Part III Cosmology 2011-12

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Examples sheet I

1. **Newtonian cosmology** If the Newtonian gravitational force F exerted by a mass M on a unit mass at distance r is generalised to include a cosmological constant, Λ, so that

$$F = -\frac{GM}{r^2} + \frac{\Lambda r}{3},$$

show that a Newtonian cosmology containing matter with zero pressure is governed by the equations

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3} \\ \frac{\dot{a}^2}{a^2} &= \frac{8\pi G\rho}{3} - \frac{K}{a^2} + \frac{\Lambda}{3} \end{aligned}$$

where $r = a(t)r_0$ and K is a constant. Show that a static cosmological model (the 'Einstein static universe') can exist with $a = a_s, \dot{a} = \ddot{a} = 0$ and find a_s in terms of K and Λ. What condition on K is required? By introducing a small perturbation with $a = a_s(1 + \delta(t))$, and $\delta(t) << 1$, determine whether $a = a_s$ is a stable solution of these equations.

2. **Newtonian potential** Consider a thin spherical shell of radius a and mass per unit area σ whose centre at O lies at a distance r from an external point P. If the the gravitational potential at r is $\Phi(r)$ verify that the external potential due to the shell will be equal to that due to a point of the same mass as the shell ($4\pi a^2\sigma$) located at O if

$$\Phi(r) = Ar^{-1} + Br^2 + C$$

where A, B and C are constant.

[Hint: Show $M(a)\Phi(r) + 2\pi\sigma a\lambda(a) = \frac{2\pi\sigma a}{r}\int_{r-a}^{r+a} x\Phi(x)dx$ where $\lambda(a)$ is a constant which can be added without altering the force law.]

3. **Spaces of constant curvature** A 3-dimensional space of constant curvature K has Riemann curvature (Latin indices run 1,2,3)

$$Rabcd = K(g_{ac}g_{bd} - g_{ad}g_{bc})$$

where K is constant. Show that this requires the Ricci tensor to satisfy $R_{bd} = 2Kg_{bd}$.

If this 3-dimensional space of constant curvature is isotropic about every point it must be spherically symmetric about every point and have metric

$$g_{ab}dx^a dx^b = e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $\lambda(r)$ is an undetermined function. Use the condition $R_{bd} = 2Kg_{bd}$ on the Ricci tensor to show that

$$e^{-\lambda(r)} = 1 - Kr^2.$$

Show that under the coordinate transformation $r = \bar{r}/(1 + \frac{1}{4}K\bar{r}^2)$ this space metric transforms to

$$(1 + \frac{1}{4}K\bar{r}^2)^{-2}[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2).$$

4. **Distance measures** (a) Cosmological distances can be inferred from the *luminosity distance* $d_{\rm L}$ defined by $F = L/4\pi d_{\rm L}^2$ where L is the absolute luminosity of a comoving 'standard candle' and F is its measured luminosity (energy flux $\Delta E/\Delta t$). Include geometric, redshift and time dilation effects, to show that $d_{\rm L} = a_0 r(1+z)$. Consider the relationship between $d_{\rm L}$ and the proper separation ℓ_0 in a flat matter-dominated universe ($a \propto t^{2/3}$). For a radial photon ($d\theta = d\phi = 0$), show that $\ell_0 = 3t_0(1 - 1/\sqrt{1+z})$ and hence

$$d_{\rm L} = \ell_0(1 - \ell_0/3t_0)^{-2}\,.$$

What happens as $\ell_0 \to 3t_0$? (b) The *angular diameter distance* $d_{\rm A} = D/\theta$ relates the observed angular size θ of an object to its known physical size D. Show that $d_{\rm A}$ is related to the luminosity distance by $d_{\rm L} = (1+z)^2 d_{\rm A}$. Now consider objects of a fixed physical size D in a flat matter-dominated universe. Show that the angular diameter of these objects at first decreases with distance, but then becomes larger beyond a critical distance. What is the redshift corresponding to this critical distance (assume k =0, Λ=0 and P=0)?

5. **Observational Hubble law** Consider a Taylor expansion of the scale factor $a(t)$ about $a(t_0)$ and use this to express the redshift as

$$z = H_0(t_0 - t) + \tfrac{1}{2}(2 + q_0)\, H_0^2\, (t_0 - t)^2 + \ldots,$$

where the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$ (the subscript 0 denotes today). Invert this expression and then eliminate t in favour of r by considering radial photon propagation ($d\theta = d\phi = 0$), that is, find

$$r = a(t_0)^{-1}\left[(t_0 - t) + \frac{1}{2}H_0(t_0 - t)^2 + \ldots\right].$$

Thus, use the luminosity distance $d_{\rm L} = a_0 r(1+z)$ to find Hubble's law in terms of measurable quantities,

$$H_0 d_{\rm L} = z + \tfrac{1}{2}(1 - q_0)z^2 + \ldots$$

Sketch d_L versus z for the flat matter-dominated universe and compare with that for an accelerating universe with $q_0 \approx -0.5$ (suggested by recent supernova Ia observations).

6. **The matter-radiation transition** Consider the evolution of a flat FRW universe which today has two energy density components, radiation with density parameter $\Omega_{\rm r,0}$ and matter with density parameter $1-\Omega_{\rm r,0}$. Write the Friedmann equation in conformal time $d\eta = dt/a(t)$, using the parameters H_0 and $\Omega_{\rm r,0}$ and setting the scale factor today equal to unity. Show that the equation is solved by

$$a(\eta) = A\eta^2 + B\eta$$

where $A = \frac{1}{4}H_0^2(1-\Omega_{\rm r,0})$ and $B = H_0\sqrt{\Omega_{\rm r,0}}$. Verify that this solution takes the expected asymptotic forms at early times, where $a \propto t^{1/2}$, as well as at late times where $a \propto t^{2/3}$. Now assume $\Omega_{\rm r,0} \ll 1$. Show that the scale factor at equal matter and radiation density, $a_{\rm eq} \approx 1/z_{\rm eq} \approx \Omega_{\rm r,0}$. Show that the corresponding conformal time $\eta_{\rm eq} \approx (\sqrt{2}-1)\sqrt{\Omega_{\rm r,0}}\eta_0 \approx (\sqrt{2}-1)\eta_0/\sqrt{z_{eq}}$, where η_0 is the conformal time today. Hence calculate the angular scale subtended by the causal horizon at matter-radiation equality. Finally, estimate in Mpc the comoving scale today which corresponds to the horizon at matter-radiation equality (an important length scale for structure formation).

7. **General perfect-fluids** Integrate the Friedmann equation with $k = \Lambda = 0$ for a fluid with general equation of state $P = (\gamma-1)\rho$, with γ constant. Show that

$$a \propto t^{2/3\gamma}.$$

Show that for $w < -1/3$ the universe is accelerating. What happens to the solution when $\gamma = 0$? If $k \neq 0$ and $\Lambda = 0$ show that for $3\gamma \neq 2$ all the Friedmann models are solutions of the simple harmonic oscillator equation

$$\frac{d^2y}{d\eta^2} = -\frac{k(3\gamma-2)^2 y}{4}$$

where $y = a^{(3\gamma-2)/2}$ and η is the conformal time defined by $d\eta = dt/a$.If $a(0) = 0$ sketch these solutions for $k = +1$.

8. **Closed universe** Consider a closed FRW model ($k > 0$) with Λ=0 which is filled with pressure-free matter (P=0). Normalise (a_0=1) to find that $k = H_0^2(\Omega_{\rm m,0}-1)$. Write down the Friedmann equation in conformal time and show that

$$a(\eta) = \frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0}-1)}\left\{1-\cos\left[\sqrt{k}\,\eta\right]\right\}$$

provides the required solution. Integrate to obtain the proper time

$$t(\eta) = H_0^{-1}\frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0}-1)^{3/2}}\left\{\sqrt{k}\,\eta - \sin\left[\sqrt{k}\,\eta\right]\right\}\,.$$

Show that the universe collapses to a 'big crunch' at $t_{BC} = \pi H_0^{-1}\Omega_{\rm m,0}(\Omega_{\rm m,0}-1)^{-3/2}$.

9. **Accelerating universe** Consider flat (k =0) FRW models with pressure-free matter (P=0) and a non-zero cosmological constant $\Lambda \neq 0$, that is, with $\Omega_{\rm m,0} + \Omega_{\Lambda,0} = 1$.

(a) Scale-factor and age of the universe: Show that the normalised (a_0=1) solution can be written in the form

$$a(t) = \left(\frac{\Omega_{\rm m,0}}{1-\Omega_{\rm m,0}}\right)^{1/3} \left(\sinh\left[\tfrac{3}{2}H_0(1-\Omega_{\rm m,0})^{1/2}t\right]\right)^{2/3} .$$

[Hint: integrate the Friedmann equation with respect to proper time t]. Verify that this has the correct limits as $\Omega_{\rm m,0} \to 1$ and $\Omega_{\rm m,0} \to 0$. Hence show that the age of the universe t_0 in these models is

$$t_0 = \tfrac{2}{3}H_0^{-1}(1-\Omega_{\rm m,0})^{-1/2}\sinh^{-1}\left[(1/\Omega_{\rm m,0}-1)^{1/2}\right] ,$$

and roughly sketch this as a function of $\Omega_{\rm m,0}$.

(b) Λ-domination and acceleration: Show that the energy density of the universe becomes dominated by the cosmological constant term at a redshift

$$1+z_\Lambda = \left(\frac{1-\Omega_{\rm m,0}}{\Omega_{\rm m,0}}\right)^{1/3} ,$$

but that it begins accelerating earlier, at $1+z = 2^{\frac{1}{3}}(1+z_\Lambda)$.

(c) Causal structure and future: Show that the furthest object with which we can communicate is today at a physical distance (comoving also with $a_0 = 1$)

$$r_1 = \int_0^1 \frac{H_0^{-1}dx}{\sqrt{1-\Omega_{\rm m,0}+\Omega_{\rm m,0}x^3}} .$$

[Hint: Use $dt = da/\dot{a}$, use the Friedmann equation for $\dot{a}$ and change variables to $x = 1/a$.] Argue that this implies the existence of a future event horizon (for all $\Omega_{\rm m,0} < 1$). By integrating back in time, show that the redshift $1 + z_{\rm eh}$ of these objects can be found by equating

$$\int_1^{1+z_{\rm eh}} \frac{dx}{\sqrt{1-\Omega_{\rm m,0}+\Omega_{\rm m,0}x^3}} = \int_0^1 \frac{dx}{\sqrt{1-\Omega_{\rm m,0}+\Omega_{\rm m,0}x^3}} .$$

For the 'concordance' model ($\Omega_{\rm m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$) we find $z_{\rm eh} \approx 1.8$ and so the many galaxies and quasars observed beyond this will be forever inaccessible. What caveats might affect this conclusion?