Problems 2: Geodesics, curvature, Einstein's equations and variational principles

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2.1 Parallel transport

A vector field Y is parallely propagated (with respect to the metric connection) along a geodesic curve with tangent vector X in a (pseudo-)Riemannian space. Show that the magnitudes of the vectors and the angle between them are constant along the geodesic.

On the unit sphere a unit vector Y is initially tangent to the meridian $\phi = 0$ at a point on the equator. It is then moved by parallel propagation first along the equator to the point at longitude ϕ_0, from there along a meridian to the North pole, and thence along another meridian to its original position. By how much has it changed, and why?

2.2 Geodesics as extremal curves

Consider a curve $x^a(s)$ and set $\dot{x}^a = \mathrm{d}x^a/\mathrm{d}s$. Let

$$F = \left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 = -g_{ab}\dot{x}^a\dot{x}^b,$$

where g_{ab} is the spacetime metric. Show that the variational principle $0 = \delta\int \mathrm{d}\tau = \delta\int \dot{\tau}\,\mathrm{d}s = \delta\int\sqrt{F}\,\mathrm{d}s$ generates timelike geodesics of the metric connection. Show also that the principle $0 = \delta\int F\,\mathrm{d}s$ generates all geodesics of the metric connection.

Obtain the equations of all of the geodesics of the spacetime with metric

$$ds^2 = t^{-2}(-\mathrm{d}t^2 + \mathrm{d}x^2),$$

and the form of the general timelike geodesic.

[*Hint:* you should use the symmetries of the Lagrangian F, and you will probably find the indefinite integrals

$$\int \frac{\mathrm{d}t}{t\sqrt{1+C^2t^2}} = \frac{1}{2}\ln\left(\frac{\sqrt{1+C^2t^2}-1}{\sqrt{1+C^2t^2}+1}\right), \qquad \int\frac{\mathrm{d}s}{\sinh^2 s} = -\coth s,$$

useful.]

2.3 Newtonian connection and tidal forces

A Newtonian particle experiencing gravity, and no other forces, is said to be *freely falling*, and its path is taken to be a geodesic. Starting from $\ddot{\mathbf{x}} = -\nabla\phi$ (where ϕ is the Newtonian potential) introduce space-time coordinates $x^i = (t, \mathbf{x})$ and obtain the Newtonian connection

$${\Gamma_{\mathrm{N}}}^{\alpha}{}_{00} = \phi_{,\alpha}, \qquad {\Gamma_{\mathrm{N}}}^{i}{}_{jk} = 0 \quad \text{otherwise}$$

(α, β range over $1, 2, 3$). Compute the Newtonian curvature tensors

$$-R_{\mathrm{N}}{}^{\alpha}{}_{00\beta} = R_{\mathrm{N}}{}^{\alpha}{}_{0\beta0} = \phi_{,\alpha\beta}, \qquad R_{\mathrm{N}}{}^{i}{}_{jkl} = 0 \quad \text{otherwise.}$$
$$R_{\mathrm{N}\,00} = \nabla^2\phi, \qquad R_{\mathrm{N}\,ij} = 0 \quad \text{otherwise.}$$

Suppose we choose coordinates with spatial origin attached to a particular freely falling particle P, so that its path is $x^i = (t, 0, 0, 0)$, and the tangent to the path has components $X^i = (1, 0, 0, 0)$. Let Z be a connecting vector. Show that

$$(\nabla_X \nabla_X Z)^{\alpha} = -\phi_{,\alpha\beta} Z^{\beta}. \tag{*}$$

Now work through the same calculation in the Newtonian theory of gravitation. Note that for P, $\ddot{\mathbf{x}} = 0 \Longrightarrow \phi_{,\alpha}(\mathbf{0}) = 0$. Let Q be a neighbouring particle with position vector $\delta\mathbf{x}$ where $|\delta\mathbf{x}|$ is small. Then

$$\delta\ddot{\mathbf{x}} = -\varphi_{,\alpha}(\delta\mathbf{x}) = -\phi_{,\alpha\beta}(\mathbf{0})\delta\mathbf{x}^{\beta} + \dots .$$

Note the similarity to ($*$). Recall from problem 1.4 that because $\phi_{,\alpha}(\mathbf{0}) = 0$, $\phi_{,\alpha\beta}(\mathbf{0})$ are the components of a $\binom{1}{2}$ tensor. This is often called the *tidal force tensor*, because the inhomogeneity in the terrestial potential caused by the moon produces ocean tides.

2.4 Newtonian metric and connection

Propose a definition of a metric tensor for Newtonian space-time. Is the Newtonian connection metric?

2.5 The contracted Bianchi identities

Starting from the Bianchi identities $R^a{}_{b[cd;e]} = 0$, deduce the contracted Bianchi identities $G^a{}_{b;a} = 0$ where

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}$$

is the Einstein tensor.

2.6 Component counting for the Riemann tensor

How many independent components does the Riemann tensor have in pseudo-Riemannian manifolds of two, three and four dimensions? Show that in two dimensions

$$R_{abcd} = \tfrac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}).$$

Discuss Einstein's theory in two dimensions.

2.7 The Weyl tensor

In a space of n dimensions define a tensor

$$C_{abcd} = R_{abcd} + \alpha(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) + \beta R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

where α and β are constants. Show that C_{abcd} has the same symmetries as R_{abcd}.

The coefficients α and β are chosen to set $C^a{}_{bad} = 0$. Determine them. With this extra condition C_{abcd} is called the *Weyl tensor*. Show that it vanishes if $n = 2, 3$.

Setting $n = 4$, how many independent components do R_{ab} and C_{abcd} have? What does the Weyl tensor represent physically? Show that in vacuo

$$\nabla^a C_{abcd} = 0.$$

2.8 The weak energy condition

Physically reasonable matter with energy-momentum tensor T^{ab} is expected to satisfy the *weak energy condition*, i.e.

$$T_{ab} U^a U^b \geqslant 0$$

for all timelike U^a. As an observer with a fixed 4-velocity V^a you can measure the components of T_{ab} in your rest frame. You note that T_{ab}, regarded as a symmetric matrix, has one timelike eigenvector W^a with eigenvalue ρ and three spacelike eigenvectors $X^a_{(\alpha)}$ with eigenvalues $p_{(\alpha)}$. How would you test whether this matter satisfied the weak energy condition?

2.9 Variational principle for a perfect fluid

A *perfect fluid* with 4-velocity u^a is described by a number density n, energy density ρ, pressure p and specific entropy s all measured in its rest frame. The corresponding flux densities

$$N^a = nu^a, \qquad T^{ab} = (\rho + p) u^a u^b + p g^{ab}, \qquad S^a = nsu^a,$$

satisfy the conservation laws

$$N^a{}_{;a} = 0, \qquad T^{ab}{}_{;b} = 0, \qquad S^a{}_{;a} = 0.$$

Another useful quantity is the specific enthalpy $\epsilon = (\rho + p)/n$. Only two of the scalars ρ, p, ϵ and s are independent and it is customary to take these to be ϵ and s. The functional form of p is given by an equation of state $p = p(\epsilon, s)$, and the remaining quantities are determined by the first law of thermodynamics

$$\mathrm{d}p = n\, \mathrm{d}\epsilon - nT\, \mathrm{d}s,$$

where T is the temperature. You may assume (Pfaff's theorem) that an arbitrary covector field Ω_a may be written in the form

$$\Omega_a = \chi_{,a} + \alpha\beta_{,a} + \theta s_{,a}$$

where α, β, χ and θ are scalar fields, three *Clebsch potentials* and the *thermasy*.

Consider the action

$$S = \int \sqrt{-g} \left[p(\epsilon, s) - \frac{n}{2\epsilon} \left(g^{ab} \Omega_a \Omega_b + \epsilon^2 \right) \right] \mathrm{d}^4 x.$$

Verify that the quantities which can be varied independently are n, ϵ, χ, α, β, θ, s and g_{ab} and obtain the equations of motion of the system.

[*Hint:* $u_a = \Omega_a / \epsilon$]

2.10 Brans-Dicke theory

The action for the *Brans-Dicke* theory of gravity is given by

$$S = \frac{1}{16\pi}\int\sqrt{-g}\Big[R\phi - \omega\frac{\phi_{,a}\phi^{,a}}{\phi} + 16\pi L_{\mathrm{matter}}\Big]\,\mathrm{d}^4x,$$

where ϕ is a scalar field and ω is a coupling constant. Ordinary matter is included in the action L_{matter}. How are the Einstein field equations modified, and what is the field equation for ϕ?

[For the background see [MTW] §39.2.]

2.11 The Palatini procedure for electrodynamics

Consider the following action for vacuum electrodynamics

$$S = \frac{1}{8\pi}\int\sqrt{-g}\Big[\tfrac{1}{2}F_{ab}F^{ab} - \big(A_{b,a} - A_{a,b}\big)F^{ab}\Big]\,\mathrm{d}^4x,$$

which is formally equivalent to the action given in the lectures if the usual relation between A_a and F_{ab} is assumed. Here though you should assume that A_a and F_{ab} are formally independent fields which are also independent of g_{ab}. Use the Palatini procedure to obtain the field equations.