Mathematical Tripos Part III Lent Term 2009
Black Holes: Examples Sheet 2 H.S. Reall

1. A general static, spherically symmetric metric in d dimensions, can be written

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_{d-2}^2,$$

where $d\Omega_{d-2}^2$ is the metric on a unit $(d-2)$-sphere. Assume that $A(r)$ and $B(r)$ are analytic functions of r such that both have a simple zero at $r = r_+ > 0$ and are positive for $r > r_+$.

(a) Show that radial null geodesics are given by $t \pm r^* =$ constant, where

$$r^* \equiv \int_{r_0}^{r} \frac{dx}{\sqrt{A(x)B(x)}},$$

with $r_0 > r_+$ an arbitrary constant. Show that $r^* \to -\infty$ as $r \to r_+$.

(b) Obtain the metric in ingoing Eddington-Finkelstein coordinates. Explain why this metric can be analytically continued through $r = r_+$.

(c) The timelike Killing field is $k \equiv \partial/\partial t$ in static coordinates. Show that $k = \partial/\partial v$ in EF coordinates, and that $r = r_+$ is a Killing horizon of k. What is the surface gravity?

(d) Check that your formula gives the correct answer for the Schwarzschild solution, and use it to determine the surface gravity of the Schwarzschild-Tangherlini solution from Examples sheet 1.

(e) What happens if A and B both have a zero of order $p > 1$ at $r = r_+$ instead of a simple zero?

2. Consider a particle with 4-velocity U in a stationary, asymptotically flat, space-time with timelike Killing vector field k. We have seen how $\mathcal{E} = -k \cdot U$ has the interpretation of "energy per unit mass measured at infinity" if the particle moves on a geodesic. For non-geodesic motion, this equation is used to *define* "energy per unit mass measured at infinity."

(a) Consider a unit mass particle P following an orbit of k at radius $r = r_P > 2M$ in the Schwarzschild spacetime. Assume that the force making this particle accelerate comes from a radial massless inelastic string, whose other end is held by an observer Q at infinity. If Q pulls the string through proper distance δS then what is the change δr_P in r_P ?

(b) What is the change $\delta\mathcal{E}$ in the energy of P measured by Q? This must equal the work $F\delta S$ done by Q where F is the force that the string exerts on Q, i.e., the tension at Q. Calculate F. Show that $F \to \kappa$ as $r_P \to 2M$. What is the force measured by P as $r_P \to 2M$?

3. Consider a spacetime with metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

where $f(r)$ is a smooth positive function of $r \geq 0$, with the following properties: (i) $f(r)$ is an analytic function of r^2 near $r = 0$ with $f(0) = 1$; (ii) $f(r) = 1 - 2M/r$ for $r > R$ where $R > 3M$ is a constant; (iii) $f(r)/r^2$ is monotonically decreasing for $r \leq R$.

(a) Show that (i) implies that this space-time is regular near $r = 0$ by converting to Cartesian coordinates.

(b) Determine the effective potential governing null geodesics. Hence show that all such geodesics start and end at $r = \infty$.

(c) Prove that this space-time is asymptotically simple and hence deduce that the Kruskal spacetime is asymptotically flat.

4. (a) Prove that, if a vector field ξ preserves the Maxwell field (i.e. $\mathcal{L}_\xi F = 0$) then locally there exists a scalar potential Φ such that $i_\xi F = d\Phi$. (*Hint:* Q2 of examples sheet 1.)

(b) The equation of motion of a particle of charge q and 4-velocity U^a is $U^b \nabla_b U^a = (q/m) F^a{}_b U^b$. Let ξ be a Killing vector field that preserves the Maxwell field. Show that $\xi \cdot U - (q/m)\Phi$ is conserved along the particle's worldline.

(c) Deduce that, for a particle of mass m moving in the equatorial plane ($\theta = \pi/2$) of a Reissner-Nordstrom black hole (with $Q > 0$, $P = 0$), the quantities $\mathcal{E} = (\Delta/r^2) dt/d\tau + qQ/(mr)$, and $h = r^2 d\phi/d\tau$ are constant. Hence show that the radial motion is determined by the equation

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{\Delta(r)}{r^2}\left(1 + \frac{h^2}{r^2}\right) = \left(\mathcal{E} - \frac{qQ}{mr}\right)^2 .$$

(d) What is the physical interpretation of the case $q/m = Q/M = 1$, $\mathcal{E} = 1$, $h = 0$?

(e) *The Penrose process.* A particle P_1 falls from $r = \infty$ towards the black hole. Just before it crosses the event horizon, it decays into two other particles P_2 and P_3 where P_2 has charge $q < 0$. The decay happens such that P_2 initially has $dr/d\tau \approx 0$. P_2 subsequently falls into the black hole and P_3 escapes to $r = \infty$. Let $E_i \equiv m_i \mathcal{E}_i$ denote the energy of P_i (which has mass m_i). Show that $E_1 > 0$ and $E_2 < 0$. Hence, by energy conservation, $E_3 > E_1$, i.e., the particle returning to infinity has more energy than the initial particle! This is consistent because P_2 has carried negative energy into the black hole. Hence energy (and charge) are extracted from the black hole in this process.

5. (a) Let Σ be an Einstein-Rosen bridge, i.e., a surface of constant t, in the Kruskal spacetime. The geometry of Σ can be visualized by embedding it into four-dimensional Euclidean space R^4. In cylindrical polar coordinates, the metric of R^4 is $ds^2 = dR^2 + R^2 d\Omega^2 + dz^2$. Consider a surface $R = R(\rho)$, $z = z(\rho)$. Show that $R(\rho)$ and $z(\rho)$ can be chosen so that this surface has the same metric as Σ (use isotropic coordinates on Σ). Give a sketch of this embedding of Σ into flat space (suppressing the coordinate θ). Could someone in region I travel across the bridge to visit region IV?

   (b) Show that the geometry of a surface of constant r in region II (or III) of the Kruskal manifold is the same as that of an infinite cylinder embedded in R^4.

6. By replacing the time coordinate t by one of the radial null coordinates

$$u = t + \frac{M}{\lambda}, \qquad v = t - \frac{M}{\lambda}$$

show that the singularity at $\lambda = 0$ of the Robinson-Bertotti (RB) metric

$$ds^2 = -\lambda^2 dt^2 + M^2 \left(\frac{d\lambda}{\lambda}\right)^2 + M^2 d\Omega^2$$

is merely a coordinate singularity. Show also that $\lambda = 0$ is a degenerate (i.e. $\kappa = 0$) Killing horizon with respect to $\partial/\partial t$. By introducing the new coordinates (U, V), defined by

$$u = \tan\left(\frac{U}{2}\right) \qquad v = -\cot\left(\frac{V}{2}\right)$$

obtain the maximal analytic extension of the RB metric and deduce its Penrose diagram (more precisely: deduce the Penrose diagram of the adS_2 part of the RB metric).

7. Let $\mathcal{E}$ and h be the energy and and angular momentum per unit mass of a zero charge particle in free fall within the equatorial plane, i.e on a timelike ($\sigma = 1$) or null ($\sigma = 0$) geodesic with $\theta = \pi/2$, of a Kerr-Newman black hole. Show that the particle's Boyer-Lindquist radial coordinate r satisfies

$$\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}^2 ,$$

where λ is an affine parameter, and the effective potential V is given by

$$V(r) = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)\left(\sigma + \frac{h^2}{r^2}\right) + \frac{2a\mathcal{E}h}{r^3}\left(2M - \frac{e^2}{r}\right) + \frac{a^2}{r^2}\left[\sigma - \mathcal{E}^2\left(1 + \frac{2M}{r} - \frac{e^2}{r^2}\right)\right] .$$