Mathematical Tripos Part III M.J. Perry

GENERAL RELATIVITY: Examples 2 Michaelmas Term 2018

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1. Consider the alternating tensor in a spacetime of dimension d. Show that

$$\epsilon^{a_1 a_2 \ldots a_p b_1 \ldots b_{d-p}} \; \epsilon_{a_1 a_2 \ldots a_p c_1 \ldots c_{d-p}} = -p! \; \delta^{b_1 \ldots b_{d-p}}_{c_1 \ldots c_{d-p}}$$

where $\delta^{b_1 \ldots b_{d-p}}_{c_1 \ldots c_{d-p}}$ is the generalised Kronecker delta that takes the value $+1$ if $c_1 \ldots c_{d-p}$ are distinct integers and $b_1 \ldots b_{d-p}$ is an even permutation of them, takes the value -1 if $c_1 \ldots c_{d-p}$ are distinct integers and $b_1 \ldots b_{d-p}$ is an odd permutation of them and zero otherwise.

Show also that

$$\nabla_b \epsilon_{a_1 a_2 \ldots a_d} = 0.$$

2. Suppose that Φ is a p-form and that $d\Phi$ is integrated over a $(p+1)$-dimensional surface Σ with boundary $\partial\Sigma$. Show, by using a judicious choice of co-ordinates or otherwise, that

$$\int_\Sigma d\Phi = \int_{\partial\Sigma} \Phi.$$

3. The action describing the motion of a charged particle of mass m and charge q can be written as

$$\int ds \;\; (-m\sqrt{-g_{ab}\dot{x}^a \dot{x}^b} + qA_a \dot{x}^a)$$

where A_a is the electromagnetic vector potential related to the field strength tensor by $F_{ab} = \partial_a A_b - \partial_b A_a$. Derive the analogue of the geodesic equation assuming that s is the proper time and that a dot means the derivative with respect to s.

4. A vector field Y is parallely propagated (with respect to the Levi-Civita connection) along an affinely parametrized geodesic with tangent vector X in a Riemannian space. Show that the magnitudes of the vectors and the angle between them are constant along the geodesic.

Let Y be a unit vector on the unit sphere that is initially tangent to the line $\phi = 0$ at a point on the equator. It is then moved by parallel propagation first along the equator to the location $\phi = \phi_0$, from there along the line $\phi = \phi_0$ to the North pole, and then back along the line $\phi = 0$ to its original position. By how much has it changed, and why?

5. Suppose one is in a d-dimensional spacetime. Show that the symmetries of the Riemann tensor imply that the number of its independent components is $\frac{1}{12}d^2(d^2-1)$.

How many independent components does the Weyl tensor have?

6. Use the Bianchi identity to derive the *Penrose equation* for a vacuum spacetime obeying $R_{ab} = 0$.

$$\nabla^e \nabla_e R_{abcd} = 2R_{aedf} R_b{}^e{}_c{}^f - 2R_{aecf} R_b{}^e{}_d{}^f - R_{abef} R_{cd}{}^{ef}\,.$$

Suppose now that the cosmological constant is non-zero, how would this equation be modified, if at all?

7. Show that in a vacuum spacetime with zero cosmological constant, that

$$\nabla^a C_{abcd} = 0.$$

Suppose now that the cosmological constant is non-zero, how would this equation be modified, if at all?

8. In d spacetime dimensions a conformal transformation of the metric is made so that

$$\hat{g}_{ab} = \Omega^2 g_{ab}.$$

Show that

$$R(\hat{g}) = \Omega^{-2} R(g) - 2(d-1)\Omega^{-3}\Box_g \Omega - (d-1)(d-4)\Omega^{-4}\nabla_a \Omega \nabla_b \Omega g^{ab}.$$

In four spacetime dimensions ϕ obeys a modified version of the massless wave equation

$$\Box_g \phi - \frac{1}{6} R(g)\phi = 0$$

in the metric g_{ab}. Show that in the metric $\hat{g}_{ab}$, $\Omega^{-1}\phi$ obeys

$$\Box_{\hat{g}} \phi - \frac{1}{6} R(\hat{g})\phi = 0$$

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Generalise this result to d spacetime dimensions.

9. Consider the unit sphere with metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Show that

$$\frac{\partial}{\partial\phi} \quad \text{and} \quad \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

are both Killing vectors. Find a third Killing vector. Are there any more?

10. Show that any Killing vector k_a obeys

$$\nabla_a \nabla_b k_c = R_{abc}{}^{d} k_d.$$

11. Suppose that we try to find an analogue of de Sitter space but with a negative cosmological constant. Again, one can assume that it is conformally flat but now in order to find the conformal factor, you need to pick out a space direction rather than the time direction, say x, so that metric is of the form

$$ds^2 = \Omega^2(-dt^2 + dx^2 + dy^2 + dz^2).$$

Find Ω. This spacetime is usually called anti-de Sitter space.

Show that the Riemann tensor is given by

$$R_{abcd} = c(g_{ac}g_{bd} - g_{ad}g_{bc})$$

and determine c. Hence show directly that the spacetime obeys the appropriate Einstein equations.

Hardish and optional: Invent, based on what you already know from lectures, or indeed elsewhere, a series of co-ordinate transformations that put the metric into the form

$$-d\tau^2 + \cos^2\tau(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2))$$

in the case $\Lambda = -3$. Observe that the spatial sections are hyperboloids of negative curvature and not spheres.