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1. Show that the second order terms in the expansion of the Ricci tensor around Minkowski spacetime are

$$R_{ab} = \frac{1}{4}\partial_a h_{cd}\partial_b h^{cd} + \frac{1}{2}h^{cd}(\partial_a\partial_b h_{cd} + \partial_c\partial_d h_{ab} - \partial_c\partial_a h_{bd} - \partial_c\partial_b h_{ad})$$
$$+ \frac{1}{2}\partial^d h_b{}^c(\partial_d h_{ca} - \partial_c h_{da}) - \frac{1}{2}(\partial_c h^{cd} - \frac{1}{2}\partial^d h)(\partial_a h_{db} + \partial_b h_{da} - \partial_d h_{ab}).$$

2. (a) Use the linearized Einstein equation to show that in the absence of any energy-momentum tensor

$$\langle \eta^{ab} R^{(2)}_{ab}[h] \rangle = 0 \tag{1}$$

(b) Show that

$$\langle t_{ab} \rangle = \frac{1}{32\pi}\langle \partial_a \bar{h}_{cd}\partial_b \bar{h}^{cd} - \frac{1}{2}\partial_a \bar{h}\partial_b \bar{h} - 2\partial_c \bar{h}^{cd}\partial_{(a}\bar{h}_{b)d}\rangle \tag{2}$$

(c) Show that $\langle t_{ab} \rangle$ is gauge invariant.

3. Two bodies, with masses m_1, m_2, move in an elliptical Newtonian orbit with semi-major axis a and eccentricity e. Choosing coordinates so that the orbit lies in the (x, y) plane with the centre of mass at the origin, the bodies have positions $\mathbf{x_1} = (r_1 \cos\psi, r_1 \sin\psi, 0)$ and $\mathbf{x_2} = (-r_2 \cos\psi, -r_2 \sin\psi, 0)$ where

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r \qquad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) r$$

The equation of the ellipse is

$$r(t) = \frac{a(1 - e^2)}{1 + e\cos\psi(t)}$$

and the angular velocity is

$$\dot{\psi}(t) = \frac{\sqrt{(m_1 + m_2)a(1 - e^2)}}{r(t)^2}$$

Treating the bodies as non-relativistic point masses (in the sense of question 7 of examples sheet 3), compute the corresponding energy-momentum tensor, the quadrupole moment Q_{ij}, and the metric perturbation $\bar{h}_{ij}$. (If you find the calculation too long then consider the simplified case of a circular orbit $e = 0$, or simplify further still by setting $e = 0$ and $m_1 = m_2$.) Show that the time average of the total power radiated in gravitational waves is

$$\langle P \rangle = \frac{32}{5}\frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} f(e), \qquad f(e) = \frac{1 + (73/24)e^2 + (37/96)e^4}{(1 - e^2)^{7/2}}$$

Note that $f(e)$ increases rapidly as $e \to 1$: $f(0.6) \sim 10$, $f(0.8) \sim 100$, $f(0.9) \sim 1000$. So highly eccentric orbits emit more gravitational radiation than circular ones. [This calculation is taken from P.C. Peters and J. Mathews, "Gravitational radiation from point masses in a Keplerian orbti," Physical Review, **131**, 435-439, 1963.]

4. As the above system emits gravitational radiation, it loses energy and angular momentum which causes the shape of the orbit to change gradually. We can model this by allowing a and e to be slowly varying functions of time. The energy of a Newtonian orbit is $E = -m_1 m_2/(2a)$ so a decreases over time. Setting $dE/dt = -\langle P \rangle$ gives an expression for da/dt. For a circular orbit ($e = 0$), use this to show that $a(t)$ reaches zero at a time

$$T = \frac{5a(0)^4}{256 m_1 m_2 (m_1 + m_2)}$$

Consider two black holes in a circular orbit, each with mass $30M_\odot$, which gives $m_1 = m_2 \approx 45$km. What is the time to merger if the initial distance between them is 1 astronomical unit (AU)? (1AU is the Earth-Sun distance: 1.5×10^8km) For comparison, the age of the Universe is about 14×10^9 years. How far apart are the black holes when the time to merger is 1 year? For comparison, the radius of the Sun is about 7×10^5km.

[The general case of an elliptical orbit with evolving a and e is considered in P.C. Peters, "Gravitational Radiation and the Motion of Two Point Masses," Physical Review **136**, B1224-B1232, 1964. Emission of gravitional radiation causes e to decrease so the orbit becomes more circular over time.]

5. Consider the above system after emission of gravitational radiation has caused the orbit to become circular. How is the frequency f of the gravitational waves related to $\dot{\psi}$? Use your expression for da/dt from the previous question to derive an expression for $\ddot{\psi}$, the rate of change of the angular velocity due to emission of gravitational radiation. Eliminate a to obtain an expression relating $\dot{\psi}$, $\ddot{\psi}$, m_1 and m_2. Hence show that by measuring f and $\dot{f}$ it is possible to determine the *chirp mass*

$$M_{\text{chirp}} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

[This gives an estimate of the total mass of the system. One can then estimate the distance to the binary using the amplitude of the gravitational radiation.]

6. Show that $R_{abcd}(\Gamma) = R_{abcd}(\omega)$.

7. It has been suggested that an action for the vacuum Einstein equations can be written as

$$I = \int \eta_{\mu\nu\rho\sigma} (\Omega^{\mu\nu}(\omega) \wedge E^\rho \wedge E^\sigma)$$

where $\eta_{\mu\nu\rho\sigma}$ is the alternating symbol, $\Omega(\omega)$ is the curvature 2-form made from the spin connection ω and E^μ are the basis 1-forms. By treating $\omega^\mu{}_\nu$ and E^μ as independent variables, show that extrema of this action reproduce the vacuum Einstein equations.