Mathematical Tripos Part III Dr O. Rinne

GENERAL RELATIVITY Michaelmas 2008

Problems 4: The Schwarzschild metric, other exact solutions and conformal infinity

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4.1 The Shapiro time delay effect

This question looks at the time delay caused to radar signals bounced off planets or satellites. Recall that, ignoring curvature, a light ray is given by $r\sin\phi = b$ where b is the distance of closest approach to the origin. By differentiating this show that $r^2\,\mathrm{d}\phi^2 = b^2\,\mathrm{d}r^2/(r^2-b^2)$. Now consider light rays moving in the equatorial plane $\theta = \frac{1}{2}\pi$ of Schwarzschild spacetime. Ignoring $\mathcal{O}(M/r)^2$ terms show that

$$\mathrm{d}t = \pm\frac{r}{\sqrt{r^2-b^2}}\left(1+\frac{2M}{r}-\frac{Mb^2}{r^3}\right)\mathrm{d}r.$$

Show that in this approximation the time taken to move from the position of closest approach to r is given by

$$\Delta t = \sqrt{r^2-b^2} + 2M\ln\left(r/b+\sqrt{r^2/b^2-1}\right) - \frac{M}{r}\sqrt{r^2-b^2},$$

and identify the first term on the right hand side. The total time taken, made up of four terms like the above, has been verified experimentally.
[*Hint:* $\int \mathrm{d}x/\sqrt{x^2-1} = \ln\left(x+\sqrt{x^2-1}\right)$ and $\int \mathrm{d}x/(x^2\sqrt{x^2-1}) = \sqrt{x^2-1}/x$.]

4.2 Light bending in Schwarzschild spacetime

Proceeding as in the lectures, construct the Lagrangian L for geodesic motion in Schwarzschild spacetime. Show that without loss of generality we may set $\theta \equiv \pi/2$, and that

$$(1-2M/r)\dot{t} = E, \qquad r^2\dot{\phi} = h$$

are conserved quantities whose interpretation should be given. For light rays $L = 0$. Show that this implies

$$\dot{r}^2 = E^2 - \frac{h^2}{r^2}\left(1-\frac{2M}{r}\right).$$

Because we are interested only in the shape of the orbits here, it is convenient to introduce $u = r_0/r$ where r_0 is some suitable scale length, $\epsilon = 2M/r_0 \ll 1$, and to regard u as a function of ϕ, $u' = \mathrm{d}u/\mathrm{d}\phi = \dot{u}/\dot{\phi}$. Show that

$$u'^2 = \frac{r_0^2}{d^2} - u^2 + \epsilon u^3,$$

where $d = h/E$ is a distance. Make the Ansatz $u = u_0(\phi) + \epsilon u_1(\phi) + \mathcal{O}(\epsilon^2)$, and solve successively for u_0 and u_1, finding (choosing a suitable origin for ϕ),

$$u = \frac{r_0}{d}\sin\phi + \epsilon\Big[a\cos\phi + \frac{1}{2}\frac{r_0^2}{d^2}(1+\cos^2\phi)\Big] + O(\epsilon^2),$$

where a is a constant. Show that to lowest order this represents a straight line distance d from the origin for $0 \leqslant \phi \leqslant \pi$. Show that the first order correction gives a deflection (towards the origin) $\Delta\phi = 2\epsilon r_0/d = 4M/d$. Taking M and d to be the mass and radius of the Sun gives $\Delta\phi = 1''.75$. This was measured in a solar eclipse in 1919, and is observational evidence for GR.

4.3 The precession of the perihelion

This question examines planetary orbits in linearized Schwarzschild spacetime. Proceed as in question 4.2, but set $L = -1$ to find

$$u'^2 = \frac{r_0^2}{h^2}(E^2 - 1) + \frac{r_0^2}{h^2}\epsilon u - u^2 + \epsilon u^3.$$

Newtonian theory is obtained by deleting the cubic term. To solve it set $u_{\mathrm{N}} = u_0 + v$, where $u_0 = \frac{1}{2}\epsilon(r_0/h)^2$ is chosen to eliminate the linear term. Hence show

$$u_{\mathrm{N}} = u_0\big[1 + e\sin(\phi - \phi_0)\big],$$

where e and ϕ_0 are constants, and e need not be evaluated explicitly. This is a conic section of eccentricity e and for planetary orbits we assume $e^2 < 1$.

Proceed similarly with the cubic term included. The linear term vanishes if u_0 satisfies a quadratic equation, and the solution closest to the Newtonian one is wanted. Show

$$v'^2 = K^2 - \omega^2 v^2 + \epsilon v^3,$$

where K is an irrelevant constant and $\omega^2 = 1 - 3\epsilon u_0$. This can be solved exactly in terms of elliptic functions, but to get the essential feature delete the cubic term, obtaining $v = (K/\omega)\sin\omega(\phi - \phi_0)$. This is an ellipse but of new angular frequency ω. On each orbit the major axis shifts through an angle $\Delta\phi = 2\pi/\omega - 2\pi \approx 3\pi\epsilon u_0$. For an ellipse, semi-major axis a, eccentricity $e > 0$, we have $r_{\max} = a(1+e)$, $r_{\min} = a(1-e)$. Using the formula for u_N show that $r_{\max}^{-1} + r_{\min}^{-1} = 2u_0/r_0$ and hence that

$$\Delta\varphi = \frac{6\pi M}{a(1-e^2)},$$

which can be tested against observation. The predicted value for Mercury is $43''$ per century, consistent with the observed value $43''.1 \pm 0.5$. Again linearized theory is confirmed.

4.4 Last news from the event horizon

An intrepid relativist embarks on a (purely radial) journey into a distant Schwarzschild black hole. He describes his trip in a radio broadcast to his colleagues back home. Just before he crosses the event horizon, they find that his broadcast frequency is becoming redshifted enormously with a time dependence $\exp(-t/T)$, where T is a constant. How can they deduce the mass of the black hole from this?

[*Hints:* Work in outgoing Eddington-Finkelstein coordinates u, r. Show first that

$$\frac{\lambda_o}{\lambda_e} = \frac{\Delta t_o}{\Delta\tau} = \frac{\Delta u_o}{\Delta\tau} = \frac{\Delta u_e}{\Delta\tau} = U^u,$$

where e and o refer to emitter and observer, and τ and U^a are the emitter's proper time and 4-velocity. Next show that $U_u = -E = \text{const}$. Find U^u and U^r from the relations $-E = g_{ua}U^a$ and $g_{ab}U^aU^b = -1$. Thus obtain $\mathrm{d}u/\mathrm{d}r = U^u/U^r \approx -2/F$ near the horizon, where $F \equiv 1 - 2M/r$. Integrate this to find $F \propto \exp(-u/4M)$. Finally substitute this in the expression for U^u.]

4.5 Geodesics and conformal structure of de Sitter spacetime

Consider radial geodesics in de Sitter spacetime, for which the line element may be taken to be

$$ds^2 = -\mathrm{d}t^2 + \cosh^2 t\,\mathrm{d}\chi^2.$$

Show that the geodesics $x^a(\lambda)$ satisfy

$$\cosh^2 t\,\dot{\chi} = K, \qquad \cosh^2 t\,\dot{t}^2 = K^2 - L\cosh^2 t,$$

where $\dot{t} \equiv \mathrm{d}t/\mathrm{d}\lambda$ etc., K and L are constants, and L should be identified. Make changes of both dependent and independent variables, setting $v(\chi) = \tanh t$, and show that

$$v'^2 = M^2 - v^2,$$

where $v' \equiv \mathrm{d}v/\mathrm{d}\chi$, and the constant M should be identified. Hence determine and sketch the geodesics (timelike, null and spacelike) passing through $x^a = (0, 0)$. Sketch the Penrose diagram for this spacetime.

4.6 Geodesics and conformal structure of anti-de Sitter spacetime

Consider radial geodesics in anti-de Sitter spacetime, for which the line element may be taken to be

$$ds^2 = -\cosh^2 r\,\mathrm{d}t^2 + \mathrm{d}r^2.$$

Using an approach similar to the one used for de Sitter spacetime, determine and sketch the geodesics. Sketch the Penrose diagram for this spacetime.

4.7 Conformal structure and horizons of FRW spacetimes

Using "conformal time" τ, the FRW line element may be written as

$$ds^2 = a^2(\tau)\big[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + f^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\big],$$

where

$$f(\chi) = \begin{cases} \sin\chi, & K = 1, \\ \chi, & K = 0, \\ \sinh\chi, & K = -1. \end{cases}$$

In each of the three cases identify a suitable conformally related spacetime and sketch the Penrose diagrams. What can be said about horizons in FRW spacetimes?
[*Hint:* You will need to be careful about the ranges of τ and χ. For the $K = -1$ case the change of coordinates

$$\begin{aligned} T &= \tan^{-1}\big(\tanh\tfrac{1}{2}(\tau+\chi)\big) + \tan^{-1}\big(\tanh\tfrac{1}{2}(\tau-\chi)\big), \\ R &= \tan^{-1}\big(\tanh\tfrac{1}{2}(\tau+\chi)\big) - \tan^{-1}\big(\tanh\tfrac{1}{2}(\tau-\chi)\big) \end{aligned}$$

will be useful.]