**Mathematical Tripos Part III** Lent Term 2018
**Black Holes: Examples Sheet 2** J.E. Santos

1. A general static, spherically symmetric metric can be written

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2,$$

where $d\Omega^2$ is the metric on a unit 2-sphere. Assume that $A(r)$ and $B(r)$ are analytic functions of $r$ such that both have a simple zero at $r = r_+ > 0$ and are positive for $r > r_+$.

(a) Show that radial null geodesics are given by $t \pm r^* =$ constant, where

$$r^* \equiv \int_{r_0}^{r} \frac{dx}{\sqrt{A(x)B(x)}},$$

with $r_0 > r_+$ an arbitrary constant. Show that $r^* \to -\infty$ as $r \to r_+$.

(b) Obtain the metric in ingoing Eddington-Finkelstein coordinates. Explain why this metric can be analytically continued through $r = r_+$.

2. Consider a particle with 4-velocity $U$ in a stationary, asymptotically flat, space-time with timelike Killing vector field $k$. $E = -k \cdot U$ has the interpretation of "energy per unit mass measured at infinity" if the particle moves on a geodesic. For non-geodesic motion, this equation is used to *define* "energy per unit mass measured at infinity."

(a) Consider a unit mass particle $P$ following an orbit of $k$ at radius $r = r_P > 2M$ in the Schwarzschild spacetime. Assume that the force making this particle accelerate comes from a radial massless inelastic string, whose other end is held by an observer $Q$ at infinity. If $Q$ pulls the string through proper distance $\delta S$ then what is the change $\delta r_P$ in $r_P$ ?

(b) What is the change $\delta E$ in the energy of $P$ measured by $Q$? This must equal the work $F\delta S$ done by $Q$ where $F$ is the force that the string exerts on $Q$, i.e., the tension at $Q$. Calculate $F$. Show that $F \to 1/(4M)$ as $r_P \to 2M$. What is the force measured by $P$ as $r_P \to 2M$?

3. Use isotropic coordinates to prove that a surface of constant $t$ in the Schwarzschild spacetime is an asymptotically flat end with $K_{ab} = 0$.

4. (a) Let $(M, g)$ be the 2d Einstein static Universe with metric $ds^2 = -dt^2 + d\phi^2$ where $\phi \sim \phi + 2\pi$. Let $S$ be the surface $t = 0$. Determine $D^+(S)$ and $J^+(S)$. (b) Do the same where $(M, g)$ is now the spacetime obtained by deleting the point $t = \phi = 0$ from the Einstein static Universe. (c) Do the same for the Kruskal spacetime where $S$ is the surface $t = 0$ in region I.

5. A perfect fluid has stress tensor $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where $\rho$ is the energy density, $p$ the pressure, and $u^a$ the 4-velocity of the fluid. Show that

(a) the dominant energy condition is obeyed if, and only if, $\rho \geq |p|$;

(b) the weak energy condition is obeyed if, and only if, $\rho \geq 0$ and $\rho + p \geq 0$;

(c) the null energy condition is obeyed if, and only if, $\rho + p \geq 0$;

(d) the strong energy condition is obeyed if, and only if, $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

A cosmological constant has $p = -\rho$. Which energy conditions does it violate? (Consider both signs for $\rho$.)

Copyright © 2018 University of Cambridge. Not to be quoted or reproduced without permission.

6. Consider two Lorentzian metrics on a manifold $M$ related by a conformal transformation $\bar{g} = \Omega^2 g$ where $\Omega$ is a positive function on $M$.

   (a) Show that $g$ and $\bar{g}$ have the same null geodesics.

   (b) Show that the Ricci tensor of $g$ is related to the Ricci tensor of $\bar{g}$ by

$$R_{ab} = \bar{R}_{ab} + 2\Omega^{-1}\bar{\nabla}_a\bar{\nabla}_b\Omega + \bar{g}_{ab}\bar{g}^{cd}\left(\Omega^{-1}\bar{\nabla}_c\bar{\nabla}_d\Omega - 3\Omega^{-2}\partial_c\Omega\partial_d\Omega\right)$$

   where $\bar{\nabla}$ is the Levi-Civita connection associated with $\bar{g}$.

   (c) Let $\psi$ be a solution of the equation

$$g^{ab}\nabla_a\nabla_b\psi + \xi R\psi = 0$$

   We say that the equation is *conformally covariant* if there exists a constant $p$ such that $\bar{\psi} \equiv \Omega^p\psi$ is a solution of the equation in a spacetime with metric $\bar{g} = \Omega^2 g$ whenever $\psi$ solves the equation in a spacetime with metric $g$. Determine the value of $\xi$ for which this equation is conformally covariant.

7. The Robinson-Bertotti metric is

$$ds^2 = -\lambda^2 dt^2 + M^2\left(\frac{d\lambda}{\lambda}\right)^2 + M^2 d\Omega^2$$

   This is the product $AdS_2 \times S^2$ where $AdS_2$ denotes 2d anti-de Sitter spacetime. By replacing the time coordinate $t$ by one of the radial null coordinates $u = t + M/\lambda$, $v = t - M/\lambda$ show that the singularity at $\lambda = 0$ is merely a coordinate singularity. By introducing the new coordinates $(U, V)$, defined by $u = \tan(U/2)$, $v = -\cot(V/2)$, obtain the maximal analytic extension of the RB metric and deduce its Penrose diagram (more precisely: deduce the Penrose diagram of the $AdS_2$ part of the RB metric). Is this spacetime globally hyperbolic?

8. Determine the Penrose diagram of *de Sitter spacetime* with metric

$$ds^2 = -dt^2 + H^{-2}\cosh^2(Ht)(d\chi^2 + \sin^2\chi d\Omega^2)$$

   where $H > 0$ is a constant and $0 \leq \chi \leq \pi$ ($(\chi, \theta, \phi)$ parameterize a round 3-sphere). (*Hint.* Use a coordinate transformation $t = t(\eta)$ to bring the metric to a form where it is manifestly conformal to the Einstein static Universe.)

9. Consider a vacuum spacetime that is asymptotically flat at null infinity. In lectures we introduced coordinates $(u, \Omega, \theta, \phi)$ such that $\mathcal{I}^+$ is $\Omega = 0$ and the "unphysical" metric satisfies

$$\bar{g}|_{\Omega=0} = 2dud\Omega + d\theta^2 + \sin^2 d\phi^2$$

   and, for small non-zero $\Omega$, the corrections to this are $\mathcal{O}(\Omega)$ except for the $uu$, $u\theta$ and $u\phi$ components which are $\mathcal{O}(\Omega^2)$, and the $\Omega\Omega$ component which vanishes everywhere. The physical metric is $g = \Omega^{-2}\bar{g}$. Introduce a new coordinate $r = 1/\Omega$ and determine the form of the physical metric for large $r$, keeping track of the size of the subleading corrections. You should find that the $uu$ component is $\mathcal{O}(1)$. Show that this component can be set to $-1 + \mathcal{O}(1/r)$ by a shift $r \to r + f(u, \theta, \phi)$. Finally define "asymptotically inertial" coordinates $(t, x, y, z)$ by $t = u + r$ and $(x, y, z)$ related to $(r, \theta, \phi)$ as for spherical polars. Show that the spacetime metric becomes $-dt^2 + dx^2 + dy^2 + dz^2$ with corrections that are $\mathcal{O}(1/r)$ at large $r$.

Copyright © 2018 University of Cambridge. Not to be quoted or reproduced without permission.