Mathematical Tripos Part III Lent Term 2009
Black Holes: Example Sheet 1 H.S. Reall

1. Let X be a p-form. Show that (a) dX is a $(p+1)$-form; (b) $d(dX) = 0$; (c) in the definition of dX, the partial derivative ∂_a can be replaced by the covariant derivative ∇_a without changing the result.

2. Let X be a vector field and Y be a p-form. Show that

$$\mathcal{L}_X Y = i_X dY + d\left(i_X Y\right),$$

where, for a q-form Z, $i_X Z$ is the $(q-1)$-form resulting from contracting X with the first index of Z.

3. The geodesic equation determining the trajectory of a free massive particle moving from A to B is obtained by extremizing the action

$$S = -m \int_A^B d\tau \equiv -m \int_{\lambda_A}^{\lambda_B} \sqrt{-g_{ab}(x)\dot{x}^a \dot{x}^b} d\lambda,$$

where τ is proper time, and the extremization is over timelike paths $x^a = x^a(\lambda)$ where λ takes fixed values λ_A, λ_B at A and B respectively. A dot denotes a derivative with respect to λ.

(a) Let k^a be a vector field. Consider moving infinitesimal parameter distance ϵ along the flow lines of k, i.e., the transformation $x^a \to x^a + \epsilon k^a(x)$. Show that the change in S under this transformation is

$$\delta S = \frac{m}{2} \int_{\lambda_A}^{\lambda_B} \frac{\epsilon \dot{x}^a \dot{x}^b \mathcal{L}_k g_{ab}}{\sqrt{-g_{cd}\dot{x}^c \dot{x}^d}} \, d\lambda.$$

Hence this transformation is symmetry of the action if k is a Killing vector field.

(b) Use the Noether procedure to show that the conserved quantity corresponding to this symmetry is $g_{ab}U^a k^b$, where $U^a \equiv dx^a/d\tau$ is the 4-velocity of the particle.

4. A *conformal Killing vector field* is a vector field k such that

$$\mathcal{L}_k g_{ab} = \phi g_{ab}$$

for some function ϕ. (i) Use the geodesic equation to show that the existence of a conformal Killing vector field implies the existence of a conserved quantity along *null* geodesics. (ii) Show that if k is a conformal Killing vector field for the metric g then it is also a conformal Killing vector field for the metric $\Omega^2 g$ where Ω is any positive function.

5. Consider a null geodesic incident from infinity on a Schwarzschild black hole. Let $\mathcal{E}$ and h denote the usual conserved quantities associated with the timelike Killing field and the angular Killing field $\partial/\partial\phi$.

(a) Show that the maximum value for the *impact parameter* $b \equiv |h/\mathcal{E}|$ for which the geodesic falls into the black hole is $b_{\max} = 3\sqrt{3}M$.

(b) Determine the geometrical interpretation of the impact parameter by considering ϕ as a function of r at large r.

(c) Hence show that the geodesics that fall into the hole are the same as those that would be absorbed by a perfectly absorbing disc of radius b_{max} in Minkowski space-time. (Therefore the *photon absorption cross-section* of the black hole is $\pi b_{\text{max}}^2 = 27\pi M^2$.)

6. Consider a geodesic of the Schwarzschild solution with non-vanishing h. Let $u = 2M/r$.

   (a) Show that
$$\frac{d^2u}{d\phi^2} + u - \frac{3}{2}u^2 = \frac{2M^2\sigma}{h^2}.$$

   (b) Show that a solution to this equation is
$$u = \frac{1}{3} + \frac{4\omega^2}{3} - 4\omega^2 \text{sech}^2(\omega\phi),$$
   where $\omega = (1/2)(1-12M^2\sigma/h^2)^{1/4}$. Interpret these orbits in terms of the effective potential $V(r)$. Comment on the special cases of $\omega^2 = 0, 1/8, 1/4$.

7. A photon is emitted outwards from a point P outside a Schwarzschild black hole with radial coordinate $r = r_0$ between $2M$ and $3M$. Show that if the photon is to reach infinity, then the angle its initial direction makes with the radial direction as determined by a stationary observer at P cannot exceed
$$\arcsin\sqrt{\frac{27M^2}{r_0^2}\left(1 - \frac{2M}{r_0}\right)}.$$

8. Consider spherically symmetric collapse of a star made of pressureless dust using ingoing Eddington-Finkelstein coordinates. Assume that the star starts from rest at radius $r = r_0 > 2M$. (a) Show that the radial equation is the same as in Schwarzschild coordinates. (b) Determine the proper time it takes for the star's surface to reach (i) $r = 2M$, (ii) $r = 0$. (c) Show that the star reaches $r = 0$ in finite EF coordinate time v.

9. Using Schwarzschild coordinates, show that *every* timelike curve in region II of the Kruskal manifold intersects the singularity at $r = 0$ within a proper time no greater than πM. For what curves is this bound attained?

10. The generalization of the Schwarzschild solution to d dimensional spacetime is the Schwarzschild–Tangherlini metric
$$ds^2 = -\left[1 - \left(\frac{a}{r}\right)^{d-3}\right] dt^2 + \left[1 - \left(\frac{a}{r}\right)^{d-3}\right]^{-1} dr^2 + r^2 d\Omega_{d-2}^2,$$
    where $a > 0$ is constant and $d\Omega_{d-2}^2$ is the standard round metric on S^{d-2}, defined inductively by $d\Omega_1^2 = d\phi^2$ and $d\Omega_{i+1}^2 = d\theta_i^2 + \sin^2\theta_i d\Omega_i^2$ for $i \geq 1$, with $0 \leq \phi \leq 2\pi$ and $0 \leq \theta_i \leq \pi$. Show that there are no stable bounded massive particle orbits for $d \geq 5$. Why might you have expected this result?