Solitons and Instantons
Example Sheet 1

David Stuart, April 2010

1 Write down the Lorentz boosted kink solutions to the sine-Gordon equation

$$\theta_{tt} - \theta_{xx} + \sin\theta = 0 \tag{1}$$

describing kinks moving along a straight line $X = X_0 + ut$. Define the energy and momentum to be, repectively:

$$E(t) = \int_{\mathbb{R}} \Big[\frac{1}{2}\big(\theta_t^2 + \theta_x^2\big) + 1 - \cos\theta\Big]dx, \qquad \text{and } P(t) = -\int_{\mathbb{R}} \theta_t\theta_x\, dx.$$

Compute these for the moving kink solutions, and verify that they are independent of time and satisfy the relativistic energy-momentum relationship $E^2 = P^2 + m^2$ for appropriate m (which should be found).

2 Consider the ϕ^4 model on 1+1 dimensional space-time:

$$\phi_{tt} - \phi_{xx} + \lambda\phi(\phi^2 - \frac{\mu^2}{\lambda}) = 0 \tag{2}$$

to be solved for $\phi : \mathbb{R}^2 \to \mathbb{R}$ with boundary conditions

$$\phi \to \phi_\pm \equiv \pm\frac{\mu}{\sqrt{\lambda}} \qquad \text{as } x \to \pm\infty \tag{3}$$

Show that this is the Euler-Lagrange equation for the action functional

$$S[\phi] = \int\int\Big[\frac{1}{2}(\phi_t^2 - \phi_x^2) - \frac{\lambda}{4}(\phi^2 - \frac{\mu^2}{\lambda})^2\Big]dxdt,$$

and that the energy

$$E = \int \frac{1}{2}\phi_t^2 dx + V(\phi), \qquad V(\phi) = \int\Big[\frac{1}{2}\phi_x^2 + \frac{\lambda}{4}(\phi^2 - \frac{\mu^2}{\lambda})^2\Big]dx \tag{4}$$

is independent of time if ϕ solves (2).

3 Carry out the Bogomolny argument for the potential energy $V(\phi)$ in (4), and with boundary conditions (3). Hence obtain explicitly kink solutions ϕ_K for (2) and show that they minimize $V(\phi)$ amongst confiurations satisfying (3).

4 Derive the linearization of the equation (2) at the kink ϕ_K (the equation for small fluctuations about ϕ_K) in the form $\eta_{tt} + L\eta = 0$, with L a second order differential operator. Calculate also the Taylor expansion close to ϕ_K for the potential energy $V(\phi)$ in (4), and show that the same operator appears as the operator L in the linearized equation.

5 Write down the operator L from the previous question explicitly, and show that it is a special case of the class of solvable potentials appearing in Morse and Feshbach, Methods of

Theoretical Physics Vol. II, equation 12.3.22 in §12.3. Hence write down the eigenvalues corresponding to bound states, and discuss the evolution of small fluctuations at the linearized level. Compare this with the case of small fluctuations about the sine-Gordon kinks.

6 Consider the action for (1):

$$S[\theta] \equiv \int \big[\frac{1}{2}(\theta_t^2 - \theta_x^2) - (1 - \cos\theta)\big]\,dxdt \tag{5}$$

Define the effective Lagrangian $S_{eff}[X] \equiv S[\theta_K(\cdot - X(t))]$, for low energy dynamics of the kink. Carry out the integrals over x explicitly to derive the form of this Lagrangian. What are the consequences for low energy kink dynamics (assuming that this Lagrangian does indeed give a good approximation to (1) at low energies)?

7 Consider the action for the sine-Gordon equation on a background space-time $\mathbb{R}^2$ with metric $e^{2\rho^\epsilon}(dt^2 - dx^2)$:

$$S[\theta] \equiv \int \big[\frac{1}{2}(\theta_t^2 - \theta_x^2) - e^{2\rho^\epsilon}(1 - \cos\theta)\big]\,dxdt. \tag{6}$$

Derive the correponding Euler-Lagrange equation of motion. Assume that $\rho^\epsilon(t,x) = \rho(\epsilon t, \epsilon x)$, where ρ is a smooth function with all derivatives bounded. Show that when $\epsilon = 0$, so that $\rho^\epsilon(t,x) = \rho(0,0) = \rho_0$ is a constant, there are uniformly moving kink solutions with

$$\theta(t,x) = \theta_K(\gamma\lambda(x - X)), \quad \theta_t(t,x) = -\gamma\lambda\dot{X}\theta_K'(\gamma\lambda(x - X)) \tag{7}$$

as long as $\dot{X} = u$ is constant. Here $\gamma = (1 - u^2)^{-1/2}$ is the Lorentz contraction factor, and $\lambda = e^{\rho_0}$.

Define an effective Lagrangian for small ϵ by substituting the expressions (7) into (6), but with $\lambda(t) = e^{\rho(\epsilon t, \epsilon X(t))}$, replacing $\rho(\epsilon t, \epsilon x)$ by $\rho(\epsilon t, \epsilon X(t))$ and integrating over x. Explain why this is plausible for small ϵ. Obtain the explicit form for this Lagrangian and its corresponding equation of motion, and interpret physically.

8 Try to find a special form for the linear operator $L = -\frac{d^2}{dx^2} + 1 - 2\mathrm{sech}^2(x)$ which we derived in discussing the linearzation of (1) at the kink, using the Bogomolny decomposition of V. Use this to derive as much information about the eigenvalues and eigenfunctions of L as you can. [*You may find it useful to take a look at question 8, sheet I and qu 3, sheet II of the IB undergraduate quantum mechanics course*]

9 For the equation

$$\phi_{tt} - \phi_{xx} + U'(|\phi|^2)\phi = 0 \tag{8}$$

find a non-topological soliton solution $\phi(t,x) = e^{i\omega t}f(x)$ explicitly, in the case $U(y) = m^2 y(1-y)^2, U'(y) = m^2(1 - 4y + 3y^2)$. (Notice that U' is evaluated at $y = |\phi|^2$ in the equation). Write down the Lorentz transformations for the soliton and verify the relativistic energy momentum relation for appropriate mass M.

10 For the nonlinear Schroedinger equation

$$i\phi_t - \phi_{xx} + U'(|\phi|^2)\phi = 0 \tag{9}$$

find non-topological soliton solutions of the form $e^{-iEt}f(x)$. Transform them to obtain uniformly moving solitons, and find the energy of these moving solutions as a function of their velocity.

1 Show that the nonlinear Schroedinger equation

$$i\phi_t = -\phi_{xx} - |\phi|^2\phi + \epsilon V(\epsilon x)\phi \tag{10}$$

can be derived as the Euler-Lagrange equation for the action

$$S[\phi] = \frac{1}{2}\int \Big[\langle i\phi, \dot\phi\rangle + |\phi_x|^2 - \frac{1}{2}|\phi|^4 + \epsilon V(\epsilon x)|\phi|^2\Big] dxdt.$$

For the case $\epsilon = 0$ show that there are exact solutions $\phi(x,t) = f_E(x-X)e^{-i\theta + \frac{i}{2}u(x-X)}$ depending upon the four parameters E, θ, X, u as long as these evolve in time according to $\dot E = 0, \dot\theta = -E + \frac{1}{4}u^2, \dot X = u, \dot u = 0$, and $f_E(x) = \sqrt{2E}\,\mathrm{sech}\,(\sqrt{E}(x-X))$.

For small ϵ define and compute and effective Lagrangian and discuss the predicted dynamics of the soliton in this potential.

2 For the piecewise constant potential

$$V(x) = \begin{cases} +\infty & \text{if } |x| \geq b > a > 0 \\ 0 & \text{if } |x| \in (a,b) \\ V_0 > 0 & \text{if } |x| \leq a \end{cases}$$

consider the Schrodinger equation

$$-\frac{\hbar^2}{2}\psi'' + V(x)\psi = E\psi.$$

Compute equations determining the eigenvalues E, and calculate the difference between the ground and first excited state for small $\hbar$ (or large V_0).

3 Calculate explitly the curvature of the BPST instanton, and verify that it is (anti-)-self-dual, and also calculate explicitly the Yang-Mills action.

4 Derive the Euler-Lagrange equations for the critical points of the functional

$$V_\lambda(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2} \Big[B^2 + |D\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2\Big] d^2x, \tag{11}$$

where $B = \partial_1 A_2 - \partial_2 A_1$ and $D\Phi = (\nabla - iA)\Phi$.

5 Substitute the radial ansatz $\Phi = f_N(r)e^{iN\theta}$ and $A = N\alpha_N(r)d\theta$ into the Euler-Lagrange equations derived in the previous question and hence obtain coupled ODE's satisfied by f_N, α_N.

6 Substitute the radial ansatz from the previous question into the energy functional (11) to obtain a radial energy functional. Calculate the corresponding Euler-Lagrange equations and check they are the same as those obtained in the previous question.

7 Consider the Ginzburg-Landau energy with $\lambda = 1$. Assume that the degree (or winding number) is a negative integer:

$$N = \lim_{R \to +\infty} \frac{1}{2\pi} \int_{|x|=R} \langle i\Phi, d\Phi \rangle = \frac{1}{\pi} \int_{\mathbb{R}^2} d\Phi \wedge d\Phi < 0.$$

Carry out the Bogomolny argument with appropriate modification from the $N > 0$ case to deduce the Bogomolny bound $V_1 \geq \pi |N|$. Obtain the corresponding first order Bogomolny equations whose solvability provides configurations which saturate the Bogomolny bound.

8 Let $a(x; \mathbf{Z}), \phi(x; \mathbf{Z})$ be multi-vortex configurations which minimize V_1, and with ϕ vanishing at the N points $\{Z_1, \dots, Z_N\}$ in the plane where $\mathbf{Z} = (Z_1, \dots Z_N)$. Show that given a curve $t \mapsto \mathbf{Z}(t)$ it is possible to find a time-dependent gauge transformation $\chi(t, x)$ so that the curve

$$(A(t,x), \Phi(t,x) \equiv e^{i\chi(t,x)} \cdot \big(a(x; \mathbf{Z}(t)), \phi(x; \mathbf{Z}(t))\big)$$

satisfies the condition

$$\operatorname{div} \dot{A} - \langle i\Phi, \dot{\Phi} \rangle = 0.$$

Here $e^{i\chi} \cdot (a, \phi) \equiv (a + d\chi, e^{i\chi}\phi)$, is the usual action of the gauge group, and $\dot{f} = \partial_t f$.

9 Carry out the Bogomolny argument for the functional

$$W(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \Big[\frac{1}{|\Phi|^2} B^2 + |D\Phi|^2 + \frac{|\Phi|^2}{4} (1 - |\Phi|^2)^2 \Big] d^2x, \tag{12}$$

and obtain the first order Bogomolny equations whose solutions minimize W.

10 The Ginzburg-Landay energy functional on the hyperbolic disc $D = \{z : |z| < 1\} \subset \mathbb{C}$ is

$$\begin{aligned} V(A, \Phi) &= \frac{1}{2} \int_D \Big[e^{2\rho} B^2 + |D\Phi|^2 + \frac{e^{2\rho}}{4} (1 - |\Phi|^2)^2 \Big] d^2x, & (13) \\ &= \frac{1}{2} \int_D \Big[e^{-2\rho} (\partial_x A_2 - \partial_y A_1)^2 + |D\Phi|^2 + \frac{e^{2\rho}}{4} (1 - |\Phi|^2)^2 \Big] d^2x, & (14) \end{aligned}$$

where the metric is $g = e^{2\rho}(dx^2 + dy^2)$, $z = x + iy$, $B = e^{-2\rho}(\partial_x A_2 - \partial_y A_1)$ and $D\Phi = (\nabla - iA)\Phi$ as usual. Carry out the Bogomolny argument to obtain the first order Bogomolny equations for the minimizers.

11 In the previous question consider the case $e^{2\rho} = \frac{8}{(1-|z|^2)^2}$, and define ψ by

$$\psi = u - \ln(1 - z\bar{z}) + \ln 2$$

where $u = \ln |\Phi|$. Show that $\Delta\psi = e^{2\psi}$ (Liouville equation). Verify that this latter equation has solution

$$\psi = \frac{1}{2} \ln \frac{4|g'(z)|^2}{(1 - |g(z)|^2)^2}$$

for any holomorphic function $g : D \to D$, and hence obtain radially symmetric vortex solutions (i.e. those having the form given in qu. 5 in polar coordinates with $z = re^{i\theta}$.)

1 For maps $\Phi : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ work out directly the Euler-Lagrange equation of motion for the energy functional

$$V(\Phi) = \frac{1}{2}\int_{\mathbb{R}^2} |d\Phi|^2 d^2x. \tag{15}$$

(Here S^2 is the unit sphere $\sum_{a=1}^{3}(\Phi^a)^2 = 1$ in $\mathbb{R}^3$.) Check that the Lagrange multiplier method gives the same result.

2 For $\phi \in \mathbb{R}^3$ of unit length introduce spherical coordinates $\phi = (\cos\theta_1 \sin\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_2)$ with $|\phi| = 1$. Calculate $\phi \cdot (d\phi \wedge d\phi)$, and relate this to the area element on the unit sphere in standard spherical co-ordinate form, i.e. $\sin\theta_2 d\theta_1 \wedge d\theta_2$. (Here $\phi = \phi(\theta_1, \theta_2)$ so that $d\phi = \sum_{i=1}^{2} \frac{\partial\phi}{\partial\theta_i} d\theta_i$.) Use this and the change of variables formula to interpret the formula for the degree

$$N = \frac{1}{8\pi}\int_{\mathbb{R}^2} \Phi \cdot d\Phi \wedge d\Phi$$

where $\Phi : \mathbb{R}^2 \to S^2$ is a field for the σ-model and $d\Phi = \sum_{j=1}^{2} \frac{\partial\Phi}{\partial x^j} dx^j$.

3 Introduce complex co-ordinates on the sphere $\sum_{a=1}^{3}(\Phi^a)^2 = 1$ in $\mathbb{R}^3$ by stereographic projection:

$$w^1 + iw^2 = \frac{(\Phi^1 + i\Phi^2)}{1 + \Phi^3}.$$

Write the Bogomolny equation $d\Phi = \pm * \Phi \wedge d\Phi$ associated to (15) in terms of w. Write also the energy functional (15) in terms of w and carry out the Bogomolny argument directly in terms of w.

4 Derive the Euler-Lagrange equations for the critical points of the Yang-Mills-Higgs functional

$$V_\lambda(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^3} \Big[B^2 + |D\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2\Big] d^3x \tag{16}$$

with Φ $su(2)$ valued, the connection one-form $A_j dx^j$ is also $su(2)$ valued and $B = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field, and the curvature is $F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$, also $su(2)$ valued, and $|A|^2 = \langle A, A\rangle$ and $\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}\,(AB)$.

5 For $\lambda = 0$ verify that the Prasad-Sommerfeld solution does indeed solve the equations of motion derived in the previous question.

6 Show that any minimizer of (16) with $\lambda = 0$ must satisfy $\int |B|^2 d^3x = \int |D\Phi|^2 d^3x$.

7 Work out the Derrick scaling argument for the Yang-Mills-Higgs energy functional on $\mathbb{R}^n$:

$$V_\Lambda(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^n} \Big[|F|^2 + \mu|D\Phi|^2 + \frac{\Lambda}{4}(1-|\Phi|^2)^2\Big] d^n x \tag{17}$$

with μ, Λ non-negative real numbers.

(i) Show that for pure Yang-Mills ($\mu = \Lambda = 0$) the only possibility for nontrivial finite energy critical points is if $n = 4$.

(ii) Show that if $n > 4$ there are no nontrivial finite energy critical points.

(iii) Show that if $n = 4$ the only finite energy critical points have A a pure Yang-Mills critical point.

8 For the Prasad Sommerfeld solution compute $\langle \Phi, B_j\rangle$. Compute

$$\frac{1}{4\pi}\int_{|x|=R} \langle \Phi, B_j\rangle \cdot n_j \, d^2\Sigma$$

where $n_j = x_j/|x|$.

9 For the equation

$$\phi_{tt} - \Delta\phi + U'(|\phi|^2)\phi = 0 \tag{18}$$

for $\phi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, show that non-topological solitons $\phi(t,x) = e^{i\omega t} f(x)$ with f real and positive, are obtained from solutions of

$$-\Delta f + U'(f^2)f - \omega^2 f = 0.$$

For the case $U(y) = y - y^2/2 + y^3/3$ show that for $\omega = 0$ solutions do not exist (Derrick). Show however that for $\omega^2 > 13/16$ the potential $G_\omega(y) = (1-\omega^2)y - y^2/2 + y^3/3$ becomes negative as y increases from zero. This implies that for each $\omega^2 \in (13/16, 1)$ there is a positive radial solution: in fact for $n \geq 3$, if $\frac{1}{2}U(|\phi|^2) = \sum_{j=1}^N a_j|\phi|^{2j}$, with $a_1 > 0$ and $N \leq n/(n-2)$ (and $a_N > 0$ if $N = n/(n-2)$) and U negative somewhere, the equation $-\Delta\phi + U'(|\phi|^2)\phi = 0$ has a positive radial solution decreasing to zero exponentially (see Berestycki-Lions ARMA 82 (1983) 313-345 if you want to see a proof but the proof is not needed for this course).

10 For (18) consider the case $\frac{1}{2}U(|\phi|^2) = \frac{|\phi|^2}{2} - \frac{|\phi|^{p+1}}{p+1}$ Write the nontopological soliton solutions $e^{i\omega t}\phi_\omega$, and show by rescaling that the solutions for different ω are all related by rescaling:

$$\phi_\omega(x) = (1-\omega^2)^{\frac{1}{p-1}} f(\sqrt{1-\omega^2}\,x)$$

where $-\Delta f + f = f^p$. Deduce that the solitons exist and the stability condition

$$-\partial_\omega[Q(\phi_\omega, i\omega\phi_\omega)] > 0$$

where $Q(\phi,\psi) = \int \langle i\phi, \psi\rangle d^n x$ is the Noether charge, holds if

$$\frac{1}{1+\frac{4}{p-1}-n} < \omega^2 < 1, \qquad 1 < p < 1 + \frac{4}{n}.$$