1. Obtain the Kerr-Newman solution in Kerr coordinates. When is there a regular event horizon? What is the surface gravity? Show that the area of the event horizon of a Kerr-Newman black hole is

$$A = 8\pi \left[M^2 - \frac{e^2}{2} + \sqrt{M^4 - e^2 M^2 - J^2} \right] .$$

2. Let E denote the maximum energy that can be extracted from a Kerr black hole in the Penrose process. The *efficiency* of this process is $\eta \equiv E/M$ where M is the initial mass of the black hole. What is the largest possible value of η?

3. (a) Calculate the ADM mass of the Reissner-Nordstrom solution.

   (b) Calculate the electric and magnetic charges, the Komar mass, and the Komar angular momentum of the Kerr-Newman solution.

4. In the Kerr geometry, consider two spacelike surfaces Σ, Σ' which both extend from i^0 to $\mathcal{H}^+$ with Σ' lying entirely to the future of Σ. Let H and H' denote the intersections of Σ and Σ' with $\mathcal{H}^+$. Let $\mathcal{N}$ denote the portion of $\mathcal{H}^+$ from H to H'. Let $J_a = -T_{ab}k^b$ be the conserved energy-momentum 4-vector.

   (a) Show that

$$E(\Sigma') - E(\Sigma) = \int_{\mathcal{N}} \star J, \qquad (1)$$

   where $E(\Sigma) \equiv -\int_\Sigma \star J$ is the total energy of matter fields on Σ, and similarly for $E(\Sigma')$. What is the physical interpretation of this formula?

   (b) Explain why the orientation of $\mathcal{N}$ used in this formula is given by $dv \wedge d\theta \wedge d\chi$ in Kerr coordinates (the orientation of spacetime is given by $dv \wedge dr \wedge d\theta \wedge d\chi$).

   (c) Show that

$$(\star J)_{v\theta\chi} = (r_+^2 + a^2) \sin\theta \xi^a J_a.$$

   (d) Assume that matter obeys the dominant energy condition. Explain why $E(\Sigma') \leq E(\Sigma)$ for a Schwarzschild black hole (i.e. $a = 0$) but why this is not necessarily true for a Kerr black hole.

   (e) Now take the matter to be a massless real scalar field, with energy-momentum tensor $T_{ab} = \partial_a\Phi\partial_b\Phi - (1/2)g_{ab}(\partial\Phi)^2$. Consider a mode of this field with frequency ω and azimuthal quantum number ν, i.e., $\Phi = \mathrm{Re}[\Phi_0(r,\theta)\exp(-i\omega v + i\nu\chi)]$. Show that the RHS of equation (1) is positive for $0 < \omega < \nu\Omega_H$. (Note that $\xi \cdot k = 0$ on $\mathcal{H}^+$ because $\mathcal{H}^+$ must be invariant under an isometry hence any Killing field must be tangent to $\mathcal{H}^+$.)

   This example shows that energy can be extracted from a black hole by scattering waves off it. This is called *superradiant scattering.*

5. Four-dimensional anti-de Sitter space-time (adS$_4$) with radius of curvature ℓ has metric

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2,$$

   where $U(r) = 1 + r^2/\ell^2$.

   (a) Show that, along a null geodesic with affine parameter λ, $r \to \infty$ and $t \to$ constant as $\lambda \to \pm\infty$.

   (b) Construct the conformal compactification of adS$_4$ by defining a new radial coordinate χ by $r = \ell \tan\chi$.

   (c) Show that adS$_4$ is asymptotically simple. Is is globally hyperbolic?

   (d) The Schwarzschild-adS metric, given by setting $U(r) = 1 - 2M/r + r^2/\ell^2$ above, is the unique spherically symmetric solution of the Einstein equation with a negative cosmological constant. Show that

Schwarzschild-adS is weakly asymptotically simple. (*Hint.* Use the same kind of argument as was used on Examples Sheet 2 to show that Kruskal is weakly asymptotically simple.)

(e) Let $M(r)$ and $\bar{M}(r)$ denote the Komar mass associated with a sphere of constant r and t in the Schwarzschild-adS and adS metrics respectively. Show that M and $\bar{M}$ both diverge as $r \to \infty$ but $M(r) - \bar{M}(r)$ has a finite limit. (Note that r is invariantly defined in a spherically symmetric spacetime, so this prescription for calculating the mass is coordinate-independent.)

(f) Consider Schwarzschild-adS with $M > 0$. Show that there is a Killing horizon of $\partial/\partial t$ at $r = r_+ > 0$ where $U(r_+) = 0$. Plot the surface gravity κ as a function of M. How does this differ from the corresponding plot for a Schwarzschild black hole?

6. A perfect fluid has stress tensor $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where ρ is the energy density, p the pressure, and u^a the 4-velocity of the fluid. Show that

(a) the dominant energy condition is obeyed if, and only if, $\rho \geq |p|$;

(b) the weak energy condition is obeyed if, and only if, $\rho \geq 0$ and $\rho + p \geq 0$;

(c) the null energy condition is obeyed if, and only if, $\rho + p \geq 0$;

(d) the strong energy condition is obeyed if, and only if, $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

A cosmological constant has $p = -\rho$. Which energy conditions does it violate? (Consider both signs for ρ.)

7. This example works through the proof of the zeroth law of black hole mechanics. Let $\mathcal{N}$ be a Killing horizon of a Killing vector field ξ with surface gravity κ.

(a) If we know that $A = 0$ on $\mathcal{N}$ for some tensor $A_{a_1 \ldots a_p}$ then $A \cdot B \equiv A_{a_1 \ldots a_p} B^{a_1 \cdots a_p} = 0$ on $\mathcal{N}$ for any tensor $B^{a_1 \cdots a_p}$. Hence $\mathcal{N}$ is a surface of constant $A \cdot B$, so $d(A \cdot B)$ is normal to $\mathcal{N}$ hence $\xi \wedge d(A \cdot B) = 0$ on $\mathcal{N}$. (i) Show that this implies $\xi_{[a} \nabla_{b]} A_{c_1 \ldots c_p} = 0$ on $\mathcal{N}$. (ii) Taking $A_a = \xi^b \nabla_b \xi_a - \kappa \xi_a$, use this (and the formula $\nabla_a \nabla_b \xi_c = R^d{}_{abc} \xi_d$) to show

$$\xi_a \xi_{[d} \nabla_{c]} \kappa + \kappa \xi_{[d} \nabla_{c]} \xi_a = \left(\xi_{[d} \nabla_{c]} \xi^b \right) \nabla_b \xi_a + \xi^b \xi_{[d} R^e{}_{c]ba} \xi_e \qquad \text{on } \mathcal{N}. \tag{2}$$

(b) Using Frobenius' theorem, show that

$$\xi_c \nabla_a \xi_b = -2 \xi_{[a} \nabla_{b]} \xi_c \qquad \text{on } \mathcal{N}. \tag{3}$$

Hence show that $\left(\xi_{[d} \nabla_{c]} \xi^b \right) \nabla_b \xi_a = \kappa \xi_{[d} \nabla_{c]} \xi_a$ on $\mathcal{N}$, so equation (2) reduces to

$$\xi_a \xi_{[d} \nabla_{c]} \kappa = \xi^b \xi_{[d} R^e{}_{c]ba} \xi_e \qquad \text{on } \mathcal{N}. \tag{4}$$

(c) Set $A_{abc} = \xi_c \nabla_a \xi_b + 2 \xi_{[a} \nabla_{b]} \xi_c$ and use the result of (a)(i) and equation (3) to show that

$$\xi_c \xi_{[d} \nabla_{e]} \nabla_a \xi_b = -2 \left(\xi_{[d} \nabla_{e]} \nabla_{[b} \xi_{|c|} \right) \xi_{a]} \qquad \text{on } \mathcal{N}.$$

and hence

$$\xi_c \xi_{[d} R^f{}_{e]ab} \xi_f = 2 \xi_{[d} R^f{}_{e]c[b} \xi_{a]} \xi_f \qquad \text{on } \mathcal{N}.$$

(d) Contract this equation on the indices c and e, show that the LHS vanishes and the resulting equation can be written

$$-\xi_{[a} R_{b]}{}^f \xi_f \xi_d = \xi_{[a} R^f{}_{b]cd} \xi^c \xi_f \qquad \text{on } \mathcal{N}.$$

Hence show that equation (4) reduces to

$$\xi_{[d} \nabla_{c]} \kappa = -\xi_{[d} R_{c]}{}^f \xi_f \qquad \text{on } \mathcal{N}.$$

As (will be) explained in lectures, if the Einstein equation and the dominant energy condition are satisfied, then the RHS vanishes and hence κ is constant on the horizon.