

Part III Applications of Differential Geometry to Physics, Sheet One

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1. Show, by exhibiting the coordinate charts, that the real projective space $\mathbb{R}\mathbb{P}^n$ is a manifold. Show that $\mathbb{R}\mathbb{P}^n$ may be regarded as the n -sphere S^n with antipodal points identified. Prove that $\mathbb{R}\mathbb{P}^3 \cong SO(3)$. Show also that $\mathbb{R}\mathbb{P}^n \cong O(n+1)/O(n)$.

The complex projective space $\mathbb{C}\mathbb{P}^n$ is defined analogously to $\mathbb{R}\mathbb{P}^n$, as a set of one-dimensional *complex* subspaces in \mathbb{C}^{n+1} . Prove that, as real manifolds, $\mathbb{C}\mathbb{P}^1 \cong S^2$.

Remark. Thus S^2 has an atlas with holomorphic transition functions which makes it a *complex manifold*. It is known that no other sphere apart from S^6 is a complex manifold. It is still not known whether S^6 is a complex manifold.

2. Show that the Lie algebra of $SO(n) = \{A \in GL(n, \mathbb{R}), A^T A = \mathbf{1}\}$ may be identified with antisymmetric $n \times n$ matrices.

Let J be a $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

and let $Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}), A^T J A = J\}$. Compute the dimensions of $SO(n)$ and $Sp(2n, \mathbb{R})$. What is the Lie algebra of $Sp(2n, \mathbb{R})$?

3. Starting from the definition of the Lie derivative show that

$$\mathcal{L}_V(W) = [V, W]$$

if V and W are vector fields. Use the Leibniz rule to establish the Cartan formula

$$\mathcal{L}_V \Omega = d(V \lrcorner \Omega) + V \lrcorner d\Omega,$$

where Ω is a p -form.

Show that, if Ω is a one-form, then

$$d\Omega(V, W) = V(W \lrcorner \Omega) - W(V \lrcorner \Omega) - [V, W] \lrcorner \Omega.$$

4. Consider the matrix representation of the Euclidean group $E(2)$ in two dimensions

$$\begin{pmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

to find a basis of right- and left-invariant one forms and the dual vector fields. How is this matrix representation related to the action of $E(2)$ on \mathbb{R}^2 discussed in Lectures?

The location of a motor car with rear wheel drive may be specified by giving the coordinates (a, b) of the centre of the front axle and the angle θ that the axis of the car makes with the a -axis. Show that the configuration space of the car may be regarded as $E(2)$. If l is the distance between the mid-points of the rear and front axles, show that the vector field \mathbf{V}_ψ associated with driving forward the front wheels making a constant angle $\frac{\pi}{2} - \psi$ to the axis of the car is given by

$$\mathbf{V}_\psi = \cos \psi \cos \theta \frac{\partial}{\partial a} - \cos \psi \sin \theta \frac{\partial}{\partial b} + \sin \psi \frac{1}{l} \frac{\partial}{\partial \theta}$$

Show that a basis for $\mathfrak{e}(2)$ is given by **Steer** = $\mathbf{V}_{\frac{\pi}{2}}$, **Drive** = \mathbf{V}_0 , and **Left** = $[\mathbf{Steer}, \mathbf{Drive}]$. Calculate the commutation relations. Show in particular how, in the UK, parking may be achieved by a succession of infinitesimal steering and driving.

5. Consider three one-parameter groups of transformations of \mathbb{R}

$$x \rightarrow x + \varepsilon_1, \quad x \rightarrow e^{\varepsilon_2} x, \quad x \rightarrow \frac{x}{1 - \varepsilon_3 x},$$

and find the vector fields V_1, V_2, V_3 generating these groups. Deduce that these vector fields generate a three-parameter group of transformations

$$x \rightarrow \frac{ax + b}{cx + d}, \quad ad - bc \neq 0.$$

Show that the vector fields V_α generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and thus deduce that $\mathfrak{sl}(2, \mathbb{R})$ is a subalgebra of the infinite dimensional Lie algebra $\text{vect}(\Sigma)$ of vector fields on $\Sigma = \mathbb{R}$. Find all other finite dimensional subalgebras of $\text{vect}(\Sigma)$.

Let $f : \Sigma \rightarrow \mathbb{R}$ be a smooth function. Consider a map $\pi : C^\infty(\Sigma) \rightarrow C^\infty(T^*\Sigma)$ given by

$$\pi(f)(x, p) = pf(x), \quad \text{where } (x, p) \in T^*\Sigma$$

and show that this map gives a homomorphism between $\text{vect}(\Sigma)$ and the Lie algebra of Poisson bracket on $T^*\Sigma$.

Remark. The Poisson bracket on $T^*\Sigma$ admits a deformation to the so called *Moyal bracket* (if you want to, look it up on Wikipedia) which makes quantisation possible. On the other hand the algebra $\text{vect}(\Sigma)$ can be centrally extended to the Virasoro algebra as discussed in lectures, but is otherwise rigid.

6. The dilatations \mathbb{R}_+ and translations \mathbb{R}^4 combine as the semi-direct product $\mathbb{R}_+ \ltimes \mathbb{R}^4$ to act on $y^\mu \in \mathbb{E}^{3,1}$, the Minkowski space-time, as

$$\begin{pmatrix} y^\mu \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \lambda & x^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^\mu \\ 1 \end{pmatrix}. \quad (1)$$

Show that

$$ds^2 = \frac{1}{\lambda^2} (d\lambda^2 + \eta_{\mu\nu} dx^\mu dx^\nu)$$

is a left-invariant metric on $\mathbb{R}_+ \ltimes \mathbb{R}^4$. By considering the embedding into $\mathbb{E}^{4,2}$ given by

$$X^6 + X^5 = \frac{1}{\lambda}, \quad X^6 - X^5 = \lambda + \frac{\eta_{\mu\nu} x^\mu x^\nu}{\lambda}, \quad X^\mu = \frac{x^\mu}{\lambda},$$

with X^6 an extra timelike coordinate and X^5 an extra spacelike coordinate, show that $\mathbb{R}_+ \ltimes \mathbb{R}^4$ with this metric is one half of five-dimensional Anti-de-Sitter space-time AdS_5 .

Show that, despite being a group manifold, $\mathbb{R}_+ \ltimes \mathbb{R}^4$ equipped with this metric is geodesically incomplete.

Remark. This construction is currently quite popular because it is the basis of the *AdS/CFT correspondence*.

7. Let $A \in SO(3)$. Find the vector fields generating the action $\mathbf{x} \rightarrow A\mathbf{x}$ of $SO(3)$ on \mathbb{R}^3 . Show that this action restricts to $S^2 \subset \mathbb{R}^3$, and that the symplectic form $d(\cos \theta) \wedge d\psi$ on S^2 , where (θ, ψ) are spherical polars,

is preserved by the action. Deduce that the action on the two-sphere is generated by Hamiltonian vector fields, and find the corresponding Hamiltonians. Verify that these Hamiltonians form a Lie algebra with a Poisson bracket, which is isomorphic to the Lie algebra of $SO(3)$.

8. A Poisson structure on \mathbb{R}^{2n} is an anti-symmetric matrix ω^{ab} whose components depend on the coordinates $x^a \in \mathbb{R}^{2n}, a = 1, \dots, 2n$ and such that the Poisson bracket

$$\{f, g\} = \sum_{a,b=1}^{2n} \omega^{ab}(x) \frac{\partial f}{\partial x^a} \frac{\partial g}{\partial x^b}$$

satisfies the Jacobi identity.

Show that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

Assume that the matrix ω is invertible with $W := (\omega^{-1})$ and show that the antisymmetric matrix $W_{ab}(\xi)$ satisfies

$$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0, \tag{2}$$

or equivalently that the two-form $W = (1/2)W_{ab}dx^a \wedge dx^b$ is closed.

[Hint: note that $\omega^{ab} = \{x^a, x^b\}$.] Deduce that if $n = 1$ then any anti-symmetric invertible matrix $\omega(x^1, x^2)$ gives rise to a Poisson structure (i.e. show that the Jacobi identity holds automatically in this case).

Remark. The invertible antisymmetric matrix W which satisfies (2) is called a symplectic structure. We have therefore deduced that symplectic structures are special cases of Poisson structures.

9. The metric of hyperbolic 3-space H^3 in Beltrami coordinates is given by

$$ds^2 = \frac{d\mathbf{r}^2}{1-r^2} + \frac{(\mathbf{r} \cdot d\mathbf{r})^2}{(1-r^2)^2}.$$

Let

$$\mathbf{M} = \mathbf{p} - \mathbf{r}(\mathbf{p} \cdot \mathbf{r}), \quad \mathbf{L} = \mathbf{r} \times \mathbf{p},$$

so that $\mathbf{M} \cdot \mathbf{L} = 0$. Show that the Hamiltonian for geodesic motion is given by

$$H = \frac{1}{2}(\mathbf{M}^2 - \mathbf{L}^2).$$

Obtain the Poisson brackets

$$\begin{aligned}\{L_i, L_j\} &= \epsilon_{ijk} L_k, \\ \{M_i, M_j\} &= -\epsilon_{ijk} L_k, \\ \{L_i, M_j\} &= \epsilon_{ijk} M_k.\end{aligned}$$

Hence show that both \mathbf{L} and \mathbf{M} are constants of the motion. Identify the associated Killing vector fields and compute their Lie brackets. Show that

$$\mathbf{M} = \frac{\dot{\mathbf{r}}}{1 - r^2}$$

and hence that the geodesics are straight lines in Beltrami coordinates. What is the geometrical significance of the condition $\mathbf{M} \cdot \mathbf{L} = 0$?

Check that the Poisson algebra of $L_{ij} = \epsilon_{ijk} L_k$ and L_{0i} is that of the Lorentz Lie algebra $\mathfrak{so}(3, 1)$, and show that H and $\mathbf{M} \cdot \mathbf{L}$ are quadratic Casimirs.

10. The set of oriented lines in Euclidean space \mathbb{R}^{n+1} may be parametrized in terms of their unit tangent vector \mathbf{t} and the vector \mathbf{p} joining the an arbitrary origin 0 to the point P of nearest approach of the line to this origin. Identify the space of oriented lines as TS^n - the tangent bundle to the n -dimensional sphere.

Now Consider $n = 2$.

- (a) Show that points $P \in \mathbb{R}^3$ corresponds to maps L_P from S^2 to TS^2 which should be constructed. Let $\tau : TS^2 \rightarrow TS^2$ be a fixed-point-free map obtained by reversing the orientation of each straight line. Show that a two-sphere in TS^2 corresponding to $P \in \mathbb{R}^3$ is preserved by τ .
- (b) Describe the action and orbits of rotations about O on TS^2 . How does the Euclidean group $E(3)$ act? What happens if we consider *unoriented* lines?

Remark. You have established a *mini-twistor correspondence* between points in \mathbb{R}^3 and spheres in TS^2 . If a complex atlas (see Question 1) is used on S^2 , then TS^2 becomes a complex manifold, and holomorphic functions on this manifold give rise to solutions of linear and non-linear PDEs on \mathbb{R}^3 (like the Bogomolny equations for magnetic monopoles).