Structure Formation

EXAMPLES III

1. Projections of statistically-homogeneous and isotropic random fields.

(a) Consider a time-independent random field $f(\mathbf{x})$ that is statistically homogeneous and isotropic and has zero mean. The projection of this field on the surface of the sphere $|\mathbf{x}| = r$ defines a field $f(\hat{\mathbf{n}}) \equiv f(r\hat{\mathbf{n}})$. Using the Rayleigh plane-wave expansion,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{x}}) ,$$

where $j_l(kr)$ are spherical Bessel functions, show that the spherical multipoles of $f(\hat{\mathbf{n}})$ are related to the Fourier transform of $f(\mathbf{x})$ by

$$f_{lm} = 4\pi i^l \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} f(\mathbf{k}) j_l(kr) Y_{lm}^*(\hat{\mathbf{k}}) .$$

(b) Hence show that $f(\hat{\mathbf{n}})$ is statistically isotropic with angular power spectrum

$$C_l = 4\pi \int d\ln k \, \mathcal{P}_f(k) j_l^2(kr) ,$$

where $\mathcal{P}_f(k)$ is the power spectrum of $f(\mathbf{x})$.

2. Power spectrum of randomly-placed haloes.

(a) A crude model for the non-Gaussian distribution of dark matter at late times considers all dark matter to be located in haloes with identical density profiles $\kappa(\mathbf{x})$ and mass $m = \int d\mathbf{x}\, \kappa(\mathbf{x})$. Imagine laying these down at random with an average number density n. If, in some realisation of this random process, there are a total of N such haloes in some large volume V, the mass density within the volume is

$$\rho(\mathbf{x}) = \sum_{i=1}^{N} \kappa(\mathbf{x} - \mathbf{x}_i) ,$$

where $\mathbf{x}_i$ is the position of the centre of the ith halo. Taking the $\{\mathbf{x}_i\}$ to be uniformly distributed in V, and the total number N to be a Poisson process with mean nV, show that the two-point correlation function of $\rho(\mathbf{x})$ is

$$\xi(\mathbf{x}, \mathbf{x}') = \int d^3\mathbf{y} \kappa(\mathbf{x} - \mathbf{y}) \kappa(\mathbf{x}' - \mathbf{y}) .$$

[Note that $\rho(\mathbf{x})$ does not have zero mean so the correlation function is defined as $\xi(\mathbf{x}, \mathbf{x}') \equiv \langle \rho(\mathbf{x}) \rho(\mathbf{x}') \rangle - \langle \rho(\mathbf{x}) \rangle \langle \rho(\mathbf{y}) \rangle$.]

(b) Show that the two-point correlation function is consistent with statistical homogeneity and, for spherically-symmetric $\kappa(\mathbf{x})$, statistical isotropy.

(c) By expressing the two-point correlator in Fourier space in terms of $\xi(\mathbf{x}, \mathbf{x}')$, show that the power spectrum of $\rho(\mathbf{x})$ is

$$\mathcal{P}_\rho(k) = 4\pi n k^3 |\kappa(k)|^2 ,$$

for spherical profiles with Fourier transform $\kappa(k)$.

(d) Argue that the power spectrum behaves as k^3 on scales large compared to the characteristic size of the haloes.

3. **Growth of density perturbations at late times**.

(a) Outline the derivation of the equation for the evolution of the fluctuations in the matter density,

$$\partial_t^2 \delta_m + 2H\partial_t \delta_m - 4\pi G \bar{\rho}_m \delta_m = 0 \,,$$

using Newtonian theory. (This equation also holds in the full relativistic treatment where δ_m is defined in the comoving gauge.)

(b) Define the growth function $D(t)$ by $\delta_m(t, \mathbf{x}) = D(t)\delta_m(t_0, \mathbf{x})$, where t_0 is the current time, so that $D(t_0) = 1$. Use the Friedmann equations in a flat universe with pressure-free matter and a cosmological constant, and the evolution equation above for δ_m, to show that $u \equiv D/H(t)$ satisfies

$$\frac{d^2 u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} = 0 \,,$$

where z is the redshift.

(c) Show that this equation gives a solution $D(z) \propto H(z)$ (the "decaying mode") and another ("growing mode") with

$$D(z) \propto H(z) \int_z^\infty \frac{1+z'}{H^3(z')}\, dz' \,.$$

(d) Verify that for vanishing cosmological constant these solutions recover the expected behaviours $D \propto a^{-3/2}$ and $D \propto a$ respectively. Show further that for a non-zero cosmological constant, the asymptotic form of the growing mode in the future ($z \to -1$) is constant, i.e. the growth of structure halts.

4. **Newtonian theory in the Lagrangian description**.

(a) In the *Lagrangian description* of matter-dominated Newtonian cosmology, we express the comoving position, $\mathbf{x}$, of a particle at time t in terms of its position, $\mathbf{q}$, at some initial time, t_i, as

$$\mathbf{x} = \mathbf{q} + \boldsymbol{\Psi}(\mathbf{q}, t) \,.$$

Show (by mass conservation) that if the matter density is homogeneous at the initial time t_i, with value $\bar{\rho}(t_i)$, then

$$\rho(\mathbf{x}, t) = \frac{\bar{\rho}(t_i)}{|\partial \mathbf{x}/\partial \mathbf{q}|} \,,$$

where the Jacobian $|\partial \mathbf{x}/\partial \mathbf{q}|$ is evaluated at the corresponding $\mathbf{q}$.

(b) Working to first order in ψ, show that the density contrast is

$$\delta(\mathbf{x}, t) \approx -\nabla_{\mathbf{q}} \cdot \boldsymbol{\Psi} \,.$$

(c) Show further that the peculiar velocity, $\mathbf{v} = a d\mathbf{x}/dt$, can be written as

$$\mathbf{v}(\mathbf{x}, t) = a\partial_t \boldsymbol{\Psi}(\mathbf{q}, t) \,,$$

and verify that the linearised continuity equation (1.2.21 in the notes) is satisfied. Argue that if the peculiar velocity is irrotational (i.e. there are no vector modes), then $\boldsymbol{\Psi}$ can be written as a gradient in linear theory: $\boldsymbol{\Psi}(\mathbf{q}, t) \propto \nabla_{\mathbf{q}} \Psi(\mathbf{q}, t)$.

(d) By considering the linear growth of $\delta(\mathbf{x}, t)$ in an Einstein-de Sitter universe, show that, neglecting the decaying mode,

$$\mathbf{x} = \mathbf{q} - a(t)\nabla_{\mathbf{q}} \Psi(\mathbf{q}) \,.$$

(Note that the comoving displacement of a fluid element is determined by the time-independent function $\Psi(\mathbf{q})$ and the linear growth rate $a(t)$. This prescription turns out to be valid even in the non-linear regime for a one-dimensional perturbation and, coupled with the result in part (a), allows one to compute the evolution of the non-linear overdensity. In the *Zel'dovich approximation*, one assumes that the prescription above for the comoving displacement holds generally. This can be used to give a qualitative picture of three-dimensional, asymmetrical non-linear collapse.)

5. Spherical collapse.

(a) A closed, dust-filled universe has Hubble parameter H_* and density parameter Ω_* at some time t_* after the big bang. (Note that only two of these are independent.) Choosing the scale factor to be a_* at this instant, show that

$$\frac{a}{a_*} = \frac{\Omega_*}{2(\Omega_* - 1)}(1 - \cos\theta) , \qquad H_* t = \frac{\Omega_*}{2(\Omega_* - 1)^{3/2}}(\theta - \sin\theta) ,$$

where the (development) parameter $\theta = H_* a_* (\Omega_* - 1)\eta$ is proportional to the conformal time η.

(b) Show that the density contrast relative to an Einstein-de Sitter universe at the same time from its big bang is

$$\delta(t) = \left(\frac{3\pi}{4}\right)^2 \left(\frac{t}{t_{\rm max}}\right)^2 \left(\frac{a_{\rm max}}{a}\right)^3 - 1 ,$$

where $a_{\rm max} \equiv a_* \Omega_* / (\Omega_* - 1)$ and $H_* t_{\rm max} \equiv \pi\Omega_* / [2(\Omega_* - 1)^{3/2}]$ are the scale factor and time, respectively, when the closed universe starts to re-collapse.

(c) Show that for $\theta \ll 1$ (i.e. early times), the scale factor is

$$a \approx \frac{a_{\rm max}}{4}\left(\frac{6\pi t}{t_{\rm max}}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6\pi t}{t_{\rm max}}\right)^{2/3}\right] .$$

(*Hint: first express* θ *as a power series in* $(6\pi t/t_{\rm max})^{1/3}$.) Hence show that the density contrast at early times is

$$\delta(t) \approx \frac{3}{20}\left(\frac{6\pi t}{t_{\rm max}}\right)^{2/3} .$$

Note that this is just the linear growth rate in an Einstein-de Sitter universe.

(d) A simple model for the development of a spherical, non-linear structure is to carve out a spherical region of an Einstein-de Sitter universe and to concentrate the mass uniformly in a smaller region centred on the original sphere. The inner region develops like a closed universe while the outer region continues as Einstein-de Sitter. Use the results above to show that the extrapolated linear density contrast at $t = 2t_{\rm max}$, when the inner region collapses, is $3(12\pi)^{2/3}/20 \approx 1.68$. If, instead, we assume that the inner region virialises after collapsing to $a_{\rm max}/2$, forming a bound structure then with a time-independent density, show that the density contrast of this structure relative to the surrounding Einstein-de Sitter universe at time $2t_{\rm max}$ is $\delta \approx 177$.