ geometric invariance

$$U \to \frac{aU+b}{cU+d} \qquad a,b,c,d \in \mathbb{Z} \qquad ad - bc = 1 \tag{3.6}$$

which is just a manifestation of the invariance under deformations of the torus T_2. In string theory there is a further $SL(2,\mathbb{Z})$ invariance of the spectrum and partition function associated to the T field $T \to (aT+b)/(cT+d)$ which includes the 'large' to 'small' size duality ($a = d = 0, b = -c = 1$). This is the generalisation of $R \to 1/R$ duality for a circle and is called T *duality.* The $c = 0, a = d = 1$ case reflects the shift symmetry for antisymetric tensors, $B \to B + k$, $k \in \mathbb{Z}$. Furthermore in string theory compactifications there is further a symmetry exchanging the U and T fields $U \leftrightarrow T$ which is called *mirror symmetry.* Some of these results can be generalised. For a T_d torus for $d = D-4$, the moduli space generalises to $O(d,d,\mathbb{R})/O(d)^2$.

Moduli fields give rise to beautiful mathematics. However they are problematic for several reasons. The fact that they can be changed arbitrarily means that we do not know the size and shape of the extra dimensions. But as we have seen before knowing the size of the extra dimensions is crucial to determine the physical quantities such as the Planck mass. We also know that most values of the volume are incompatible with experiments since we know that only very small volumes are allowed to explain why we have not observed more than four dimensions. Furthermore having scalar fields with flat potentials imply these massless particles will mediate long-range interactions that would give rise to a fifth force-type of new forces for which there are very strong experimental constraints. Therefore theories of extra dimensions with moduli fields are ruled out by experiments and therefore we need to look for solutions of the field equations which have all moduli stabilised. This is the major challenge for theories of extra dimensions.

3.2 Freund-Rubin Compactifications

Notice that in the toroidal case the only field that has a non-trivial background is the metric $\langle G_{MN}\rangle$. But we know there are usually many other bosonic fields in extra dimensions that can take non trivial values without breaking Lorentz invariance in four dimensions. In particular the scalars and field strengths of antisymmetric tensors can be non-vanishing in general and setting them to zero is very arbitrary.

Let us consider the simplest such a case. Starting with six dimensional gravity-Maxwell theory:

$$S = -\int d^6x\sqrt{|G|}\left({}^{(6)}R + F^{MN}F_{MN}\right) \tag{3.7}$$

The Maxwell field F_{MN} may be non-vanishing but the only components that can be different from zero are F_{mn}, $m, n = 4, 5$ since nonzero values for $F_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ would break Lorentz invariance. If we want maximal symmetry in four dimensions we can write the background fields as:

$$\langle G_{mn}\rangle = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix}, \qquad \langle F_{MN}\rangle = f\begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{mn} \end{pmatrix}$$

Here $g_{\mu\nu}$ is the metric of a maximally symmetric four-dimensional spacetimes (Minkowski, de Sitter or anti-de Sitter), g_{mn} the metric of a maximally symmetric compact two dimensional space (sphere), f an arbitrary constant and ϵ_{mn} the Levi-Civita tensor in two dimensions. Plugging these expressions in the field equations give solutions with the compact space a two-dimensional sphere of radius R. The nontrivial value of the Maxwell field on a sphere is actually the same as a magnetic monopole flux that has to be quantised from the Dirac quantisation condition:

$$\int_{S^2} F = N, \qquad N \in \mathbb{Z} \tag{3.8}$$

Plugging this in the field equations (and doing a Weyl transformation of the metric to have the standard Einstein-Hilbert term in four dimensions) gives rise a potential for the radius R:

$$V(R) \sim \frac{N^2}{R^6} - \frac{1}{R^4} \tag{3.9}$$

The first term coming from the $F^{MN}F_{MN}$ term of the action and the second term from the curvature. This potential has a minimum at $R \sim N$ and therefore fixes the size of the extra dimensions! The value of the potential at the minimum is negative $\langle V \rangle \sim -1/N^4$ which means a negative value of the vacuum energy. Therefore the four dimensional space is not Minkowski but anti-de Sitter.

Notice that fluxes of the electromagnetic field provided the key ingredient to stabilise the size of the extra dimension. As usual the fluxes are quantised but the value of the integer N is not fully determined. each value gives rise to a different solution and therefore to a different four-dimensional universe. Notice also that the fact that the antisymmetric field strength tensor F_{MN} has two indices allowed a natural factorisation of the six dimensional space between a four dimensional and a two dimensional space, in this case $AdS_4 \times S^2$. This type of compactification is called 'spontaneous compactification' to make the analogy with the spontaneous symmetry breaking in gauge theories (in this case the symmetry breaking could be the symmetries of full six-dimensional space broken to the independent symmetries of the four and two dimensional spaces) and was introduced by Freund and Rubin in the 1980's. Adding extra terms to the action, such as a positive cosmological constant adds extra terms to the scalar potential and allows the possibility of a minimum with vanishing vacuum energy, that is $\mathcal{M}_4 \times S^2$ with $\mathcal{M}_4$ the four dimensional Minkowski spacetime. This is the case for compactification of a supergravity theory in six dimensions found by Salam and Sezgin in 1984. Their solution also preserves $\mathcal{N} = 1$ supersymmetry and contrary to toroidal compactifications the spectrum is chiral in four dimensions. For these reasons it has attracted much attention over the years.

Generalisations of the Freund-Rubin ansatz to higher dimensions provide interesting compactifications. In particular, starting from $D = 11$ supergravity that has a three-form field with rank-four field strength F_{MNPQ}, assuming maximal symmetric spaces this allows for $F_{\mu\nu\rho\sigma} \propto \epsilon_{\mu\nu\rho\sigma}$ gives rise to a factorisation of the eleven-dimensional spacetime into either $AdS_7 \times S^4$ or $AdS_4 \times X_7$ where X_7 is a maximally symmetric space, such as a seven-sphere S^7. Another generalisation os for IIB supergravity in ten dimensions that has a rank-five field strength $F_{M_1 \cdots M_5}$ that gives rise to backgrounds such as $AdS_5 \times S^5$ which has been the starting point of the celebrated AdS/CFT correspondence in which this compactification is claimed to be equivalent to a four-dimensional non-gravitational conformal field theory.

3.3 Calabi-Yau Compactifications

Starting from (chiral) $\mathcal{N} = 1$, $D = 10$ supergravity (type I) motivated by string compactifications we can search for compactifications that preserve $\mathcal{N} = 1$ supersymmetry in $D = 4$ in order to have chiral theories and still benefit from the properties of supersymmetric theories, as for addressing the hierarchy problem and also having a flat four-dimensional spacetime. These requirements put strong constraints on the nature of the compact six-dimensional manifolds. They have to be Ricci flat ($R_{mn} = 0$) just as torii but the defining property is that to preserve $\mathcal{N} = 1$ supersymmetry they have to be manifolds with $SU(3)$ holonomy group. Roughly speaking the holonomy group G is the group defined by parallel transporting a vector around a closed trajectory on the corresponding manifold and the resulting vector is related to the original one by a G transformation. In six dimensions G is a subgroup of the rotation group $SO(6)$. Being holonomy $SU(3)$ defines the manifold to be a Calabi-Yau manifold.

The knowledge of Calabi-Yau manifolds is limited since it is known they admit Ricci flat metrics but explicit metrics are not known since in particular the manifold has no isometries. This makes them not suitable for the Kaluza-Klein realisation of gauge symmetries. The origin of gauge symmetries in four dimensions then comes from gauge symmetries already existing in the extra dimensional theory. A great amount of knowledge has been accumulated about these manifolds mostly on the topological side, using techniques of algebraic topology.

It is beyond the scope of these notes to discuss further the details of Calabi-Yau manifolds. However a few relevant properties can be mentioned. The two-dimensional version of a Calabi-Yau manifold is the two-torus T_2 we have discussed. We know that T_2 has two types of topologically non-trivial one-cycles (dual to each other). Being six-dimensional Calabi-Yau manifold have topologically non-trivial cycles of dimension $2, 3, 4$. The 2-and 4-cycles are dual to each other and two types of 3-cycles (there are not non-trivial 1 or 5 cycles). The size of the three cycles are the *complex structure* moduli U_a with $a = 1, \cdots, h_{12}$. The size of the 2- or 4-cycles are the *Kähler moduli* T_i with $i = 1, \cdots, h_{11}$. Here h_{12} and h_{11} are the Hodge numbers of the manifold. They determine the Euler number $\chi = 2(h_{12} - h_{11})$. Their typical values are of order $\simeq 10^3 - 10^4$ for the known Calabi-Yau manifolds. This gives us an idea of the number of solutions of the field equations that these manifolds provide: the number of Calabi-Yau manifolds is not even known to be finite. For each Calabi-Yau there are many fluxes that can be turned on like

$$\int_{\gamma^i} H_3 = N_i \tag{3.10}$$

for each three-index antisymmetric tensor field strength H_{mnp} and each non-trivial 3-cycle γ_i, $i = 1, \cdots h_{12}$. This is the main source of what is known as the string landscape with a number of solutions for each Calabi-Yau manifold estimated to be of order 10^{500} or more. When all moduli are stabilised each of these solutions will come with a different value of the vacuum energy (or cosmological constant). This has been proposed as a concrete way to address the cosmological constant problem. The idea is that the density of solutions with different values of the cosmological constant is high enough so that whatever value of the cosmological constant is obtained after all quantum corrections to the vacuum energy are computed, there will be a vacuum with the right value of the vacuum energy to give the total vacuum energy of the order of the observed one. This is an unusual 'solution' to one of the main problems of physics. It only indicates that what we thought it was a major question to be answered by first principles (what is the value of the cosmological constant) happens to be only an environmental fact related to our own universe, within what is usually (mis) named the anthropic principle. This may be disappointing for theoretical physicists searching for proper explanations of nature. It has some scientific merit in the sense that Weinberg predicted the current observed value of the cosmological constant by these kind of arguments almost 10 years before the discovery of the acceleration of the universe.

For the purpose of this course it has a merit in the sense that the main motivation for supersymmetry at low energies is the solution to the hierarchy problem. For which the main criticism is that since we do not know what solves the cosmological constant problem, any proposal to address the hierarchy problem neglecting the cosmological constant problem is not well justified. If the cosmological constant is only an environmental question then it is justified to neglect this problem in addressing the hierarchy problem. However it may be argued that probably the hierarchy problem could also be environmental. This is an unsolved issue but fortunately for this case experimental searches in the near future may give us an idea.

3.4 Final Remarks

This is the end of these lectures. We have seen that both supersymmetry and extra dimensions provide the natural way to extend the spacetime symmetries of standard field theories.

They both have a set of beautiful formal properties, but they also address important unsolved physical questions such as the hierarchy problem for instance.

For supersymmetry we can say that it is a very elegant and unique extension of spacetime symmetry:

- It may be realized at low energies, the energy of SUSY breaking of 1 TeV is within experimental reach (hierarchy, unification, dark matter)
- It may be an essential ingredient of fundamental theory (M theory, strings).
- It is a powerful tool to understand QFTs, especially non-perturbatively (S-duality, Seiberg-Witten, AdS/CFT).

Extra dimensions are in some sense competing proposals to address the hierarchy problem both from the warped and unwarped cases. They are also important ingredients of fundamental theories (string/M theories). It is then compelling to study the physical implications of supersymmetric theories in extra dimensions.

However the lack of evidence of new physics from LHC already puts all the proposals to address the hierarchy problem (supersymmetry, extra dimensions and other alternatives) in tension with experiments. Some amount of fine tuning may be needed and therefore the whole naturalness issue may be under question.

Both supersymmetry and extra dimensions may be subject to be further tested soon in experiments. Notice that they are both basic ingredients of string theory (that addresses the problem of quantum gravity) and as such deserve further study, but they may be relevant only at higher energies than those available in the near future, we need to remain patient. Independent of any experimental verification they have expanded our understanding of physical theories which is a good argument to continue their study.

Bibliography

[1] F. Quevedo, S. Krippendorf and O. Schlotterer, *Cambridge Lectures on Supersymmetry and Extra Dimensions,* arXiv:1011.1491 [hep-th].

[2] S. Weinberg, *The quantum theory of fields, Volume I Foundations,* CUP (1995).

[3] For a recent discussion of dualities see: J. Polchinski, *Dualities of Fields and Strings,* arXiv:1412.5704 [hep-th].

[4] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity,* Phys. Rev. D **59** (1999) 086004 [arXiv:hep-ph/9807344].

[5] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension,* Phys. Rev. Lett. **83** (1999) 3370 [arXiv:hep-ph/9905221].

[6] P. C. West, *Supergravity, brane dynamics and string duality,* arXiv:hep-th/9811101.

[7] S. Weinberg, *The quantum theory of fields, Volume III Foundations,* CUP (1998).

[8] J. Polchinski, *Superstring Theory and beyond, Volume II,* CUP 2005.

[9] J. A. Strathdee, *Extended Poincaré Supersymmetry,* Int. J. Mod. Phys. A **2** (1987) 273.