Mathematical Tripos Part III Lent Term 2019
Black Holes: Example Sheet 1 J.E. Santos

1. Let V^a be a vector field. Show that
$$\nabla_a V^a = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right)$$

2. Let X be a vector field and Y be a p-form. Show that
$$\mathcal{L}_X Y = i_X dY + d\left(i_X Y\right),$$
where, for a q-form Z, $i_X Z$ is the $(q-1)$-form resulting from contracting X with the first index of Z.

3. A *conformal Killing vector field* is a vector field k such that
$$\mathcal{L}_k g_{ab} = \phi g_{ab}$$
for some function ϕ. (i) Use the geodesic equation to show that the existence of a conformal Killing vector field implies the existence of a conserved quantity along *null* geodesics. (ii) Show that if k is a conformal Killing vector field for the metric g then it is also a conformal Killing vector field for the metric $\Omega^2 g$ where Ω is any positive function.

4. (a) Use the Tolman-Oppenheimer-Volkoff equations for dm/dr and $d\Phi/dr$ to eliminate ρ and p from the equation for dp/dr. Show that the resulting equation can be written as
$$\left(e^{-\Psi}r^{-1}\zeta'\right)' = \left(\frac{m}{r^3}\right)' e^{\Psi}\zeta$$
where $\zeta = e^{\Phi}$ and a prime denote d/dr.
(b) Use $d\rho/dr \leq 0$ to argue that $(m/r^3)' \leq 0$. (Hint: the formula relating m and ρ is the same as the formula for the mass of a ball of matter of radius r in Euclidean space. In Euclidean space m/r^3 is proportional to the average density.)
(c) It follows that $\left(e^{-\Psi}r^{-1}\zeta'\right)' \leq 0$. By integrating this equation from r_1 to r and using the equation for $d\Phi/dr$, show that, for any $r_1 \leq r$,
$$\zeta'(r_1) \geq \left(1-\frac{2m(r)}{r}\right)^{-1/2}\left(\frac{m(r)}{r^3}+4\pi p(r)\right)\zeta(r)r_1\left(1-\frac{2m(r_1)}{r_1}\right)^{-1/2}$$
(d) $(m/r^3)' \leq 0$ implies $m(r_1)/r_1 \geq m(r)r_1^2/r^3$ for $r_1 \leq r$. Use this to show
$$\int_0^r r_1\left(1-\frac{2m(r_1)}{r_1}\right)^{-1/2} dr_1 \geq \frac{r^3}{2m(r)}\left[1-\left(1-\frac{2m(r)}{r}\right)^{1/2}\right]$$
(e) Deduce that
$$\zeta(0) \leq \zeta(r)\left\{1-\left(\frac{1}{2}+\frac{2\pi r^3 p(r)}{m(r)}\right)\left[\left(1-\frac{2m(r)}{r}\right)^{-1/2}-1\right]\right\}$$
(f) By definition ζ is everwhere positive. In particular $\zeta(0) > 0$. Deduce that
$$\frac{m(r)}{r} < \frac{2}{9}\left\{1-6\pi r^2 p(r)+\left[1+6\pi r^2 p(r)\right]^{1/2}\right\}$$

5. Consider a static, spherically symmetric, star made from incompressible matter, i.e., constant density ρ. Let M and R be the mass and radius of the star. Obtain an expression for the pressure p_c at the centre of the star in terms of ρ, and M/R. Show that such stars can get arbitrarily close to saturating Buchdahl's inequality $M/R < 4/9$.

6. Starting from the Schwarzschild solution in Schwarzschild coordinates, define a new coordinate T by $dT = dt + F(r)dr$. (i) Show that one can choose $F(r)$ so that in coordinates (T, r, θ, ϕ), a surface of constant T is flat. (ii) Show that the metric in the new coordinates can be extended across $r = 2M$. You should find two choices of sign for $F(r)$: what do these correspond to? (These coordinates are called Painlevé-Gullstrand coordinates.)

7. Consider a null geodesic incident from infinity on a Schwarzschild black hole. Let E and h denote the conserved quantities associated with the timelike Killing field and the angular Killing field $\partial/\partial\phi$.

   (a) Show that the maximum value for the *impact parameter* $b \equiv |h/E|$ for which the geodesic falls into the black hole is $b_{\rm max} = 3\sqrt{3}M$.

   (b) Determine the geometrical interpretation of the impact parameter by considering ϕ as a function of r at large r.

   (c) Hence show that the geodesics that fall into the hole are the same as those that would be absorbed by a perfectly absorbing disc of radius $b_{\rm max}$ in Minkowski space-time. (Therefore the *photon absorption cross-section* of the black hole is $\pi b_{\rm max}^2 = 27\pi M^2$.)

8. Consider a toy model for gravitational collapse in which we model the collapsing star as a ball of dust, i.e., a perfect fluid with vanishing pressure. This implies the fluid velocity satisfies the geodesic equation. By continuity, particles on the surface of the star will follow geodesics of the Schwarzschild metric. Assume that the star starts from rest at radius $r = r_0 > 2M$. Using ingoing Eddington-Finkelstein coordinates, determine the proper time it takes for the star's surface to reach (i) $r = 2M$, (ii) $r = 0$. (c) Show that the star reaches $r = 0$ in finite EF coordinate time v.

9. Using Schwarzschild coordinates, show that *every* timelike curve in region II of the Kruskal manifold intersects the singularity at $r = 0$ within a proper time no greater than πM. For what curves is this bound attained?

10. (a) Let Σ be an Einstein-Rosen bridge, i.e., a surface of constant t, in the Kruskal spacetime. The geometry of Σ can be visualized by embedding it into four-dimensional Euclidean space. In cylindrical polar coordinates, the metric is $ds^2 = dR^2 + R^2 d\Omega^2 + dz^2$. Consider a surface $R = R(\rho)$, $z = z(\rho)$. Show that $R(\rho)$ and $z(\rho)$ can be chosen so that this surface has the same metric as Σ (use isotropic coordinates on Σ). Give a sketch of this embedding of Σ into flat space (suppressing the coordinate θ). Could someone in region I travel across the bridge to visit region IV?

    (b) Show that the geometry of a surface of constant r in region II (or III) of the Kruskal manifold is the same as that of an infinite cylinder embedded in R^4.

11. ($\star\star$) Consider the 4−dimensional action

$$I = \frac{1}{16\pi} \int \mathrm{d}x^4 \sqrt{-g} \left(R - 2\nabla^a \phi \nabla_a \phi - e^{-2\phi} F^{ab} F_{ab} \right), \quad (\dagger)$$

where R is the Ricci scalar, ϕ is a scalar field and F_{ab} is the usual electromagnetic strength tensor, given in terms of A_a as $F_{ab} = \partial_a A_b - \partial_b A_a$.

(a) What are the equations of motion derived from (†)?

(b) Show that the equations of motion derived in (a) admit spherically symmetric black hole solutions of the form

$$\mathrm{d}s^2 = -V(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + R(r)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\,, \quad A = Q\cos\theta \mathrm{d}\varphi\,, \quad \phi(r) = \phi_0 - \frac{1}{2}\log\left(1 - \frac{Q^2 e^{-2\phi_0}}{rM}\right).$$

where

$$R^2(r) = r\left(r - \frac{Q^2}{M}e^{-2\phi_0}\right) \quad \text{and} \quad V(r) = 1 - \frac{2M}{r}\,.$$

12. ($\star\star$) Consider the following line element

$$\mathrm{d}s^2 = \ell^2\left(-r^{2z}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2\mathrm{d}x^i\mathrm{d}x^j\delta_{ij}\right),$$

where $i, j \in \{2, \ldots, d-1\}$. Show that, in these coordinates, all the components of the Riemann tensor are finite, and therefore that all curvature invariants constructed from the Riemann tensor are also finite. Prove that, nevertheless, if $z \neq 1$ there is a curvature singularity at $r = 0$ due to diverging tidal forces.