Mathematical Tripos Part III Dr O. Rinne

GENERAL RELATIVITY Michaelmas 2008

Problems 1: Equivalence principle, tensors, commutators, metric and connection

Please send comments/amendments etc. to `o.rinne@damtp.cam.ac.uk`

1.1 The equivalence principle and tidal forces

An elevator is freely falling in the Newtonian gravitational field of the Earth. There are two particles in the elevator, both of mass m, separated by a distance η in the vertical direction and initially at rest relative to the elevator. One of them carries an electric charge q, the other one is neutral. There is a constant vertical electric field E in the elevator. Find the equation of motion for the separation of the two particles, including both electrical and tidal effects due to the inhomogeneity of the gravitational field. Estimate the change in the separation after a time $\sim L$, where L is the height of the elevator. Is the outcome of the experiment compatible with the strong equivalence principle?

1.2 Bach brackets

Let V^{ab} be the components of an arbitrary $\binom{2}{0}$ tensor, and let S_{ab}, A_{ab} be components of symmetric and antisymmetric $\binom{0}{2}$ tensors, i.e., $S_{ab} = S_{ba}$, $A_{ab} = -A_{ba}$. Show that

$$V^{ab}S_{ab} = V^{(ab)}S_{ab} := \tfrac{1}{2}\big(V^{ab} + V^{ba}\big)S_{ab}, \qquad V^{ab}A_{ab} = V^{[ab]}A_{ab} := \tfrac{1}{2}\big(V^{ab} - V^{ba}\big)A_{ab}.$$

1.3 The Kronecker delta

Prove that the Kronecker delta $\delta^a{}_b$ can be regarded as the components of a $\binom{1}{1}$ tensor. Can you give a basis-independent definition of the tensor?

1.4 The direct product

You are given a $\binom{2}{0}$ tensor K. Working first in some basis devise a criterion to test whether it is the *direct product* of two vectors A, B, i.e., $K^{ab} = A^a B^b$. (You can, but do not need to, use determinants.) Can you express the test in a manifestly basis-invariant manner?

Show that the general $\binom{2}{0}$ tensor in n dimensions cannot be written as a direct product, but can be expressed as a sum of many direct products.

1.5 The Hessian

Let $\mathcal{M}$ be a manifold and $f : \mathcal{M} \to R$ be a smooth function such that $\mathrm{d}f = 0$ at some point $p \in \mathcal{M}$. Let $\{x^i\}$ be a coordinate chart defined in a neighbourhood of p. Define

$$F_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}\ .$$

By considering the transformation law for components show that F_{ij} defines a $\binom{0}{2}$ tensor, the *Hessian of f at p*. Construct also a coordinate-free definition and demonstrate its tensorial properties.

1.6 Transformation of the determinant

Let g_{ab} be the components of a $\binom{0}{2}$ tensor (e.g. the metric), and regard g_{ab} also as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{ab})$. How does g transform under a change of basis?

1.7 The flat metric in spherical polar coordinates

In Cartesian coordinates x,y,z, the metric of flat 3D Euclidean space is

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Show that in spherical polar coordinates r,θ,ϕ defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = z/r, \quad \tan\phi = y/x,$$

the metric takes the form

$$ds^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

1.8 Lorentz transformations

Recall that in special relativity, two inertial coordinate systems $x^a = (t, x, y, z)$ and $\hat{x}^a = (\hat{t}, \hat{x}, \hat{y}, \hat{z})$ are related by a Poincaré transformation

$$\hat{x}^a = L^a{}_b x^b + d^a.$$

A Lorentz *boost* in the x-direction is given by $d^a = 0$ and

$$L = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Verify that the Minkowski metric is invariant under this change of coordinates. Write out the transformation of the coordinates t and x explicitly. Show that $\hat{x} = 0$ corresponds to $x = \beta t$, where β should be identified. Hence obtain the more familiar form of the Lorentz boost,

$$\hat{t} = \gamma(t - \beta x), \qquad \hat{x} = \gamma(x - \beta t),$$

where γ should be identified.

1.9 Frobenius's theorem

Let $\{e_a\}$ be a basis for vectors and set

$$[e_a, e_b] = \gamma^c{}_{ab}\, e_c.$$

(The $\gamma^c{}_{ab}$ are the *commutator components*.) Now let $\{\omega^a\}$ be the dual basis of covectors, and suppose there exist coordinates $\{x^i\}$ so that

$$e_a = e_a{}^i \frac{\partial}{\partial x^i}, \qquad \omega^a = \omega^a{}_i\, \mathrm{d}x^i,$$

where ${e_c}^j {\omega^d}_j = {\delta_c}^d$. Show first that

$$ {e_a}^i \frac{\partial {e_b}^j}{\partial x^i} - {e_b}^i \frac{\partial {e_a}^j}{\partial x^i} = {\gamma^c}_{ab}\, {e_c}^j \,, \qquad (*) $$

and deduce that

$$ {e_a}^i {e_b}^j \frac{\partial {\omega^d}_j}{\partial x^i} - {e_b}^i {e_a}^j \frac{\partial {\omega^d}_j}{\partial x^i} = -{\gamma^d}_{ab} \,, $$

and finally that

$$ \frac{\partial {\omega^d}_n}{\partial x^k} - \frac{\partial {\omega^d}_k}{\partial x^n} = -{\gamma^d}_{ab} {\omega^a}_k {\omega^b}_n \,. \qquad (\dagger) $$

In certain circumstances there may exist coordinates $\{y^a\}$ such that

$$ \omega^a = \mathrm{d}y^a, \qquad e_a = \frac{\partial}{\partial y^a}. $$

(We would then say that the bases are *coordinate induced*.) Show that if the bases are coordinate induced then $[e_a, e_b] = 0$, $\forall a, b$. Use ($\dagger$) to show also the converse, i.e. if $[e_a, e_b] = 0$ $\forall a, b$ then the basis is coordinate induced. Deduce that bases being coordinate induced is synonymous with trivial commutator components.

1.10 Transformation of the connection components and normal coordinates

Show that under a change of basis $\hat{e}_a = {(A^{-1})^b}_a e_b$ the connection components transform according to

$$ {\hat{\Gamma}^a}_{bc} = {A^a}_d {(A^{-1})^g}_b {(A^{-1})^h}_c {\Gamma^d}_{gh} + {A^a}_d\, \hat{e}_c \left({(A^{-1})^d}_b \right). $$

Given a coordinate chart $\{x^i\}$, consider a new chart $\hat{x}^i = x^i + \frac{1}{2} {Q^i}_{jk} x^j x^k$. Work out the transformation matrix ${A^i}_j = \partial \hat{x}^i / \partial x^j$ and its inverse, neglecting higher-order terms in $|x|$. Now fix a point p where wlog $x^i = 0$. Find the connection components ${\hat{\Gamma}^i}_{jk}$ in the new coordinate-induced basis. Show that ${Q^i}_{jk}$ can be chosen such that ${\hat{\Gamma}^i}_{(jk)} = 0$ at p. Thus we have constructed *normal coordinates* at p.

1.11 The divergence of a vector field

Let g_{ab} be a metric, ∇ the associated metric connection and V^a a vector field. Show that

$$ \nabla_a V^a = \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|} V^a \right). $$

Obtain an expression for the divergence of a vector in flat 3D Euclidean space in spherical polar coordinates.

[*Hint:* For any invertible matrix (m_{ij}), the determinant may be expanded as $m = M^{(i)j} m_{(i)j}$ (no sum over i), where $M^{ij} = m m^{ij}$ is a cofactor and (m^{ij}) is the inverse matrix of (m_{ij}). Now regard m as a function of the m_{ij} and differentiate.]