Structure Formation

EXAMPLES IV

1. Acoustic oscillations in the presence of baryons.

(a) Consider a photon fluid with background energy density $\bar{\rho}_\gamma$, pressure $\bar{P}_\gamma = \bar{\rho}_\gamma/3$, and density contrast δ_γ, and a baryon fluid with background energy density $\bar{\rho}_b$, negligible pressure, and density contrast δ_b, both in the Newtonian gauge. These fluids are tightly-coupled before last scattering so have the same peculiar velocity v^i_γ (also defined in the Newtonian gauge). Noting that the stress-energy tensor for the composite fluid is conserved, apply the perturbed Euler equation (3.3.84) to show that

$$\dot{v}_{i,\gamma} + \frac{\partial_i \delta_\gamma}{4(1+R)} + \frac{\mathcal{H}R}{1+R} v_{i,\gamma} + \partial_i \psi = 0\,,$$

where $R \equiv \bar{\rho}_b/(\bar{\rho}_\gamma + \bar{P}_\gamma)$. (Note how baryon inertia tends to reduce the effect of pressure gradients and also gives rise to a Hubble drag term.)

(b) By assuming that the photon and baryon fluids do not exchange energy, argue that the photon density contrast evolves as

$$\dot{\delta}_\gamma + \frac{4}{3}\frac{\partial v^i_\gamma}{\partial x^i} - 4\dot{\phi} = 0\,.$$

By combining this with the Euler equation above, show that

$$\ddot{\delta}_\gamma + \frac{\mathcal{H}R}{1+R}\dot{\delta}_\gamma - \frac{1}{3(1+R)}\nabla^2 \delta_\gamma = 4\ddot{\phi} + \frac{4\mathcal{H}R}{1+R}\dot{\phi} + \frac{4}{3}\nabla^2\psi\,.$$

Note how baryons reduce the sound speed. In Fourier space, this is the equation of an oscillator, damped by Hubble drag on the baryons and driven by gravity.

(c) Consider solving the homogeneous equation (i.e. ignore gravity initially) on sub-Hubble scales. By noting that $R \propto a$ and so varies on a Hubble time, show that the homogeneous equation can be approximated in Fourier space by

$$\partial^2_\eta(\delta_\gamma\sqrt{1+R}) + \frac{k^2}{3(1+R)}(\delta_\gamma\sqrt{1+R}) \approx 0\,.$$

Use the WKB approximation to show that this has a solution

$$\delta_\gamma \propto \frac{1}{(1+R)^{1/4}}\cos k r_s(\eta)$$

where the sound horizon $r_s(\eta) \equiv \int_0^\eta d\eta'/\sqrt{3(1+R)}$. Note how baryons reduce the sound horizon.

(d) For gravitational potentials that are constant in time, show that their effect is approximately to shift the midpoint of the oscillation of δ_γ by $-4(1+R)\psi$ and hence that the source term in the Sachs-Wolfe effect, $\delta_\gamma/4+\psi$, oscillates about a midpoint $-R\psi$. (This is how baryons enhance the compressional peaks, $n = 1, 3, \ldots$, over the $n = 2, 4, \ldots$ peaks in the CMB angular power spectrum.)

2. Curvature perturbations.

(a) Show that the quantity

$$\zeta \equiv -\phi - \frac{1}{3}\nabla^2 E - \mathcal{H}\frac{\delta\rho}{\dot{\bar{\rho}}}$$

is gauge-invariant.

(b) Give physical interpretations of ζ in the *uniform-density gauge*, in which surfaces of constant time have constant density, and in the *zero-curvature gauge*, in which surfaces of constant time have vanishing intrinsic curvature ${}^{(3)}R$.

(c) By evaluating the difference between ζ and the comoving-gauge curvature perturbation, $\mathcal{R}$, in the Newtonian gauge, show that on super-Hubble scales they differ by terms $O(k^2/\mathcal{H}^2)\mathcal{R}$ and hence are approximately equal. (*Hint: use the* 00 *and* 0*i* *Einstein equations.*).

3. **Cosmological gravitational waves**.

(a) The line element of a FRW metric with tensor (gravitational wave) perturbations is

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + 2E_{ij})dx^i dx^j] \, ,$$

where E_{ij} is symmetric, trace-free and transverse. Working to linear order in E_{ij}, show that the non-zero connection coefficients are

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + 2\mathcal{H}E_{ij} + \dot{E}_{ij} \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j + \dot{E}^i{}_j \\
\Gamma^i_{jk} &= \partial_j E^i{}_k + \partial_k E^i{}_j - \delta^{il}\partial_l E_{jk} \, .
\end{aligned}$$

(b) Show that the perturbation to the Einstein tensor has non-zero components

$$\delta G_{ij} = \ddot{E}_{ij} - \nabla^2 E_{ij} + 2\mathcal{H}\dot{E}_{ij} - 2E_{ij}(2\dot{\mathcal{H}} + \mathcal{H}^2) \, .$$

(c) Show further that for tensor perturbations, the non-zero perturbations to the energy-momentum tensor are

$$\delta T_{ij} = 2a^2 \bar{P} E_{ij} - a^2 \Pi_{ij} \, .$$

(d) Combine these results, and the zero-order Friedmann equation, to show that the perturbed Einstein equation reduces to

$$\ddot{E}_{ij} + 2\mathcal{H}E_{ij} - \nabla^2 E_{ij} = -8\pi G a^2 \Pi_{ij} \, .$$

(e) For the case where $\nabla^2 E_{ij} = -k^2 E_{ij}$ (i.e. a Fourier mode of the metric perturbation), and assuming the anisotropic stress can be ignored, show that

$$E_{ij} \propto \frac{k\eta\cos(k\eta) - \sin(k\eta)}{(k\eta)^3}$$

is a solution for a matter-dominated universe ($a \propto \eta^2$).

(f) Show that the solution tends to a constant for $k\eta \ll 1$ and argue that such a constant solution always exists for super-Hubble gravitational waves irrespective of the equation of state of the matter. For the specific solution above, show that well inside the Hubble radius it oscillates at (comoving) frequency k and with an amplitude that falls as $1/a$. (This behaviour is also general and follows from a WKB solution of the Einstein equation.)

4. **Φ^n inflation**.

(a) Consider inflation in a potential $V \propto \Phi^n$ for $n \geq 2$. Show that the slow-roll parameters are

$$\epsilon_V = \frac{n^2}{2}\left(\frac{M_{\text{Pl}}}{\bar{\Phi}}\right)^2 \qquad \text{and} \qquad \eta_V = n(n-1)\left(\frac{M_{\text{Pl}}}{\bar{\Phi}}\right)^2 .$$

(b) Inflation ends when $\bar{\rho} + 3\bar{P} = 0$, i.e. $(\partial_t\bar{\Phi})^2 = V(\bar{\Phi})$. Assuming slow-roll holds, so $3H\partial_t\bar{\Phi} \approx -V'(\bar{\Phi})$, all the way to the end of inflation, show that the value of $\bar{\Phi}$ at the end of inflation is $\bar{\Phi}_{\text{end}} = nM_{\text{Pl}}/\sqrt{3}$.

(c) The number of e-folds to the end of inflation is N. Show that

$$N = \int H(t)\,dt = \int \frac{H}{\partial_t\bar{\Phi}}\,d\bar{\Phi} \approx \frac{1}{M_{\text{Pl}}^2}\int \frac{V}{V'}\,d\bar{\Phi} \, ,$$

where the approximation assumes slow-roll. Assuming that a given cosmological scale exited the Hubble radius $N \approx 55$ e-folds before the end of inflation, and that $n \ll 6N$, show that at Hubble exit, $\bar{\Phi} \approx \sqrt{2nN}M_{\text{Pl}}$.

(d) Hence show that the spectral index of curvature fluctuations is $n_s \approx 1 - (n+2)/(2N)$. Evaluate this for $n = 2$ and $N = 55$.

(e) Show further that the tensor-to-scalar ratio $r \approx 4n/N$.