

3.1 The Leibniz rule for Lie derivatives

Let V and ω be vector and covector fields and f a real function, all defined in the neighbourhood of a point p . Show that at p

$$\mathcal{L}_X(f\omega) = (Xf)\omega + f\mathcal{L}_X\omega.$$

3.2 The Lie derivative and contraction

For any vector field X show that $\mathcal{L}_X\delta^a_b = 0$. Deduce that Lie derivation commutes with contraction, i.e., if Y is a vector field and ω a covector field, then at any point p ,

$$\mathcal{L}_X\langle\omega, Y\rangle = \langle\mathcal{L}_X\omega, Y\rangle + \langle\omega, \mathcal{L}_XY\rangle.$$

3.3 The Lie derivative of a covector

In the lectures, definitions of the Lie derivatives of scalars and vectors were given in direct manifestly chart-free terms. The definition of the Lie derivative of a covector was obtained using these and the Leibniz property. Write out a direct manifestly chart-free definition of the Lie derivative of a covector, and demonstrate its consistency with that of the lectures.

3.4 Lie derivatives in electrodynamics

Two of the Maxwell equations governing electrodynamics are

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E},$$

and within a magnetohydrodynamic fluid of high conductivity $\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = 0$, where \mathbf{v} is the velocity of the fluid. The fluid is incompressible, i.e., $\nabla \cdot \mathbf{v} = 0$. Show that in steady-state situations the 3-vector equation of motion for \mathbf{B} can be expressed succinctly as

$$\mathcal{L}_{\mathbf{v}} \mathbf{B} = 0.$$

For unsteady motions we define a non-relativistic 4-velocity

$$V = \frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial x^\alpha},$$

and a non-relativistic 4-vector magnetic field $B = (0, \mathbf{B})$. Show that the 3-vector equation of motion for \mathbf{B} can be rewritten as

$$\mathcal{L}_V B = 0.$$

[An open-ended extension is to examine your favourite branch of physics looking for other examples of the Lie derivative at work.]

3.5 The algebra of Killing vectors

Let X and Y be two vector fields. Show that

$$\mathcal{L}_X(\mathcal{L}_Y Q) - \mathcal{L}_Y(\mathcal{L}_X Q) = \mathcal{L}_{[X,Y]} Q,$$

when Q is either a function or a vector field. Deduce that the result holds if Q is a tensor field.

Demonstrate that if a space has two “independent” isometries then it has a third, and define what is meant by independent here.

Consider the unit sphere with metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

Show that

$$\frac{\partial}{\partial \phi}, \quad \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi},$$

are Killing vectors. What is the third? Are there any more?

3.6 An identity for Killing vectors

Show that Killing covectors satisfy the equation

$$\nabla_a \nabla_b K_c = R^d{}_{abc} K_d.$$

[*Hint:* use the identity $R^a{}_{[bcd]} = 0$.]

Deduce that in Minkowski spacetime the components of Killing covectors are linear functions of the coordinates.

3.7 Killing vectors and conserved quantities in Minkowski space

Consider Minkowski spacetime with metric $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$. Let K be a Killing vector. Write down Killing’s equation. Using the result of problem 6, derive the general solution, and hence obtain the *Poincaré invariance group* of special relativity. Identify the associated conserved quantities along a timelike geodesic.

3.8 The linearized Schwarzschild solution revisited

Consider the external gravitational field of a static spherically symmetric body of mass ϵM situated at the coordinate origin $x = y = z = 0$, i.e., its 4-velocity is $U^i = (1, \mathbf{0})^T$. Take $T_{ij} = \epsilon M \delta(\mathbf{x}) U_i U_j$. Assuming that the field is both weak and a function of $R = (x^2 + y^2 + z^2)^{1/2}$ only, and that M/R is small, use the traditional approach to linearized theory to derive the linearized metric. Compare it with the solution derived in the lectures using the scalar-vector-tensor decomposition of gravitational perturbations.

3.9 A cosmic string

In Cartesian coordinates the energy momentum tensor of a straight *cosmic string* aligned along the z -axis is

$$T_{ij} = \mu\delta(x)\delta(y)\text{diag}(1, 0, 0, -1),$$

where μ is a small positive constant. Terms involving powers of μ are to be ignored. Look for a static solution using the traditional approach to linearized theory, finding $h_{11} = h_{22} = -\lambda$ as the only non-zero components of the perturbed metric tensor, where $\lambda \equiv 8\mu \ln(r/r_0)$, $r = \sqrt{x^2 + y^2}$, and r_0 is a scale length.

Rewrite the line element as

$$ds^2 = dt^2 - dz^2 - (1 - \lambda)(dr^2 + r^2 d\phi^2).$$

Make a change of radial coordinate given by $(1 - \lambda)r^2 = (1 - 8\mu)r'^2$ to obtain

$$ds^2 = dt^2 - dz^2 - dr'^2 - (1 - 8\mu)r'^2 d\phi^2,$$

and change the angular coordinate to obtain

$$ds^2 = dt^2 - dz^2 - dr'^2 - r'^2 d\phi'^2.$$

Is this Minkowski spacetime? Show intuitively how a distant object may give rise to double images.

3.10 Gravitational waves from an orbiting binary

Consider two stars, each of mass m , moving in a circular Newtonian orbit of radius R in the x, y plane centred on the origin. Show that their positions may be taken to be

$$x^a = \pm(R \cos \Omega t, R \sin \Omega t, 0),$$

where $\Omega^2 = m/(4R^3)$. Compute the corresponding energy-momentum tensor, the quadrupole moment and the metric perturbation.

3.11 The Lense-Thirring effect

Consider a large thin shell of mass M and radius R which rotates slowly about the z -axis with angular velocity Ω , so that terms of $\mathcal{O}(R^2\Omega^2)$ can be neglected. Introduce a shell density

$$\rho = M\delta(r - R)/(4\pi R^2),$$

where $r^2 = x^2 + y^2 + z^2$, and a Cartesian 4-velocity

$$U^a = (1, -\Omega y, \Omega x, 0).$$

The energy-momentum tensor is

$$T^{ab} = \rho U^a U^b.$$

Show that $T_{0\alpha}$ is a solenoidal vector. Thus we can regard this problem as a superposition of two linearized perturbations, a scalar one for which only T_{00} is nonzero, and a vector one for which only $T_{0\alpha}$ is nonzero. Peruse the handout showing the curvature tensors for these two types of perturbations. First solve the linearized Einstein equations for the scalar part. How does the result differ from Newtonian theory? Next turn to the vector part. You should find that

$$B_\alpha = \omega(y, -x, 0)$$

for $r < R$, where

$$\omega = 4M\Omega/(3R).$$

Now consider a freely falling particle moving slowly with velocity $\dot{x}^a = (1, \dot{x}, \dot{y}, \dot{z})$. Obtain the geodesic equations using the connection components given on the handout. Thus show that the inertial frames are rotating with angular velocity ω with respect to the background Minkowski frame, i.e. the inertial frames are *dragged around* by the rotating shell. This is the *Lense-Thirring effect* (1918).