Mathematical Tripos Part III **Michaelmas Term 2018**
General Relativity **M. J. Perry**

Example Sheet 1

1. In Minkowski spacetime, consider an inertial frame S with co-ordinates $x^a = (t, x, y, z)$ and a second inertial frame S' with co-ordinates $x^{a'}$ related to S by a Lorentz transformation with velocity v along the y-axis. Suppose there is a wall at rest in the S' frame along the line $x' = -y'$. An observer in the S frame watches a ball with travelling in the xy-plane at speed u and at an angle θ to the x-axis. The ball bounces off the wall. Assuming that the collision is perfectly elastic, at what speed and at what angle to the x-axis does the observer in S observe the ball to be travelling after the collision?

2. A quasar is ejecting gas at a velocity v and at angle θ to the line of sight between the observer and the quasar. Projected onto the celestial sphere, the gas appears to travel perpendicular to the line of sight of the observer. Suppose that the distance between the observer and the quasar is D. The angular velocity observed is ω and this defines the apparent velocity of the ejected gas v_a by $v_a = \omega D$. Find an expresion for v_a and show that it is possible that $v_a > 1$.

   Advice: You can treat this as problem in special relativity. If you have done GR before, then you might want to try it in a $k = 0$ FRW matter-dominated universe assuming that the quasar has redshift z.

3. Let V^{ab} be an arbitrary $\binom{2}{0}$ tensor, and let S_{ab}, A_{ab} be symmetric and antisymmetric $\binom{0}{2}$ tensors, i.e., $S_{ab} = S_{ba}$, $A_{ab} = -A_{ba}$. Show that $V^{ab}S_{ab} = V^{(ab)}S_{ab}$ and $V^{ab}A_{ab} = V^{[ab]}A_{ab}$.

4. You are given a $\binom{2}{0}$ tensor K. Working first in some basis devise a criterion to test whether it is the *direct product* of two vectors A, B, i.e., $K^{ab} = A^a B^b$.

   Can you express the test in a manifestly basis-invariant manner?

   Show that the general $\binom{2}{0}$ tensor in n dimensions cannot be written as a direct product, but can be expressed as a sum of many direct products.

   Hint: You can use determinants if you wish.

5. Consider a smooth function $f(x)$ such that $df = 0$ at some point p. Define

   $$F_{ab} = \frac{\partial^2 f}{\partial x^a \partial x^b} .$$

   By considering the transformation law for a change of co-ordinates, show that F_{ab} defines a $\binom{0}{2}$ tensor, the *Hessian* of f at p.

   Find a tensorial definition of the Hessian that one can use away from the point p.

6. Let g_{ab} be a $\binom{0}{2}$ tensor. In a basis, one can regard the components g_{ab} as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{ab})$. How does g transform under a change of co-ordinates?

7. In inertial frame coordinates, the metric of Minkowski spacetime is

   $$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

(a) Show that if we replace (x, y, z) with spherical polar coordinates (r, θ, ϕ) defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \tan\phi = \frac{y}{x},$$

then the metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(b) Find the components of the metric and inverse metric in "rotating coordinates" defined by

$$\tilde{t} = t, \quad \tilde{x} = \sqrt{x^2 + y^2}\, \cos(\phi - \omega t), \quad \tilde{y} = \sqrt{x^2 + y^2}\, \sin(\phi - \omega t), \quad \tilde{z} = z,$$

where $\tan\phi = y/x$.

8. The Riemann tensor has the following symmetries

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}, \quad R_{a[bcd]} = 0.$$

Prove that the Riemann tensor has these symmetries. Before starting however, it is best to think of a strategy to do this using least effort. One way could start from the formula for the Riemann tensor in terms of the Christoffel symbols and their derivatives and the definition of the Christoffel symbols in terms of the metric and its derivatives. However, it is easier to use the fact that the Riemann tensor is a tensor and use normal coordinates in which the Christoffel symbols vanish but their derivatives do not.

As a by-product of your computation, suppose that you have constructed normal coordinates at x_0. Show

$$g_{ab}(x) = \eta_{ab} - \frac{1}{3} R_{acbd} \Delta x^c \Delta x^d + O(\Delta x^3)$$

where $\Delta x^a = (x^a - x_0^a)$.

9. The *Schwarzschild metric* in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 .$$

Find the Christoffel symbols and the 20 independent components of the Riemann tensor. Find the Ricci tensor.

Apology: I am afraid you have to do this at least once.

10. Obtain the form of the general timelike geodesic in a two-dimensional spacetime with metric

$$ds^2 = t^{-2}(-dt^2 + dx^2)\,.$$

Hint: You should use the symmetries of the Lagrangian, and you will probably find the following indefinite integrals useful:

$$\int \frac{dt}{t\sqrt{1 + C^2 t^2}} = \frac{1}{2} \ln\left(\frac{\sqrt{1 + C^2 t^2} - 1}{\sqrt{1 + C^2 t^2} + 1}\right), \quad \int \frac{ds}{\sinh^2 s} = -\coth s\,,$$