Mathematical Tripos Part III M.J. Perry

GENERAL RELATIVITY: Examples 3 Michaelmas Term 2018

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1. Suppose the metric of spacetime is given by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2)$$

where Φ depends only on x, y and z. Find the Christoffel symbols and hence the Einstein tensor. Assuming that T_{00} is the matter density ρ, show that the 00-component of the Einstein equation reproduces the Newtonian Poisson equation for gravitation. Show that geodesic motion for a massive particle reproduces Newton's law of gravitation.

2. Suppose we take the Schwarzschild metric

$$ds^2 = -Vdt^2 + \frac{dr^2}{V} + r^2 d\theta^2 + r^2 \sin^2\theta^2 d\phi^2$$

with $V = (1 - 2M/r)$ to represent the spacetime near the Earth, which has radius R. Bob and Alice are twins. Bob remains stationary and hence earthbound on the equator of the Earth. Alice is more adventurous and gets into a circular orbit at a height h above the surface of the Earth and in free fall. She passes above Bob and sends him a signal. Next time she passes above Bob she sends another signal to him and her watch measures that a time t_A has elapsed. Bob observes on his watch that a time t_B has passed between the two signals. Find $t_A - t_B$. Take the radius of the Earth to be 6.4×10^8 cm, the mass of the Earth is 6×10^{27} gm, her height above the Earth to be 10^7 cm, the speed of light is 3×10^{10} cm sec^{-1} and Newton's constant of gravitation is 6.67×10^{-8} cm^3 gm^{-1} sec^{-2} and evaluate $t_A - t_B$.

3. Suppose that Venus is directly on the opposite side of the Sun to us. The Schwarzschild metric describes the gravitational field of the Sun. Venus is at a coordinate R_V and the Earth is at R_E. There is a null geodesic that passes close to the surface of the Sun at a coordinate value R_0. Assume $R_E, R_V \gg R_0 \gg 2M$. A radio signal beamed from the Earth follows this null geodesic, is reflected by Venus and returns to Earth. Compute the round trip travel time T, in terms of R_0, as measured on Earth and compare it to $2(R_V + R_E)$. Take the mass of the Sun to be 2×10^{33}gm, its radius 7×10^{10} cm, R_V to be 1.1×10^{13}cm, R_E to be 1.5×10^{13}cm, and evaluate $T - 2R_V - 2R_E$ for the case that where the radio signal just grazes the Sun.

4. Show that it is possible for a photon to be in a circular orbit around a Schwarzschild black hole at a radius of $3M$. Is this orbit stable?

5. Suppose a light ray is incident on a Schwarzschild black hole of mass M with an impact parameter b. Show that this if $b < 3\sqrt{3}M$, then the light ray will be abosrbed by the black hole. Hence evaluate the absorption cross-section for photons.

6. A friend mischievously suggests that if you are so interested in black holes, you should jump into one. You discover a Schwarzschild black hole and hover at $10M$ and then free fall radially in. How long do you expect to survive? Would giving yourself some angular momentum help? How much? If you still fell in, would you live for a longer or shorter time?

7. It is suggested that a black hole could act as a repository of electric charge. For line elements of the form

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\theta^2 + r^2 \sin^2\theta^2 d\phi^2$$

and an electromagnetic vector potential $A_a = (Q/r, 0, 0, 0)$, find $V(r)$.

8. In a $k = 1$ FLRW Universe filled with matter, can you see the back of your own head? Assuming you can, what would be the difference in ages between what you see and how much older you are?

9. Einstein introduced the cosmological constant in order to try find a universe that is static. Assuming that the universe contains only matter and a cosmological constant, did he succeed? If he succeeded, he would have been able to relate the cosmological constant to the size of the universe a_0 and the density. He should have wondered about the stability of this universe. Then, he would have made a perturbation of the scale factor $a(t)$, making $a(t) \to a_0 + \delta a(t)$. What conclusion should he have then come to if $k = 1$?

10. Suppose the universe contains nothing but radiation. Find $a(t)$ for the three cases $k = 0, \pm 1$.