Part III Cosmology 2011-12

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Problems marked with a star (*) are optional.

Examples sheet II

1. **Geodesic motion** (a) General particle motion in an FRW spacetime is governed by the geodesic equation, $du^\mu/ds + \Gamma^\mu_{\nu\sigma}u^\nu u^\sigma = 0$, with $u^\mu = dx^\mu/ds$ and where the Christoffel symbols are given in the handout. Show that the physical momentum of any particle always falls as $|\tilde{u}| \propto a^{-1}$, where the physical velocity is related to the comoving velocity as $|\tilde{u}| = \sqrt{-g_{ij}u^i u^j} = a|\vec{u}|$. [*Hint:* Consider $u^\mu u_\mu$ in the FRW metric and show that $u^0(du^0/ds) = |\tilde{u}|(d|\tilde{u}|/ds)$ for both massive and massless particles.]

(b) Consider the distance travelled by a massive particle with a non-zero peculiar velocity $v_{\rm pec} \neq 0$. Show that the following table summarizes the nearest distance any comoving observer can remain from the particle (that is, determine whether the particle ever stops in both comoving and physical coordinates):

Universe type	*Comoving distance*	*Proper distance*
Flat ($k=0$) matter dominated ($a \propto t^{2/3}$)	finite	∞
Flat ($k=0$) radiation dominated ($a \propto t^{1/2}$)	∞	∞

Roughly estimate in megaparsecs (Mpc) how far a massive neutrino will travel by t_0 in a flat universe ($\Lambda = 0$), if it becomes non-relativistic at a time $t_{\rm nr} \approx t_{\rm eq}$, the time of equal matter-radiation.

3. **Ideal gas approximation and thermal equilibrium** If particles are in thermal equilibrium in the early universe with a number density $n \simeq gT^3$ and they take part in interactions with an asymptotically-free cross section $< \sigma v > \simeq \alpha^2 T^{-2}$ at temperature T, where α is the dimensionless coupling constant. Estimate the mean free path, λ, of a particle and the interparticle spacing, L. Show that

$$\frac{\lambda}{L} \simeq \alpha^{-2} g^{-2/3} >> 1$$

if $\alpha \gtrsim 10^{-2}$ and $g > 100$ as $T \to 10^{32}K \simeq 10^{19}GeV$.

If these interactions can be maintained in collisional equilibrium only when their interaction rate exceeds the Hubble expansion rate of the universe, show that they will only be collisionless when the temperature exceeds $T \simeq g^{1/2}\alpha^2 m_p$, where $m_p = G^{-1/2} \simeq 10^{19}GeV$ is the Planck mass.

4. **Horizon masses** The proper distance to the particle horizon in a Friedmann universe with scale factor $a(t)$ is

$$d(t) = a(t)\int^t \frac{dt'}{a(t')}$$

Calculate the total mass of radiation inside the particle horizon at any time t during the radiation era and of dust inside the horizon during the dust era of a flat Friedmann universe (ignore the dust during the radiation era and ignore the radiation during the dust era when determining $a(t)$). Also calculate the total mass of baryons in the horizon during the radiation era in terms of their present density using the same determination of $a(t)$. What are the approximate values of these masses at the time when primordial nucleosynthesis of helium-4 occurs? In a flat Friedmann radiation universe at what time is the quantum wavelength $\lambda = \hbar/Mc$ of the total mass of radiation inside the particle horizon larger than the proper diameter of the particle horizon. How many photons are in the particle horizon volume at this time? What do you think this means for the Friedmann universe model, statistical mechanics, and the general theory of relativity?

5. **Plasma condition** During the radiation era the universe is assumed to be well described as a fully ionized quasi-neutral plasma of electrons and protons, with number densities $n_e = n_p = n$.The Debeye shielding length is $\lambda_D = (k_B T/4\pi n e^2)^{1/2} = 7 \times (T/n)^{1/2} cm$. Show that the number of photons in a Debeye sphere is a constant, approximately equal to $10^3 S^{1/2} >> 1$, so the condition for a good plasma is satisfied, where $S \simeq 10^9$ is the number of photons per baryon in the universe.

6. **Light neutrinos** Assume that one neutrino species has a nonzero mass m_ν which is much smaller than the neutrino decoupling temperature $T_D \sim 1$ MeV, so that they are relativistic when they decouple. Compute the present day contribution the massive neutrinos make to the density of the universe (flat k=0, Λ=0) and show that

$$\Omega_\nu h^2 \approx (m_\nu/91.5eV)\,.$$

7. **Gravitons** Estimate the temperature today of primordial gravitons which decoupled at $T \sim m_{pl}$ in (i) the standard model with $g^* = 106.75$ for all $T > 1\,\text{TeV}$, and (ii) a large GUT model which has $g^* \approx 10^3$ for $T > 10^{16}\text{GeV}$. What was the graviton temperature at $T = 1\,\text{MeV}$? [*Hint:* Compare with the neutrino decoupling calculation.]. Show that if the gravitons begin with a Planck spectrum they retain a Planck spectrum if the universe expands isotropically. Would this be true if the expansion was anisotropic at very early times?

8. **Heavy neutrinos** Suppose a neutral and stable particle has a suppressed weak-scale interaction rate $\Gamma \approx 0.002 G_F^2 T^5$. Assume its mass is of order 1 GeV.

(i) Show that it decouples at a temperature, $T_D \approx 40$ MeV.

(ii) Compute the contribution of the particle to the energy density of the universe today.

(iii) Is it plausible such a particle could constitute the dark matter? How many particles would there be per cm^3 in the universe?

9. **Interaction rates** Assume the interaction rate of a particle species is a power law, $\Gamma \propto T^n$ and suppose T_D and t_D are the decoupling temperatures and times when $\Gamma = H$ (with $t_D << t_{\text{eq}}$). Show that for $n \geq 3$, the average number of further interactions for each particle for the remainder of cosmic history ($t_D < t < t_0$) is less than unity.

10. **Expansion rate effects** In some extensions to general relativity it is possible to have a universal scale factor $a \propto t^\alpha$ throughout its evolution

with α constant regardless of whether the dominant energy content is matter or radiation. Show that such 'power law' cosmologies can be ruled out by constraining α in two ways:

(i) *Age and Hubble parameter:* Determine α on the basis of present estimates of the Hubble parameter and the age of the universe.

(ii) *Nucleosynthesis:* Determine α by predicting the observed Helium-4 abundance, $Y_p = 0.24 \pm 0.02$. Here, you may assume $g^* = 10.75$ at the decoupling temperature T_D (which is $T_D = 0.8$ MeV for $\alpha = 0.5$), and also that all neutrons are immediately captured into ^4He nuclei at a temperature $T = 0.1$ MeV (independent of α, unlike T_D).

11. **Recombination of hydrogen** (i) The equilibrium distributions of protons, electrons and hydrogen atoms, $p + e^- \leftrightarrow H + \gamma$, can be written in the form

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp(I/T)\,,$$

where $I = 13.6\,\mathrm{eV}$ is the ionization energy of hydrogen. Show that this can be re-expressed as Saha's equation,

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2}\eta \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp(I/T)\,,$$

where the equilibrium ionization $X_e = n_e/n_B$, the baryon-to-photon ratio $\eta = n_B/n_\gamma$, with $n_B \approx n_H + n_p$ the baryon number density (ignoring helium), and assuming charge neutrality $n_e = n_p$.

(ii) In a flat universe with $\Omega_B = 0.05$ and $h = 0.65$ (i.e. find η), show that Saha's equation can be rewritten as $e^y = \lambda \mu y^{3/2}$ where $\lambda \approx 3.4 \times 10^{15}$ and $\mu = (1 - X_e)/X_e^2$.

12. **Primordial nucleosynthesis** (i) Write down an expression which shows the explicit dependence of the nucleon decoupling temperature T_d (and so the neutron-proton ratio X_n/X_p) on the number of spin degrees of freedom g^*.
(ii) Describe the dependence of the deuterium and ^3He nucleosynthesis temperatures on the baryon-to-photon ratio η.
(iii) Discuss the effect of the following suppositions on the production of Helium-4 ^4He and deuterium D during primordial nucleosynthesis:

(a) The baryon density today is larger than we estimate.
(b) The weak interaction constant is smaller than supposed.
(c) The gravitation constant G is larger than supposed.
(d) The neutron-proton mass difference was slightly larger than supposed.

13*. **Baryon inhomogeneity** At the time of primordial deuterium nucleosynthesis the baryon density of the universe is split into two phases: a fraction f of the universe with a high baryon density (corresponding to a closure density today) and a fraction $1 - f$ with much lower baryon density, where $0 \leq f \leq 1$ is a constant. Assume the radiation density is always uniform everywhere and that the baryon density mixes and becomes smooth after nucleosynthesis has occurred. Can you choose f so that the abundance of deuterium matches that observed today even though the universe contains a closure density of dark baryons. Would there be a significant effect of this inhomogeneity in the baryon density on the helium-4 abundance?

14. **Inflationary expansion** Which of the following time evolutions for the expansion scale factor can describe a period of inflation?

(a) $a(t) = sin(t)$
(b) $a(t) = \exp[t^n]; A > 0, n > 0$
(c) $a(t) = t^n; n > 0$
(d) $a(t) = \exp[A(\ln^n(t)); A > 0, n > 0.$

15. **Slow-roll inflationary model** For a scalar field ϕ with potential $V(\phi) = \mu^2\phi^2$ show that, in the slow-roll approximation the cosmological equations for the scalar field alone can be solved to give

$$\begin{aligned} \phi(t) &= \phi_i - \frac{2\mu t}{\kappa\sqrt{3}} \\ a(t) &= a_i \exp[\frac{\kappa\mu}{\sqrt{3}}(\phi_i - \frac{\mu t^2}{\kappa\sqrt{3}}) \\ H &= \frac{\kappa\mu}{\sqrt{3}}(\phi_i - \frac{2\mu t}{\kappa\sqrt{3}}) \end{aligned}$$

where $\kappa^2 = 8\pi G$. using the criterion that inflation ends when $\frac{1}{2}\dot{\phi}^2 = V$, show that when it ends ϕ has the value

$$\phi = \phi_e = \frac{\sqrt{2}}{\kappa}$$

and that the number of e-folds of inflation between a value ϕ of the scalar field and the end of inflation is

$$N(\phi) = \frac{\kappa^2\phi^2}{2} - \frac{1}{2}.$$

16. **Exact inflationary solutions** A flat Friedmann universe contains a scalar field with a potential (setting $8\pi G = 1$)

$$V(\phi) = A^2\lambda^2[(3A^2 - 2)\cosh^2(\phi/A) + 2],$$

where A, λ are constants. If the scalar field evolves with time so that

$$\phi(t) = A\ln[\tanh(\lambda t)]$$

show that the scale factor evolves as

$$a(t) \propto [\sinh(2\lambda t)]^{A^2/2}.$$

Sketch the form of the potential for the three situations, $3A^2 > 2, 3A^2 = 2$,and $3A^2 < 2$. Find the asymptotic behaviour as $t \to 0$ and $t \to \infty$. When can inflation occur and for what values of A?

17*. **Intermediate inflation** If inflation occurs with the form

$$a(t) = \exp[At^n]$$

with $0 < n \leq 1$ and $A > 0$ constants. Use the slow-roll approximation to evaluate the slow-roll parameters and find the two values of n for which the spectrum of scalar perturbations generated by inflation has the 'flat' Harrison-Zeldovich form. What are the gravitational-wave contributions expected to be in these situations? Find the potential $V(\phi)$ for which $a(t) = \exp[At^n]$ is an exact solution of the flat Friedmann model containing only this self-interacting scalar field, ϕ. Find the behaviour of the potential for large ϕ and relate it to the approximate slow-roll solution.

18*. **Slow-roll criteria** The slow roll of a scalar field ϕ requires $\dot{\phi} << H$, where H is the Hubble expansion rate. If a tensor field $\phi^{\alpha\beta}$ was envisaged to replace ϕ what difficulty would you encounter if you tried to employ an analogue of the slow-roll approximation using the covariant time derivative, with the requirement that $\phi^{\alpha\beta}_{;0} << H$.