1. Let \( \phi : \mathbb{R}^{2,1} \to S^2 \). Set
\[
\phi^1 + i\phi^2 = \frac{2u}{1 + |u|^2}, \quad \phi^3 = \frac{1 - |u|^2}{1 + |u|^2},
\]
and deduce that the Bogomolny equations
\[
\partial_i \phi^a = \pm \varepsilon_{ij} \varepsilon^{abc} \partial_j \phi^b, \quad \phi_i = 0
\]
imply that \( u \) is holomorphic or antiholomorphic in \( z = x_1 + ix_2 \).
Find the expression for the total energy
\[
E[\phi] = \frac{1}{2} \int \partial_j \phi^a \partial_j \phi^a d^2x
\]
in terms of \( u \).
By counting the pre-images or otherwise find the topological degree of \( \phi \) corresponding to \( u(z) = u_0 + u_1 z + \ldots + u_k z^k \), where \( u_0, \ldots, u_k \) are constants with \( u_k \neq 0 \).

2. Derive the \( SU(2) \) Yang–Mills theory on \( \mathbb{R}^4 \) form the action. Let \( A_a(x) \) be a solution to these equations. Show that, for any nonzero constant \( c \), the potential \( \tilde{A}_a(x) = cA_a(cx) \) is also a solution and that it has the same action.

3. Consider the map \( g : S^3 \to SU(2) \) defined by
\[
g(x_1, x_2, x_3, x_4) = x_4 + i(x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3),
\]
where \( \sigma_i \) are Pauli matrices and \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \) and find its degree. By calculating \( Tr((dg g^{-1})^3) \) at the point on \( S^3 \) where \( x_4 = 1 \), or otherwise deduce that the formula
\[
\deg(g) = \frac{1}{24\pi^2} \int_{S^3} Tr((dg g^{-1})^3)
\]
is correctly normalised.
4. Let $T_1, T_2, T_3$ form a basis of $\mathfrak{su}(2)$ such that
\[ [T_\alpha, T_\beta] = -\varepsilon_{\alpha\beta\gamma} T_\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3, \]
and let the symbols $\sigma_{ab} = -\sigma_{ba}$ where $a, b = 1, \ldots, 4$ be defined by
\[ \sigma_{a\beta} = \varepsilon_{a\beta\gamma} T_\gamma, \quad \sigma_{a4} = T_a. \]
Show that
\[ \sigma_{ab} = \frac{1}{2} \varepsilon_{ab} \sigma_{cd}, \quad \text{and} \quad \sigma_{ab} \sigma_{ac} = -\frac{3}{4} \delta_{bc} - \sigma_{bc}. \]
Identify $\Lambda^2(\mathbb{R}^4)$ with the Lie algebra $\mathfrak{so}(4)$ and deduce that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

5. Let $V = 1 + r^{-2}$, where $r^2 := \delta_{ab} x^a x^b$. Show that the one–form
\[ A = \sigma_{ab} \frac{1}{V} \frac{\partial V}{\partial x^b} dx^a \]
is a solution of the anti–self–dual Yang–Mills equations on $\mathbb{R}^4$. The one–form $A$ is singular at $r = 0$. What can you say about the behaviour of the field strength $F$ at $r = 0$?

6. Find, by explicit integration, the Chern number of the solution (1).

7. Let $F$ be a two–form on $\mathbb{R}^4$. Show, from the definition of the Hodge operator, that
   \begin{enumerate}
   \item $**F = \pm F$ depending on the signature.
   \item $F \wedge *F = F \wedge F$.
   \end{enumerate}
Show that in the $U(1)$ theory $F \rightarrow *F$ interchanges the electric and magnetic fields with factors of $\pm 1$ or $\pm i$ and determine the different cases in the corresponding signatures.
Let $F$ be a non–zero real self–dual two–form on $\mathbb{R}^4$ such that $F \wedge F = 0$. What is the signature of the underlying metric?

8. Let $A$ be a 1–form gauge potential on $\mathbb{R}^n$ with values in $\mathfrak{su}(2)$, and let $F$ be its curvature. Verify that $Tr(A), Tr(A \wedge A), Tr(A \wedge A \wedge A \wedge A)$ and $Tr(F)$ all vanish.
Verify that $C_2 = dY_3$, where $C_2$ and $Y_3$ are the second Chern form, and the Chern–Simons three–form respectively.
9. Let $A = A_i dx^i, i = 1, 2, 3$ be a gauge potential on $\mathbb{R}^3$ with values in the Lie algebra $\mathfrak{g}$. Find the Euler–Lagrange equations arising from varying the Chern–Simons functional

$$W[A] = \int_{\mathbb{R}^3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

with respect to $A$.

Now consider a one parameter family of $\mathfrak{g}$–valued one–forms $A = A(t)$ on $\mathbb{R}^3$, and define a one–form on $\mathbb{R}^4$ by $A = A + \phi dt$, where the function $\phi = \phi(x^i, t)$ takes its values in $\mathfrak{su}(2)$. Show that, in a gauge where $\phi = 0$, the anti–self–dual Yang–Mills equations on $A$ take the gradient flow form

$$\frac{dA_i(t)}{dt} = \frac{\delta W[A]}{\delta A_i}.$$

10. Consider a connection $\omega = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma$ on a principal $G$–bundle $P \to B$, where $A$ is a one–form on $B$ and $\gamma^{-1} d\gamma$ is the Maurer–Cartan form on $G$.

(a) Show that the transformation of the fibres $\gamma' = g \gamma$, where $g \in G$ depends on the coordinates on $B$, does not change $\omega$ if $A$ transforms like a gauge potential.

(b) Let $\Omega = d\omega + \omega \wedge \omega$. Show that $\Omega = \gamma^{-1} F \gamma$ for some $F$ which should be found.

(c) Let $D_a, a = 1, \ldots, \dim(B)$ be linearly independent vector fields on $P$ such that

$$D_a \omega = 0.$$

Show that $D_a = \partial_a - A^a R_a$, where $\partial_a = \partial / \partial x^a$ are vector fields on $B$ and $R_a$ are right–invariant vector fields on $G$. Demonstrate that

$$[D_a, D_b] = -F_{ab}^\alpha R_\alpha.$$