

1. Let  $\phi : M \rightarrow N$  be a diffeomorphism. Let  $\nabla$  be a covariant derivative on  $M$ . The push-forward of  $\nabla$  is a covariant derivative  $\tilde{\nabla}$  on  $N$  defined by

$$\tilde{\nabla}_X T = \phi_* (\nabla_{\phi^*(X)} (\phi^*(T)))$$

where  $X$  is a vector field and  $T$  a tensor field on  $N$ . (In words: pull-back  $X$  and  $T$  to  $M$ , evaluate the covariant derivative there and then push-forward the result to  $N$ .)

(a) Check that this satisfies the properties of a covariant derivative. (b) Show that the Riemann tensor of  $\tilde{\nabla}$  is the push-forward of the Riemann tensor of  $\nabla$ . (c) Let  $\nabla$  be the Levi-Civita connection defined by a metric  $g$  on  $M$ . Show that  $\tilde{\nabla}$  is the Levi-Civita connection defined by the metric  $\phi_*(g)$  on  $N$ .

2. (a) Use the Leibniz rule to derive the formula for the Lie derivative of a covector  $\omega$  valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

(Hint: consider  $(\mathcal{L}_X \omega)(Y)$  for a vector field  $Y$ .)

(b) Use normal coordinates to argue that one can replace partial derivatives with covariant derivatives to obtain the basis-independent result (where  $\nabla$  is the Levi-Civita connection)

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b$$

(c) Show that the Lie derivative of a metric tensor is given in a coordinate basis by

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho$$

(d) Show that this can be written in the basis-independent form

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a$$

3. (a) Let  $X$  and  $Y$  be two vector fields. Show that

$$\mathcal{L}_X (\mathcal{L}_Y Q) - \mathcal{L}_Y (\mathcal{L}_X Q) = \mathcal{L}_{[X,Y]} Q,$$

when  $Q$  is either a function or a vector field. Deduce that the result holds if  $Q$  is a tensor field.

(b) Demonstrate that if a Riemannian or Lorentzian manifold has two “independent” isometries then it has a third, and define what is meant by independent here.

(c) Consider the unit sphere with metric

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

Show that

$$\frac{\partial}{\partial \phi} \quad \text{and} \quad \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}$$

are Killing vectors. What is the third? Are there any more?

4. Let  $K^a$  be a Killing vector field and  $T_{ab}$  the energy momentum tensor. Let  $J^a = T^a_b K^b$ . Show that  $\nabla_a J^a = 0$ .  $J^a$  is a *conserved current*.
5. (a) Show that a Killing vector field  $K^a$  satisfies the equation

$$\nabla_a \nabla_b K^c = R^c_{\ b a d} K^d.$$

[*Hint*: use the identity  $R^a_{\ [bcd]} = 0$ .]

- (b) Deduce that in Minkowski spacetime the components of Killing covectors are linear functions of the coordinates.
6. Consider Minkowski spacetime in an inertial frame, so the metric is  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ .
- (a) Let  $K^a$  be a Killing vector field. Write down Killing's equation in the inertial frame coordinates.
- (b) Using the result of the previous problem, show that the general solution can be written in terms of a constant antisymmetric matrix  $a_{\mu\nu}$  and a constant covector  $b_\mu$ .
- (c) Identify the isometries corresponding to Killing fields with (i)  $a_{\mu\nu} = 0$  (ii)  $a_{0i} = 0$ ,  $b_\mu = 0$ , (iii)  $a_{ij} = 0$ ,  $b_\mu = 0$  (where  $i, j$  take values from 1 to 3).
- (d) Identify the conserved quantities along a timelike geodesic corresponding to cases (i) to (iii).
7. Consider the energy-momentum tensor describing a point mass at the origin: in "almost inertial" coordinates it is  $T_{00}(t, \mathbf{x}) = M\delta^3(\mathbf{x})$ ,  $T_{0i} = T_{ij} = 0$ . Determine the linearized gravitational field produced by this energy momentum tensor, assuming it to be independent of  $t$ . For what values of  $R = |\mathbf{x}|$  is the linear approximation valid?
8. In "almost inertial" coordinates the energy momentum tensor of a straight *cosmic string* aligned along the  $z$ -axis is

$$T_{\mu\nu} = \mu\delta(x)\delta(y)\text{diag}(1, 0, 0, -1),$$

where  $\mu$  is a small positive constant. Terms of order  $\mu^2$  are to be ignored. Look for a time-independent solution of the linearized Einstein equation, finding  $h_{11} = h_{22} = -\lambda$  as the only non-zero components of the perturbed metric tensor, where  $\lambda \equiv 8\mu \log(r/r_0)$ ,  $r = \sqrt{x^2 + y^2}$ , and  $r_0$  is an arbitrary length.

Show that the perturbed metric can be written in cylindrical polar coordinates as

$$ds^2 = -dt^2 + dz^2 + (1 - \lambda)(dr^2 + r^2 d\phi^2).$$

Make a change of radial coordinate given by  $(1 - \lambda)r^2 = (1 - 8\mu)r'^2$  to obtain

$$ds^2 = -dt^2 + dz^2 + dr'^2 + (1 - 8\mu)r'^2 d\phi^2,$$

and change the angular coordinate to obtain

$$ds^2 = -dt^2 + dz^2 + dr'^2 + r'^2 d\phi'^2.$$

Is this Minkowski spacetime? Show intuitively how a distant object may give rise to double images.

9. Consider a large thin spherical shell of mass  $M$  and radius  $R$  which rotates slowly about the  $z$ -axis (in "almost inertial" coordinates) with angular velocity  $\Omega$ , so that terms of  $\mathcal{O}(R^2\Omega^2)$  can be neglected. Introduce a shell density  $\rho = M\delta(r - R)/(4\pi R^2)$  where  $r^2 = x^2 + y^2 + z^2$ , and a 4-velocity  $u^\mu = (1, -\Omega y, \Omega x, 0)$ . The energy-momentum tensor has components  $T^{\mu\nu} = \rho u^\mu u^\nu$ . We can regard this source as a superposition of two sources, one for which only  $T_{00}$  is nonzero, and one for which only  $T_{0i}$  is nonzero.

(a) Solve the linearized Einstein equations sourced by  $T_{00}$ . Show that the result agrees with Newtonian theory.

(b) Consider the perturbation sourced by  $T_{0i}$ . Argue that the only non-vanishing components are  $h_{0i}$ , which satisfy  $\nabla^2 h_{0i} = -16\pi T_{0i}$ . Consider the combination  $h_{01} + ih_{02}$  and work in spherical polar coordinates. You should find that the RHS of the linearized Einstein equation is proportional to  $\sin\theta e^{i\phi}$ , i.e., to a spherical harmonic with  $l = m = 1$ . Since a general solution to the Laplace equation can be expanded as a sum over spherical harmonics, this implies that the solution must be of the form  $h_{01} + ih_{02} = f(r) \sin\theta e^{i\phi}$ . Hence obtain the solution

$$h_{0i} = \begin{cases} \omega(y, -x, 0) & r < R \\ \omega \frac{R^3}{r^3}(y, -x, 0) & r > R \end{cases}$$

where  $\omega = 4M\Omega/(3R)$ . Note that this decays as  $1/r^2$  at large  $r$ . This is a general result: rotation of the source affects  $h_{\mu\nu}$  at  $\mathcal{O}(1/r^2)$ , subleading compared to the  $\mathcal{O}(1/r)$  contribution arising from the total mass.

(c) Consider a free particle moving non-relativistically inside the shell. Working to first order in the perturbation, show that the geodesic equation implies  $\ddot{\mathbf{x}} = 2\mathbf{w} \wedge \dot{\mathbf{x}}$  where  $\mathbf{w} = (0, 0, \omega)$  and a dot denotes a derivative with respect to  $t$ . This is the same as the equation of a slowly moving free particle in Minkowski spacetime, using a reference frame rotating with angular velocity  $-\mathbf{w}$  with respect to an inertial frame (the RHS is the Coriolis force). Hence, inside the shell, the background Minkowski frame is rotating with angular velocity  $-\mathbf{w}$  with respect to a local inertial frame, i.e., the local inertial frames are rotating with angular velocity  $+\mathbf{w}$  with respect to the background Minkowski frame. The local inertial frames are *dragged around* by the rotating shell. This is the *Lense-Thirring effect* (1918).

10. (Optional question.) Show that the second order terms in the expansion of the Ricci tensor around Minkowski spacetime are

$$\begin{aligned} R_{\mu\nu}^{(2)}[h] &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} \\ &+ \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - \frac{1}{4}\partial^\rho h\partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho} \end{aligned}$$

11. (a) Use the linearized Einstein equation to show that, in vacuum,

$$\langle \eta^{\mu\nu} R_{\mu\nu}^{(2)}[h] \rangle = 0$$

(b) Show that

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_\mu \bar{h} \partial_\nu \bar{h} - 2 \partial_\sigma \bar{h}^{\rho\sigma} \partial_{(\mu} \bar{h}_{\nu)\rho} \rangle$$

(c) Show that  $\langle t_{\mu\nu} \rangle$  is gauge invariant.