Problem Set 4

Initial Conditions from Inflation

Warmup Problem

(a) Describe $g_\star(T)$ given the particle content of the Standard Model.

(b) Why does the neutrino temperature scale as $T_\nu \propto a^{-1}$ after decoupling?

(c) Why is today's photon temperature larger than that of neutrinos?

(d) Why is the recombination temperature much lower than the ionization energy of hydrogen?

1. Chemical Potential for Electrons

(a) Show that the difference between the number densities of electrons and positrons in the relativistic limit ($m_e \ll T$) is

$$n_e - \bar{n}_e \approx \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right] \ ,$$

where μ_e is the chemical potential.

Hint: You may use that

$$\int_0^\infty \mathrm{d}y \, \frac{y}{e^y+1} = \frac{\pi^2}{12} \ .$$

(b) The electrical neutrality of the universe implies that the number of protons n_p is equal to $n_e - \bar{n}_e$. Use this to estimate μ_e/T.

2. Neutrinos

(a) *Massive neutrinos:*

Assume that one neutrino species has a non-zero mass m_ν which is much smaller than the neutrino decoupling temperature $T_{dec} \sim 1$ MeV, so that they are relativistic when they decouple. Compute the temperature of the neutrinos relative to the cosmic microwave photons and hence estimate their number density. From their assumed mass m_ν, show that the density the neutrinos contribute in the universe corresponds to

$$\Omega_\nu h^2 \approx \frac{m_\nu}{94\,\mathrm{eV}} \ .$$

(b) *Extra neutrino species:*

Suppose that there was a fourth generation of active neutrinos in addition to ν_e, ν_μ, and ν_τ. (This possibility is excluded by the Z lifetime, but you should still be able to do the problem.) Compute $g_\star$ prior to e^+e^- annihilation, and $g_\star$ after e^+e^- annihilation. What is the final neutron abundance in this case? What will this do to the final ^{4}He abundance?

3. Relic Baryon Density

Consider massive particles and antiparticles with mass m and number densities $n(m,t)$ and $\bar{n}(m,t)$. If they interact with cross-section σ at velocity v, explain why the evolution of $n(m,t)$ is described by

$$\frac{\partial n}{\partial t} = -3\frac{\dot{a}}{a}n - n\bar{n}\langle\sigma v\rangle + P(t) \ ,$$

and identify the physical significance of each of the terms appearing in this equation.

(a) By considering the evolution of the antiparticles, show that

$$(n - \bar{n})a^3 = const.$$

(b) Assuming initial particle-antiparticle symmetry, show that

$$\frac{1}{a^3}\frac{d(na^3)}{dt} = -\langle\sigma v\rangle\Big[n^2 - n_{\rm eq}^2\Big] \ ,$$

where $n_{\rm eq}$ denotes the equilibrium number density.

(c) Define $Y \equiv n/T^3$ and $x \equiv m/T$, and show that

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}\Big[Y^2 - Y_{\rm eq}^2\Big] \ ,$$

where $\lambda \equiv m^3\langle\sigma v\rangle/H(T=m)$. If λ is constant, show that at late times Y approaches a value given by

$$Y_\infty = \frac{x_f}{\lambda} \ ,$$

where x_f is the freeze-out time. Explain the dependence of Y_∞ on $\langle\sigma v\rangle$ and sketch the schematic evolution of Y versus x for both a strongly and a weakly interacting population of annihilating particles and antiparticles. If there was a speed-up in the expansion rate of the universe caused by the addition of extra low-mass neutrino species what would happen to the abundance of surviving massive particles and why?

Now apply this to proton-antiproton annihilation. You may use that $\langle\sigma v\rangle \approx 100\,{\rm GeV}^{-2}$.

(d) Show that $T_f \approx 20$ MeV.

(e) Show that

$$\frac{n}{n_\gamma} = \frac{\bar{n}}{n_\gamma} = 10^{-19} \ .$$

How does this compare with observational data? What do you conclude about the abundances of protons and antiprotons in the early universe?

4. Primordial Nucleosynthesis

(a) Write down an expression which shows the explicit dependence of the nucleon decoupling temperature T_{dec} (and so the neutron-proton ratio X_n/X_p) on the number of relativistic species $g_\star$.

(b) Discuss the effect of the following suppositions on the production of ^{4}He during primordial nucleosynthesis:

1. The baryon density today is larger than we estimate.
2. The weak interaction constant G_F is smaller at nucleosynthesis than it is today.
3. Newton's constant G is larger than supposed.
4. The neutron-proton mass difference was slightly larger than supposed.

5. Mukhanov-Sasaki

In the lectures, we ignored metric fluctuations in deriving the dynamics of the inflaton fluctuations $f \equiv a\delta\phi$ (in spatially flat gauge). If we had included the metric fluctuations, we would have found that the Mukhanov-Sasaki equations takes the form

$$f_k'' + \left(k^2 - \frac{z''}{z}\right) f_k = 0 \ , \qquad (\star)$$

where $z^2 = 2a^2\varepsilon$.

(a) Show that at *first order* in the slow-roll parameters,

$$aH = -\frac{1}{\tau}(1+\varepsilon) \quad \text{and} \quad \frac{z''}{z} = \frac{\nu^2 - \frac{1}{4}}{\tau^2} \ ,$$

where $\nu \equiv \frac{3}{2} + \varepsilon + \frac{1}{2}\eta$.

(b) Show that the Bunch-Davies solution of $(\star)$ is

$$f_k(\tau) = \frac{\sqrt{\pi}}{2}(-\tau)^{1/2} H_\nu^{(1)}(-k\tau) \ ,$$

where $H_\nu^{(1)}$ is a Hankel function of the first kind. You may use that

$$\lim_{k\tau \to -\infty} H_\nu^{(1,2)}(-k\tau) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{-k\tau}} e^{\mp ik\tau} e^{\mp i\frac{\pi}{2}(\nu+\frac{1}{2})} \ .$$

(c) Derive the power spectrum of curvature perturbations on superhorizon scales

$$\Delta_{\mathcal{R}}^2 = \frac{1}{z^2}\Delta_f^2 \ .$$

You may use that

$$\lim_{k\tau \to 0} H_\nu^{(1)}(-k\tau) = \frac{i}{\pi}\Gamma(\nu)\left(\frac{-k\tau}{2}\right)^{-\nu} \ .$$

(d) Show that the scale-dependence of the scalar spectrum is

$$n_s - 1 \equiv \frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} = -2\varepsilon - \eta \ .$$

Write the answer in terms of the potential slow-roll parameters $\epsilon_{\rm v}$ and $\eta_{\rm v}$.

6. The Higgs as an Inflaton?

The LHC has recently discovered a Higgs-like scalar particle. It is tempting to ask if the Higgs field could have been the scalar field that drove inflation.

(a) Let the potential of the Higgs boson be

$$V(\phi) = \lambda\left(\phi^2 - v^2\right)^2 \ ,$$

where $v = 246$ GeV. Sketch the potential and indicate the regions where slow-roll inflation might occur. Compute the slow-roll parameters $\epsilon_{\rm v} \equiv \frac{1}{2}M_{\rm pl}^2\,(V'/V)^2$ and $\eta_{\rm v} \equiv M_{\rm pl}^2 V''/V$.

(b) First, consider the region $0 < \phi < v$.

Sketch $\epsilon_{\rm v}(\phi)$ and $\eta_{\rm v}(\phi)$ between $\phi = 0$ and $\phi = v$. Is there a region in which both slow-roll conditions can be satisfied simultaneously?

(c) Now, look at the regime $\phi \gg v$.

Show that $\epsilon_{\rm v}(\phi)$ and $\eta_{\rm v}(\phi)$ become independent of v. For what field values does inflation occur? Determine the field values at the end of inflation (ϕ_E) and $N_\star \sim 60$ e-folds before ($\phi_\star$). [You may assume that $\phi_\star \gg \phi_E$.]

Compute the amplitude of the power spectrum of scalar fluctuations at $\phi_\star$. Express your answer in terms of $N_\star$ and the Higgs boson mass m_H.

Estimate the value of m_H required to match the observed scalar amplitude $\Delta_s^2 = 2 \times 10^{-9}$. Compare this to the announced mass of the Higgs boson, $m_H = 125$ GeV.

(d)* Recently, a new version of Higgs inflation has been proposed. Its key ingredient is a non-minimal coupling of the Higgs to gravity. The starting point is the following action

$$S = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}f(\phi)R + \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right] \ , \qquad \text{where} \quad f(\phi) \equiv 1 + \xi\frac{\phi^2}{M_{\rm pl}^2} \ .$$

For $\xi = 0$, this corresponds to the analysis in part (c). Now we want to study $\xi \gg 1$. It is convenient to define $\tilde{g}_{\mu\nu} \equiv f(\phi)g_{\mu\nu}$, so that the action becomes that of a standard slow-roll model

$$S = \int {\rm d}^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm pl}^2}{2}\tilde{R} + \frac{1}{2}(\partial\Phi)^2 - V(\Phi)\right] \ ,$$

with potential

$$V(\Phi) \approx \frac{\lambda M_{\rm pl}^4}{4\xi^2}\left(1 - 2\exp\left[-\sqrt{\frac{2}{3}}\frac{\Phi}{M_{\rm pl}}\right]\right) \ , \qquad \text{where} \quad \frac{\Phi}{M_{\rm pl}} = \sqrt{\frac{3}{2}}\ln(f(\phi)) \ .$$

Perform a slow-roll analysis of this potential in the limit $\Phi \gg M_{\rm pl}$:

- Show that the slow-roll parameters are

$$\eta_{\rm v} = -\frac{4}{3}\,e^{-\sqrt{2/3}\,\Phi/M_{\rm pl}} \ , \quad \epsilon_{\rm v} = \frac{3}{4}\eta_{\rm v}^2 \ .$$

- Show that the scalar spectral index is

$$n_s = 1 - \frac{2}{N_\star} \ ,$$

and the tensor-to-scalar ratio is

$$r = \frac{12}{N_\star^2} \ .$$

How do these predictions compare to the Planck data?

- By considering the amplitude of scalar fluctuations, determine the required value of the non-minimal coupling ξ for $\lambda = \mathcal{O}(1)$.

7. Tensors and the Lyth Bound

(a) Show that the tensor-to-scalar ratio predicted by slow-roll inflation is

$$r \equiv \frac{\Delta_t^2}{\Delta_s^2} = \frac{8\dot{\phi}^2}{M_{\rm pl}^2 H^2} \ .$$

(b) Show that the inflaton field travels a "distance" $\Delta\phi \equiv |\phi_E - \phi_\star|$ during (observable) inflation

$$\frac{\Delta\phi}{M_{\rm pl}} = \frac{N_\star}{60}\sqrt{\frac{r}{0.002}} \ ,$$

where $N_\star$ is the total number of e-folds between the time when the CMB scales exited the horizon and the end of inflation. [You may assume that $\varepsilon \approx$ *const.* during inflation] Comment on the implication of this result for observable gravitational waves. [Realistically, we require $r > 0.001$ to have a fighting chance of detecting gravitational waves via CMB polarisation.]

(c) Derive the following relationship between the energy scale of inflation, $V^{1/4}$, and the tensor-to-scalar ratio,

$$V^{1/4} = \left(\frac{3\pi^2}{2} r\Delta_s^2\right)^{1/4} M_{\rm pl} \ .$$

Use $\Delta_s^2 = 2.5 \times 10^{-9}$ to determine $V^{1/4}$ for $r = 0.01$. How does that compare to the energy scales probed by the LHC?