Mathematical Tripos Part III Lent 2009
Black Holes: Examples Sheet 4 H.S. Reall

1. (a) Consider two well-separated, approximately Schwarzschild black holes at rest. Let their masses be M_1 and M_2. Assume that they coalesce to form a Schwarzschild black hole of mass M_3, with gravitational waves carrying off energy E in the process. Use conservation of energy and the second law of black hole mechanics to prove that the efficiency of this process, defined as $\eta = E/(M_1 + M_2)$ obeys $\eta \leq 1 - 1/\sqrt{2}$. (The radiated energy could be used to do work, hence the second law of black hole mechanics restricts the useful energy that can be extracted from black holes, cf the second law of thermodynamics.)

   (b) Consider an initial Schwarzschild black hole of mass M that evolves to form two well-separated approximately Schwarzschild black holes at rest. Use conservation of energy and the second law of black hole mechanics to show that this cannot happen. (This is a special case of a completely general result that black holes cannot bifurcate.)

2. In a stationary, axisymmetric, asymptotically flat, black hole spacetime, let Σ denote an asymptotically flat spacelike hypersurface that intersects $\mathcal{H}^+$ in a 2-sphere H. Let $\xi = k + \Omega_H m$ be the Killing field normal to the horizon. By considering the expression (for appropriate choices of orientation)

$$\int_{S^2_\infty} \star d\xi - \int_H \star d\xi = \int_\Sigma d \star d\xi \, ,$$

   derive the *Smarr relation*

$$M = -\int_\Sigma \star J' + 2\Omega_H J + \frac{\kappa A}{4\pi} \, ,$$

   where $J'_a \equiv -2\left[T_{ab} - (1/2)T g_{ab}\right]\xi^b$.

3. Let (M, g, F) be a stationary, axisymmetric, asymptotically flat, black hole solution of the Einstein-Maxwell equations. Assume that it is possible to choose a gauge so that

$$\mathcal{L}_k A = \mathcal{L}_m A = 0 \ ,$$

   The *co-rotating electric potential* is defined by

$$\Phi = -\xi^\mu A_\mu \ .$$

   Use Einstein equation, and the fact that $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ on a Killing horizon, to show that Φ is constant on the horizon. In particular, show that, for a choice of gauge for which $\Phi = 0$ at infinity, the value of Φ on the horizon is

$$\Phi_H = \frac{Q r_+}{r_+^2 + a^2}$$

   for an electrically charged rotating black hole, where $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$.

4. Let $(\mathcal{M}, g, F)$ be an asymptotically flat, stationary, axisymmetric, black hole solution of the Einstein-Maxwell equations and let Σ be a spacelike hypersurface with one boundary at spatial infinity and an internal boundary, H, at the event horizon of a black hole of charge Q. Show that the Smarr relation can be written

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q \ .$$

   [Hint: $\mathcal{L}_\xi(F^{\mu\nu}A_\nu) = 0$]

5. Show that, under a Bogoliubov transformation, the operators a', $a'^\dagger$ obey the same commutation relations as a and $a^\dagger$.

6. Show that, for a scalar Φ,

$$\nabla^2\Phi = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\partial^a\Phi\right) .$$

7. Consider a 2d cosmological spacetime with metric $ds^2 = a(\eta)^2\left(-d\eta^2 + dx^2\right)$. Assume that $a(\eta)$ takes a constant positive value a_- or a_+ for $\eta < 0$ and $\eta > 0$ respectively. (The metric is discontinuous, but we can regard it as an approximation to a metric in which a varies smoothly from a_- to a_+ in a very short time. Birrell and Davies (section 3.4) discuss this smooth case.) Let M_-, M_+ denote the regions $\eta < 0$ and $\eta > 0$ respectively. Obtain the normalized positive frequency modes of the massive Klein-Gordon equation in $M_\pm$. Assume that the scalar field is in the vacuum state in M_-. What is the expected number of particles with wavenumber k in M_+?

8. A scalar field Φ in the Kruskal spacetime satisfies the Klein-Gordon equation $\nabla^2\Phi - m^2\Phi = 0$. Assume that, in static Schwarzshild coordinates, Φ takes the form $\Phi = (1/r)R_\ell(r)e^{-i\omega t}Y_{\ell\tilde{m}}(\theta,\phi)$ where $Y_{\ell\tilde{m}}$ is a spherical harmonic.

   An instability of the black hole with respect to scalar field perturbations would be indicated by the existence of a mode that is regular on $\mathcal{H}^+$ and decaying as $r \to \infty$, with $\omega = \omega_1 + i\omega_2$ and $\omega_2 > 0$ (so that the mode grows exponentionally in time). Show that (i) the operator on the LHS of the radial equation is self-adjoint for such modes; (ii) no such instability exists.

9. Use the fact that a Schwarzschild black hole radiates at the Hawking temperature $T_H = 1/(8\pi M)$ (in units for which $\hbar$, G, c, and Boltzmann's constant all equal 1) to show that the thermal equilibrium of a black hole with an infinite reservoir of radiation at temperature T_H is unstable.

   A finite reservoir of radiation of volume V at temperature T has an energy, E_{res} and entropy, S_{res} given by

$$E_{res} = \sigma V T^4 \qquad S_{res} = \frac{4}{3}\sigma V T^3$$

   where σ is a constant. A Schwarzschild black hole of mass M is placed in the reservoir. Assuming that the black hole has entropy

$$S_{BH} = 4\pi M^2 \ ,$$

   show that the total entropy $S = S_{BH} + S_{res}$ is extremized for fixed total energy $E = M + E_{res}$, when $T = T_H$, Show that the extremum is a maximum if and only if $V < V_c$, where the critical value of V is

$$V_c = \frac{2^{20}\pi^4 E^5}{5^5\sigma}$$

   What happens as V passes from $V < V_c$ to $V > V_c$, or vice-versa?

10. The specific heat of a charged black hole of mass M, at fixed charge Q, is

$$C \equiv T_H \frac{\partial S_{BH}}{\partial T_H}\bigg|_Q \ ,$$

   where T_H is its Hawking temperature and S_{BH} its entropy. Assuming that the entropy of a black hole is given by $S_{BH} = \frac{1}{4}A$, where A is the area of the event horizon, show that the specific heat of a Reissner-Nordstrom black hole is

$$C = \frac{2S_{BH}\sqrt{M^2 - Q^2}}{(M - 2\sqrt{M^2 - Q^2})} \ .$$

   Hence show that C^{-1} changes sign when M passes through $2|Q|/\sqrt{3}$.

   Repeat the previous question for a Reissner-Nordstrom black hole. Specifically, show that the critical reservoir volume, V_c, is infinite for $|Q| \le M \le 2|Q|/\sqrt{3}$. Why is this result to be expected from your previous result for C?