1. Obtain the Kerr-Newman solution in Kerr coordinates. When is there a regular event horizon? What is the surface gravity? Show that the area of the event horizon of a Kerr-Newman black hole is

\[ A = 8\pi \left[ M^2 - \frac{\epsilon^2}{2} + \sqrt{M^4 - \epsilon^2M^2 - J^2} \right]. \]

2. Let \( E \) denote the maximum energy that can be extracted from a Kerr black hole in the Penrose process. The efficiency of this process is \( \eta \equiv E/M \) where \( M \) is the initial mass of the black hole. What is the largest possible value of \( \eta \)?

3. (a) Calculate the ADM mass of the Reissner-Nordstrom solution.
(b) Calculate the electric and magnetic charges, the Komar mass, and the Komar angular momentum of the Kerr-Newman solution.

4. In the Kerr geometry, consider two spacelike surfaces \( \Sigma, \Sigma' \) which both extend from \( i^0 \) to \( H^+ \) with \( \Sigma' \) lying entirely to the future of \( \Sigma \). Let \( H \) and \( H' \) denote the intersections of \( \Sigma \) and \( \Sigma' \) with \( H^+ \). Let \( N \) denote the portion of \( H^+ \) from \( H \) to \( H' \). Let \( J_a = -T_{ab}k^b \) be the conserved energy-momentum 4-vector.
(a) Show that

\[ E(\Sigma') - E(\Sigma) = \int_N \star J, \quad (1) \]

where \( E(\Sigma) \equiv -\int_\Sigma \star J \) is the total energy of matter fields on \( \Sigma \), and similarly for \( E(\Sigma') \). What is the physical interpretation of this formula?
(b) Explain why the orientation of \( N \) used in this formula is given by \( dv \wedge d\theta \wedge d\chi \) in Kerr coordinates (the orientation of spacetime is given by \( dv \wedge dr \wedge d\theta \wedge d\chi \)).
(c) Show that

\[ (\star J)_{\phi\theta\chi} = (r_+^2 + a^2) \sin \xi \xi^\alpha J_\alpha. \]
(d) Assume that matter obeys the dominant energy condition. Explain why \( E(\Sigma') \leq E(\Sigma) \) for a Schwarzschild black hole (i.e. \( a = 0 \)) but why this is not necessarily true for a Kerr black hole.
(e) Now take the matter to be a massless real scalar field, with energy-momentum tensor \( T_{ab} = \partial_a \Phi \partial_b \Phi - (1/2)g_{ab} (\partial \Phi)^2 \). Consider a mode of this field with frequency \( \omega \) and azimuthal quantum number \( \nu \), i.e., \( \Phi = \text{Re} \{ \Phi_0(r, \theta) \exp(-i\omega t + i\nu \varphi) \} \). Show that the RHS of equation (1) is positive for \( 0 < \omega < \nu\Omega_H \) (Note that \( \xi \cdot k = 0 \) on \( H^+ \) because \( H^+ \) must be invariant under an isometry hence any Killing field must be tangent to \( H^+ \).)
This example shows that energy can be extracted from a black hole by scattering waves off it. This is called superradiant scattering.

5. Four-dimensional anti-de Sitter space-time (adS\(_4\)) with radius of curvature \( \ell \) has metric

\[ ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2d\Omega^2, \]
where \( U(r) = 1 + r^2/\ell^2 \).
(a) Show that, along a null geodesic with affine parameter \( \lambda \), \( r \to \infty \) and \( t \to \text{constant} \) as \( \lambda \to \pm \infty \).
(b) Construct the conformal compactification of adS\(_4\) by defining a new radial coordinate \( \chi \) by \( r = \ell \tan \chi \).
(c) Show that adS\(_4\) is asymptotically simple. Is it globally hyperbolic?
(d) The Schwarzschild-adS metric, given by setting \( U(r) = 1 - 2M/r + r^2/\ell^2 \) above, is the unique spherically symmetric solution of the Einstein equation with a negative cosmological constant. Show that
Schwarzschild-adS is weakly asymptotically simple. (Hint. Use the same kind of argument as was used on Examples Sheet 2 to show that Kruskal is weakly asymptotically simple.)

(e) Let $M(r)$ and $\bar{M}(r)$ denote the Komar mass associated with a sphere of constant $r$ and $t$ in the Schwarzschild-adS and adS metrics respectively. Show that $M$ and $\bar{M}$ both diverge as $r \to \infty$ but $M(r) - \bar{M}(r)$ has a finite limit. (Note that $r$ is invariantly defined in a spherically symmetric spacetime, so this prescription for calculating the mass is coordinate-independent.)

(f) Consider Schwarzschild-adS with $M > 0$. Show that there is a Killing horizon of $\partial / \partial t$ at $r = r_+ > 0$ where $U(r_+) = 0$. Plot the surface gravity $\kappa$ as a function of $M$. How does this differ from the corresponding plot for a Schwarzschild black hole?

6. A perfect fluid has stress tensor $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where $\rho$ is the energy density, $p$ the pressure, and $u^a$ the 4-velocity of the fluid. Show that:
   (a) the dominant energy condition is obeyed if, and only if, $\rho \geq |p|$;
   (b) the weak energy condition is obeyed if, and only if, $\rho \geq 0$ and $\rho + p \geq 0$;
   (c) the null energy condition is obeyed if, and only if, $\rho + p \geq 0$;
   (d) the strong energy condition is obeyed if, and only if, $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

A cosmological constant has $p = -\rho$. Which energy conditions does it violate? (Consider both signs for $\rho$.)

7. This example works through the proof of the zeroth law of black hole mechanics. Let $\mathcal{N}$ be a Killing horizon of a Killing vector field $\xi$ with surface gravity $\kappa$.
   (a) If we know that $A = 0$ on $\mathcal{N}$ for some tensor $A_{a_1 \ldots a_p}$, then $A \cdot B = A_{a_1 \ldots a_p} B^{a_1 \ldots a_p} = 0$ on $\mathcal{N}$ for any tensor $B^{a_1 \ldots a_p}$. Hence $\mathcal{N}$ is a surface of constant $A \cdot B$, so $d(A \cdot B)$ is normal to $\mathcal{N}$ hence $\xi \wedge d(A \cdot B) = 0$ on $\mathcal{N}$. (i) Show that this implies $\xi_{[a} \nabla_{b]} A_{c_1 \ldots c_p} = 0$ on $\mathcal{N}$. (ii) Taking $A_a = \xi^b \nabla_b \xi_a - \kappa \xi_a$, use this (and the formula $\nabla_a \nabla_b \xi_c = R^{d}_{\ abc} \xi_d$) to show
   \[ \xi_a \xi_{[d} \nabla_{c]} \kappa + \kappa \xi_{[d} \nabla_{c]} \xi_a = (\xi_{[d} \nabla_{c]} \xi^{b}) \nabla_{b} \xi_a + \xi^{b} \xi_{[d} R^{e}_{ c]b} \xi_e \quad \text{on } \mathcal{N}. \] (2)
   (b) Using Frobenius’ theorem, show that
   \[ \xi_c \nabla_a \xi_b = -2 \xi_{[a} \nabla_{b]} \xi_c \quad \text{on } \mathcal{N}. \] (3)

Hence show that $\xi_{[d} \nabla_{c]} \xi^{b}) \nabla_{b} \xi_a = \kappa \xi_{[d} \nabla_{c]} \xi_a$ on $\mathcal{N}$, so equation (2) reduces to
   \[ \xi_a \xi_{[d} \nabla_{c]} \kappa = \xi^{b} \xi_{[d} R^{e}_{ c]b} \xi_e \quad \text{on } \mathcal{N}. \] (4)

(c) Set $A_{abc} = \xi_c \nabla_a \xi_b + 2 \xi_{[a} \nabla_{b]} \xi_c$ and use the result of (a)(i) and equation (3) to show that
   \[ \xi_c \xi_{[d} \nabla_{c]} \nabla_a \xi_b = -2 \left( \xi_{[d} \nabla_{c]} \nabla_{[b} \xi_{c]} \right) \xi_a \quad \text{on } \mathcal{N}. \]

and hence
   \[ \xi_c \xi_{[d} R^{f}_{c]a} \xi_f = 2 \xi_{[d} R^{f}_{c]a} \xi_{c]} \xi_f \quad \text{on } \mathcal{N}. \]

(d) Contract this equation on the indices $c$ and $e$, show that the LHS vanishes and the resulting equation can be written
   \[ -\xi_{[a} R^{f}_{b]} \xi_f \xi_d = \xi_{[a} R^{f}_{b]} \xi_{c]} \xi_f \quad \text{on } \mathcal{N}. \]

Hence show that equation (4) reduces to
   \[ \xi_{[d} \nabla_{c]} \kappa = -\xi_{[d} R^{f}_{c]} \xi_f \quad \text{on } \mathcal{N}. \]

As (will be) explained in lectures, if the Einstein equation and the dominant energy condition are satisfied, then the RHS vanishes and hence $\kappa$ is constant on the horizon.