Concepts in Theoretical Physics

Lecture 1: Equations in Physics, Simplicity, and Chaos

John D Barrow

'If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is'
John von Neumann, 1947

Schedule of Lectures

CONCEPTS IN THEORETICAL PHYSICS (A11)

Easter Term 2019

John D. Barrow

- Lecture 1: Equations in Physics, Simplicity, and Chaos (25/4)
- Lecture 2: Invariances, Constants, and Dimensions (30/4)
- Lecture 3: Action Principles (2/5)
- Lecture 4: Quantum Mechanics (7/5)
- Lecture 5: Statistical physics: entropy, demons & black holes (9/5)
- Lecture 6: General Relativity (14/5)
- Lecture 7: The Maths of Whole Universes (16/5)
- Lecture 8: Elementary Particles (21/5)

Why is Maths so Helpful?

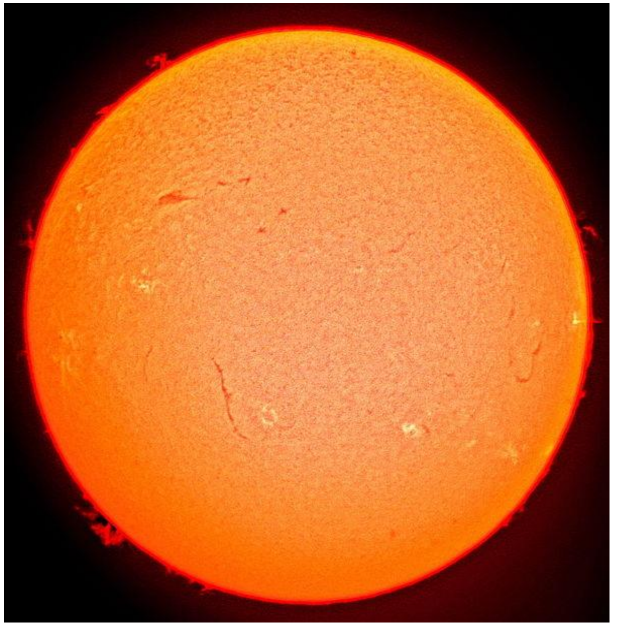
- Maths is infinite. It is the catalogue of all possible patterns. Some of these patterns are very useful descriptions of phenomena in the world. It is not therefore a mystery that maths 'works' or is 'unreasonably effective' as a description of the world because patterns must exist for there to be observers of them.
- But the surprising fact is that relatively simple patterns and a small amount of fairly elementary maths can tell us so much about the universe. We can imagine scenarios that would be much more complicated or where very few mathematical operations would be computable. This imaginary world would still be describable by maths but we wouldn't find maths so useful for carrying out useful calculations or developing algorithms.
- Sometimes Nature leads us to develop mathematics (generalised functions, fractals, chaotic dynamics, calculus, knots, non-Euclidean geometries) and sometimes pure mathematical patterns predict new physics (Riemannian geometry, complex numbers, Hilbert spaces, group theory, tensors, complex manifolds).
- In physics you will study the powerful consequences of a small part of the whole (infinite) realm of mathematics.

But Sometimes.. Not So Effective

- The effectiveness of mathematics is particularly impressive in the physical sciences, like physics. Eugene Wigner called it the 'unreasonable effectiveness of mathematics'
- However, maths is often unreasonably ineffective in the human sciences of behaviour, psychology, economics, and the study of life and consciousness.
- These complex sciences are dominated by non-linear behaviour and only started to be explored effectively by many people (rather than only huge well-funded research groups) with the advent of small personal computers (since the late 1980s) and the availability of fast supercomputers.
- Some complex science contain unpredictabilities in principle: predicting the economy changes the economy whereas predicting the weather doesn't change the weather
- The key to good mathematical modelling is using the right mathematics and realistic idealisations and approximations.
- *The Theory of Sound* vs *Mathematical Modelling of Sonic Phenomena*

Physics Problems often have symmetry or almost symmetry

- We can start with an idealised model of a star as spherical and of constant surface temperature and gradually add small variations, step by step. This is the essence of 'computability' – the same idea used again and again to improve accuracy.



The Ubiquitous Harmonic Oscillator

It has been said that what distinguished physics from biological science is that physics has the simple harmonic oscillator! Why is it so ubiquitous? Suppose we have a general force law of the form, then

$$x'' = F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + ... \simeq F(0) + xF'(0)$$

for small displacements, x. Change coordinates by a constant shift $x = X - F(0)/F'(0)$ so that the equilibrium point is at the origin $X = 0$ and

$$X'' = XF'(0) = -\omega^2 X$$

if the force is attractive $(F'(0) < 0)$.So small displacements from the equilibrium all obey the SHO eqn and

$$X(t) = A\cos(\omega t) + B\sin(\omega t).$$

The Differential Equations of Physics

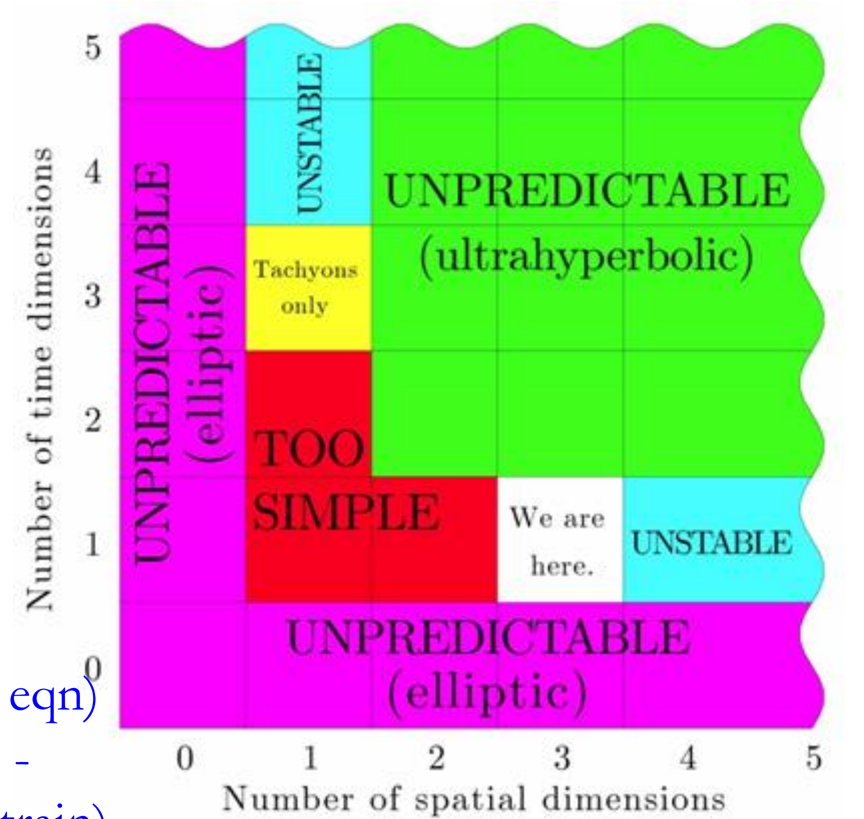
- You will encounter several ubiquitous types of partial differential equation (pde). For example,
- Wave equations: $u_{tt} = c^2 u_{xx}$ or $u_{tt} = c^2 \nabla^2 u$
- Diffusion equations: $u_t = k^2 u_{xx}$ or $u_t = k^2 \nabla^2 u$
- Poisson's equation: $\nabla^2 \phi = 4\pi G \rho$, for grav. potential ϕ
- Schrödinger's equation:

$$[\frac{-\hbar^2}{2m} \nabla^2 + V] \Psi = i \hbar \frac{\partial}{\partial t} \Psi$$

Are there any general lessons ?

Dimensions and pdes

$$\left[\sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x^i} + c\right] u = 0$$



d is the space+time dimension;

A,b,c fns of the d coordinates (time + space);

Key are the eigenvalues of the matrix A_{ij} :

Elliptic if all + or all -, (Poisson eqn)

Hyperbolic one +, rest - (or vice versa), (wave eqn)

Ultrahyperbolic at least two + and at least two -

Elliptic eqns don't predict the future (just constrain)

Ultrahyperbolic are eqns ill-posed

(ie future is not uniquely, completely and stably predicted,

Hyperbolic eqns allow well-posed initial value problem

with one time (or 1 space coordinate)

Gravitational force $F \propto 1/r^{N-1}$

$\nabla^2 \phi = 0 = \nabla . F$

in N-dimensional empty space

This is why we see so many

inverse-square laws

Most equations cannot be solved exactly

- Most integrals and differential equations cannot be solved exactly

$$\int e^{x^2}\, dx \text{ or } dy/dx = e^{xy}$$

- Duffin's 'Universal Equation' shows how complicated ordinary differential eqns can be:

$$2y'''y'^2 - 5y'''y''y' + 3y''^3 = 0$$

has solutions which follow any pre-specified function f(t) arbitrarily closely for all $t \in (-\infty, \infty)$

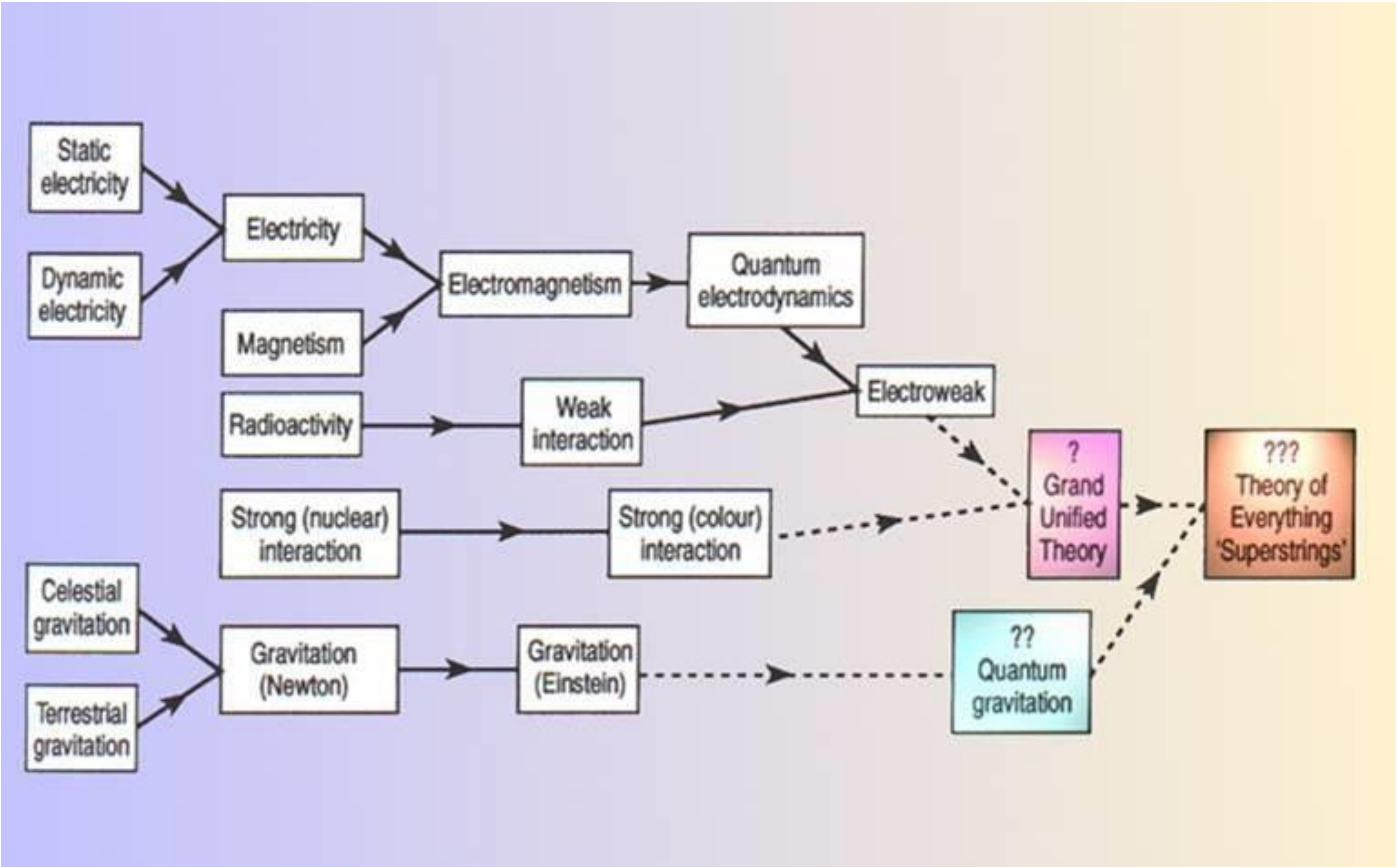
$$|y(t) - f(t)| < \varepsilon(t)$$

for any continuous function $\varepsilon(t)$.

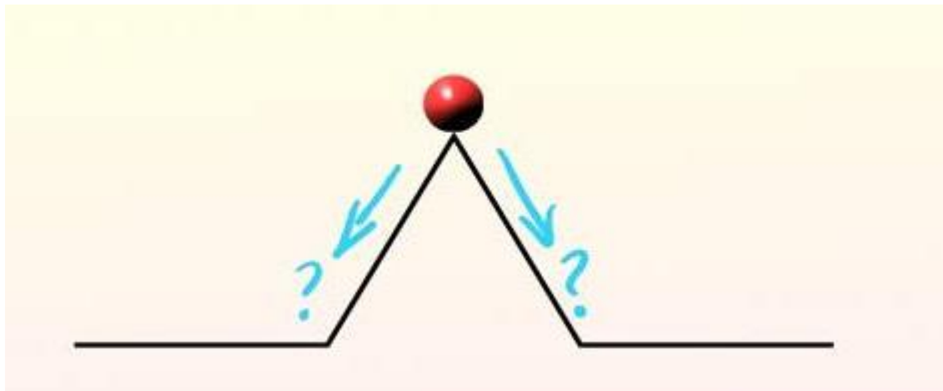
Simplicity and Complexity

- Physical problems also often have symmetry and stable equilibrium states so there are lots of computable problems whose solutions can be built up step by step by successive approximations, using the same principle over and over again.
- An uncomputable function needs a new idea at each stage of the approximation process.
- Many complex systems -- like economies, societies -- don't have obvious symmetrical approximations. They are often, so called, 'emergent' systems in which the whole is more than the sum of its parts because of the importance of the connectivities between parts.

The Story of Physics – unifications of laws



Laws (equations) *vs* Outcomes (solutions)

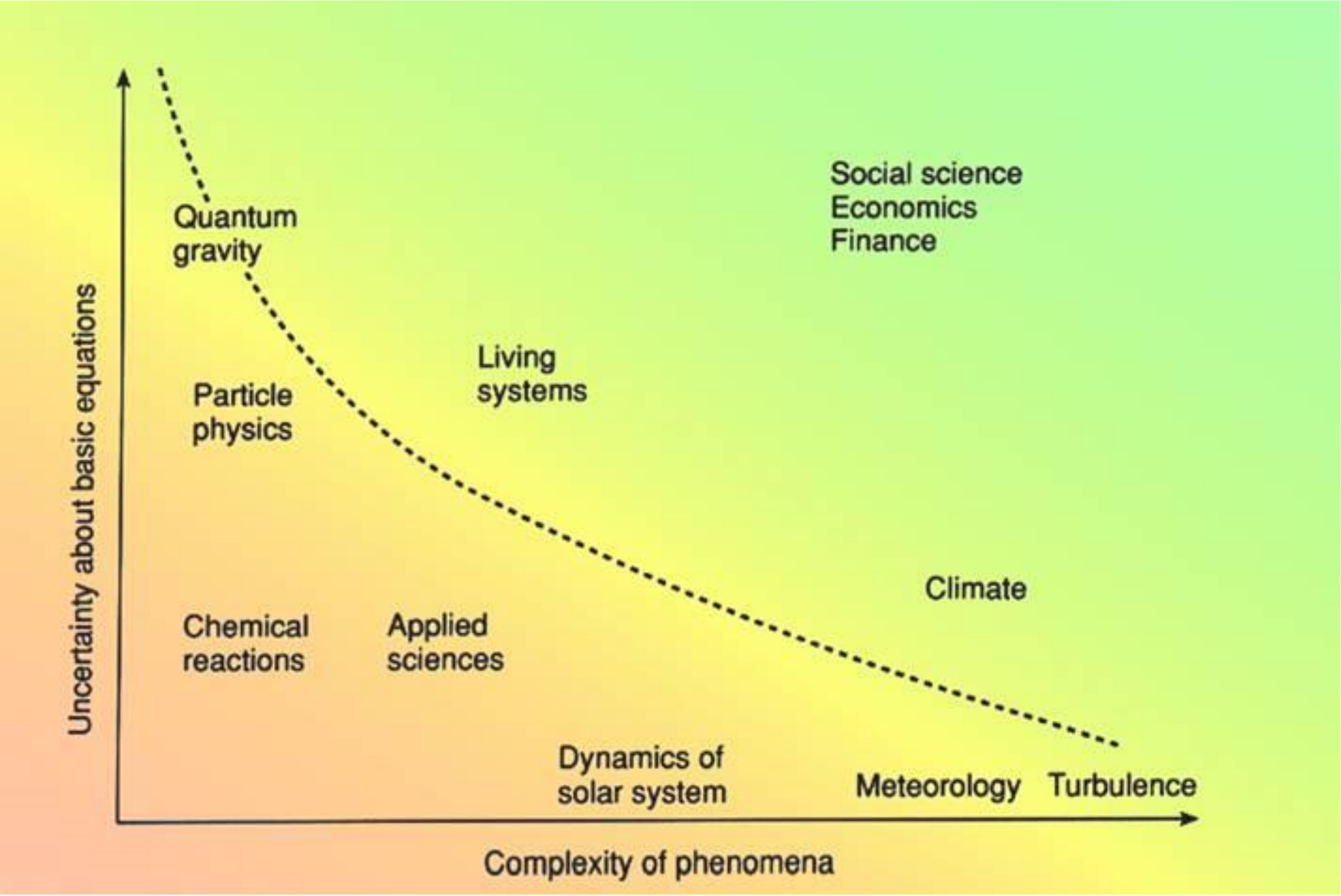


Outcomes are more complicated and less symmetrical than the laws (equations) that govern them

Solutions of equations need not possess the same symmetries as those equations

Symmetry breaking makes the world interesting

Simplicity *vs* Complexity

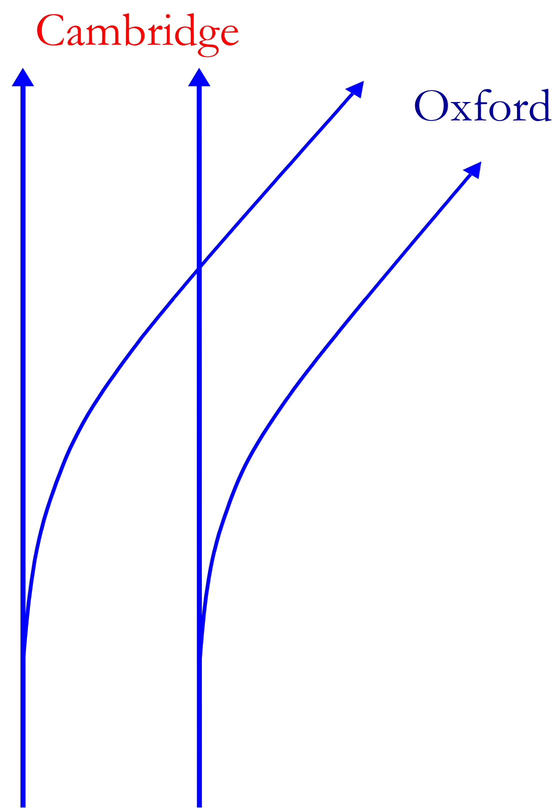


Chaos

- Chaos is exponential sensitivity to ignorance

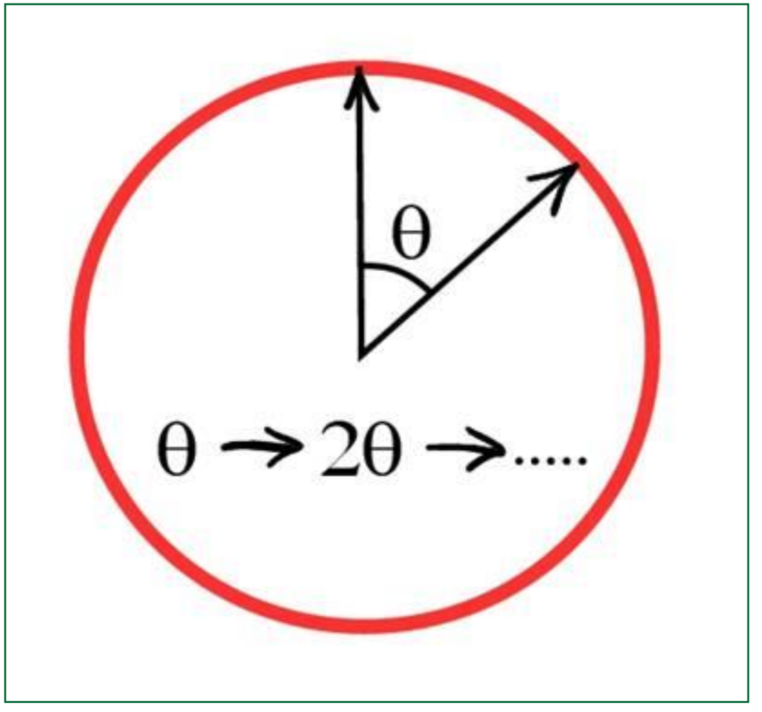
  $\delta x(t) \sim \delta x(0)\exp[\lambda t],\ \lambda>0$

- James Clerk Maxwell first noticed this in1873 when in Cambridge



"the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate, and not accurate." Maxwell, 1873

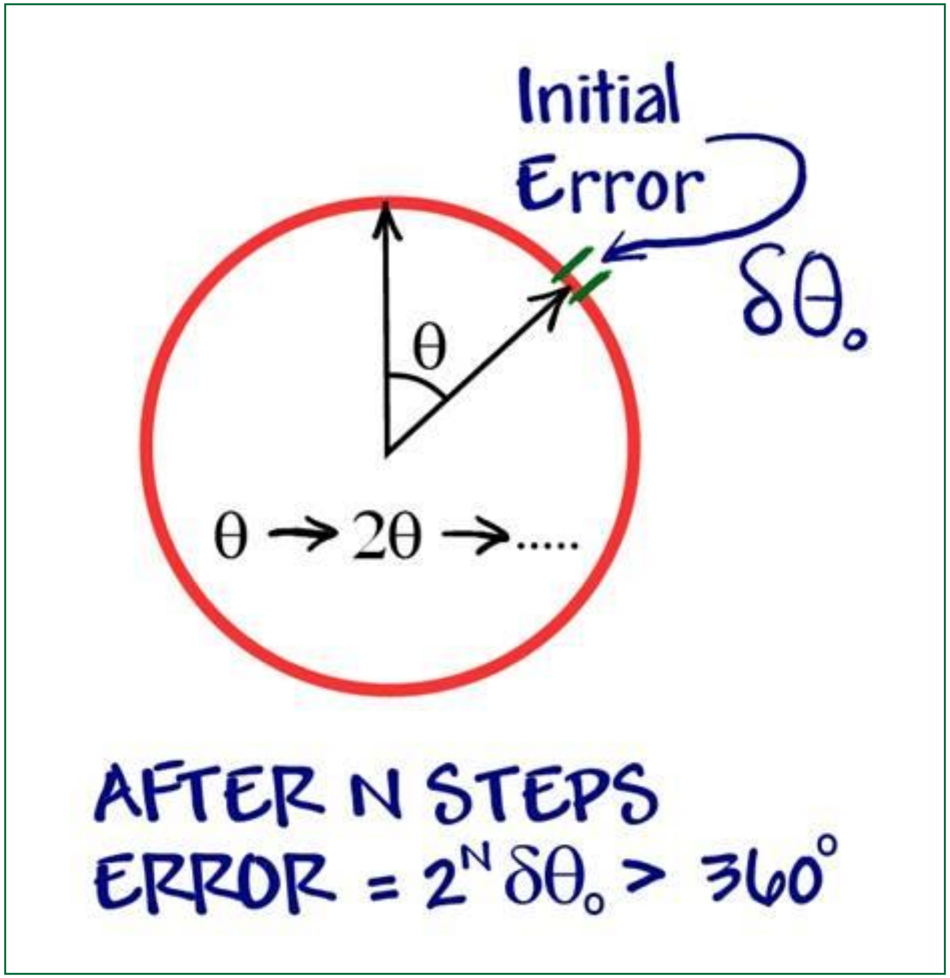
Deterministic in principle



A clock-face 'universe' governed by one law:

$$\theta_{n+1} = 2\,\theta_n$$

But not deterministic in practice…



Snooker balls and molecules

- Collision between two snooker balls (radius r, inter-collision distance d) creates

$$\theta_{n+1} = (d/r)\,\theta_n$$

So for snooker with θ_0 as small as Heisenberg's Uncertainty Principle allows $\theta > 360$ deg when $n > 13$

- For gas molecules in the room $d/r \sim 200$ leads rapidly to chaos
- But we have well-behaved averages with Boyle's Gas Law:

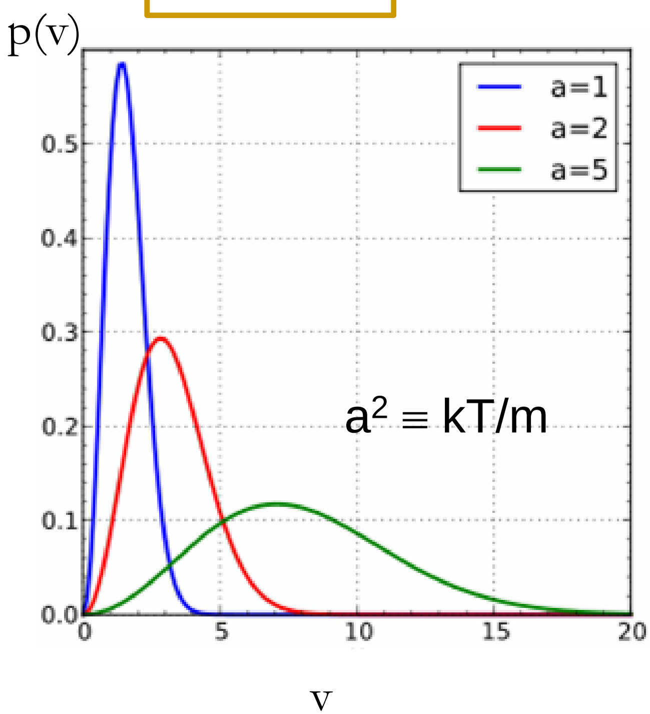
$$PV/T = \text{constant}$$

There is an equilibrium ('Maxwell-Boltzmann') probability distribution of velocities:

$$p(v) = 4\pi[m/2\pi kT]\, v^2 \exp\{-mv^2/2kT\}$$

k is Boltzmann's constant.

"Ideal gas approx"
$d >> r$



Chaos doesn't mean we can't predict anything!

How Big For Chaos to Occur?

- Differential equations need to be at least 3rd order to be chaotic (d^3y/dx^3 +…. = 0): for example
- $d^3y/dx^3 + a\, d^2y/dx^2 - (dy/dx)^2 + y = 0$, for $2.0168 < a < 2.0577$

But occurs in 1st order differential delay eqns:

$$dy/dx = \sin\{y(x - T)\},\ T \text{ const.}$$

- Difference equations can be chaotic in one dimension:

$$x_{n+1} = T(x_n) \text{ with}$$

$$|dT/dx| > 1$$

- Sensitive dependence on initial conditions
- This occurs in number theory in the continued fraction expansion digits of almost every irrational number

Continued fractions

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}}}}}}}}}$$

π = [3;7,15,1,292,1,1,3,1,14,2…..]

e = 2.718…. = [2;1,2,1,1,4,1,1,6,1,1,8,1,1,10,….]

ϕ = [1;1,1,1,1,1,1,1,……..]

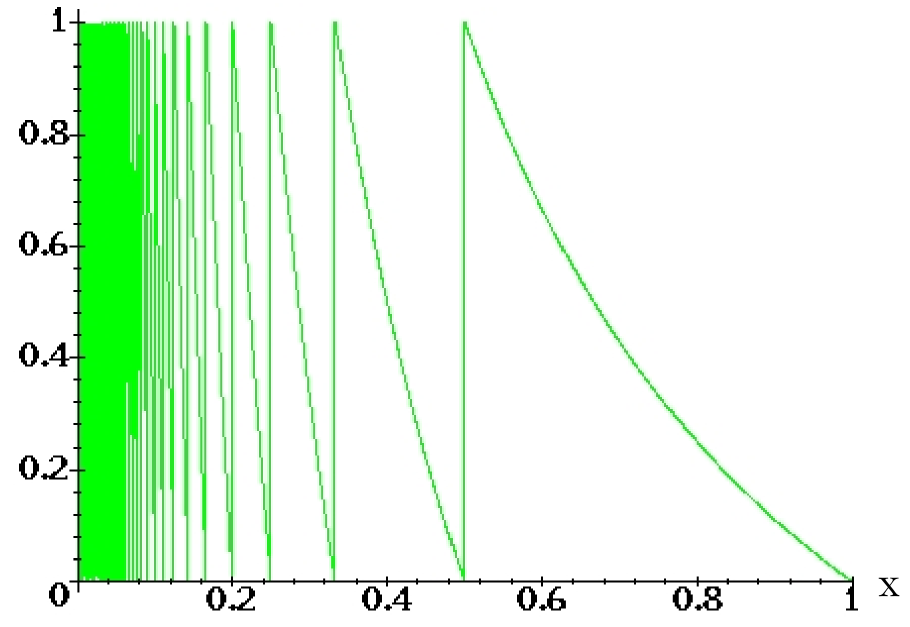
Continued fractions as chaotic dynamics

Integer part

- $x_{n+1} = 1/x_n - [1/x_n]$
- For $0 < x_n < 1$
  - $|dT/dx| = 1/x^2 > 1$
- As $n \to \infty$ the probability f of outcome x tends to
  - $p(x) = 1/[(1+x)\ln 2]$:
    - $\int_0^1 p(x)dx = 1$
- $h = \int_0^1 |dT/dx|\, p(x)dx$

$T(x) = 1/x - k$

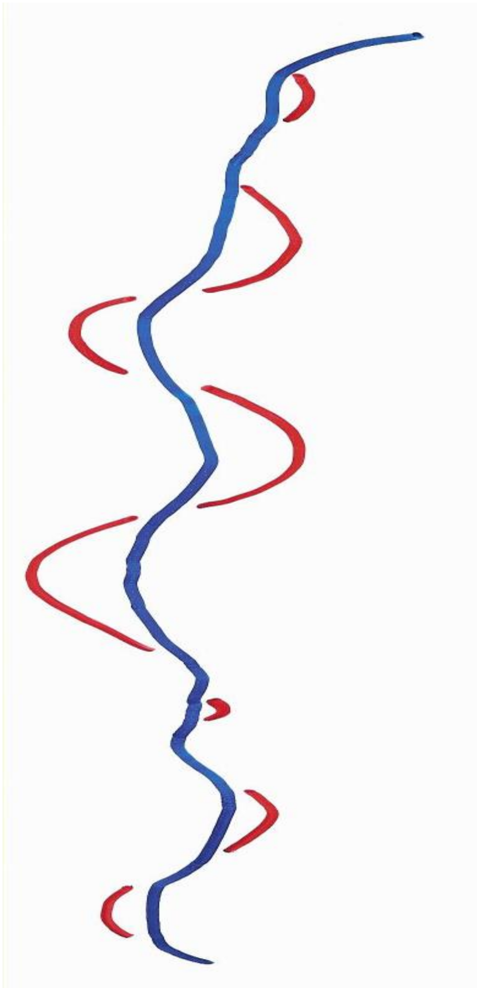
$(1-k)^{-1} < x < k^{-1}$

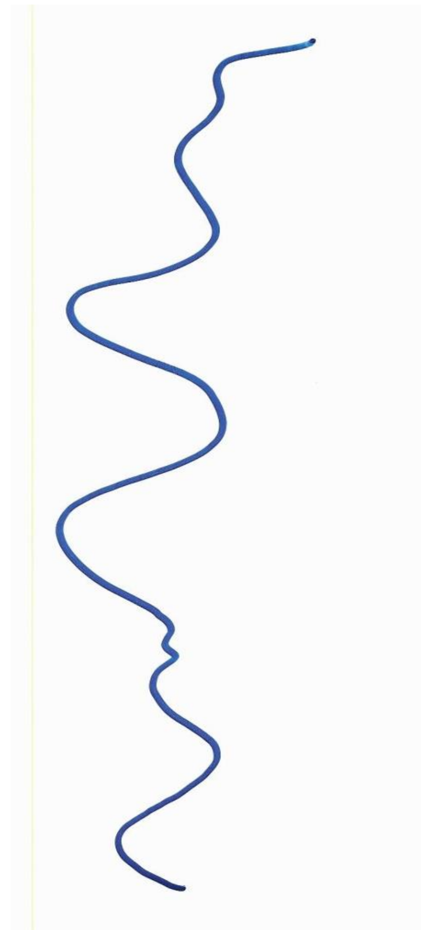


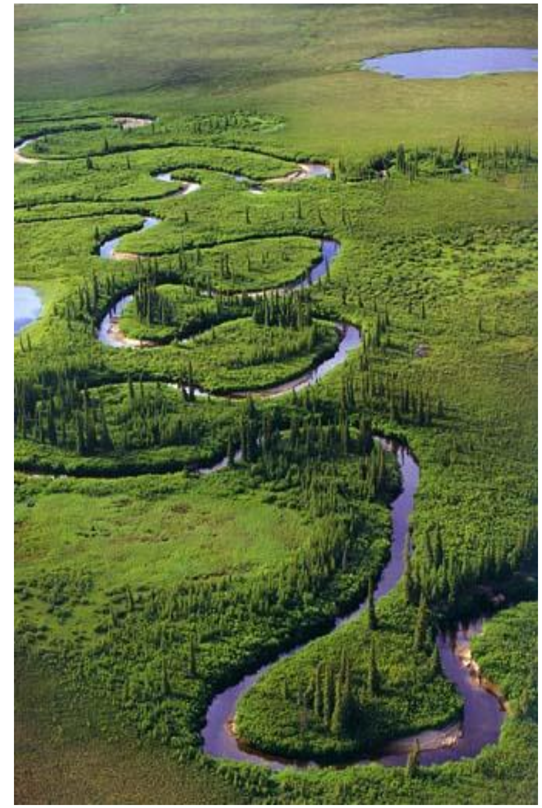
$\Delta_{n\ steps} = \Delta_{initial} \times \exp(ht)$: $h = \pi^2/[6(\ln 2)^2] \approx 3.45$

Organised complexity

Oxbow lakes: the sandpile in disguise?

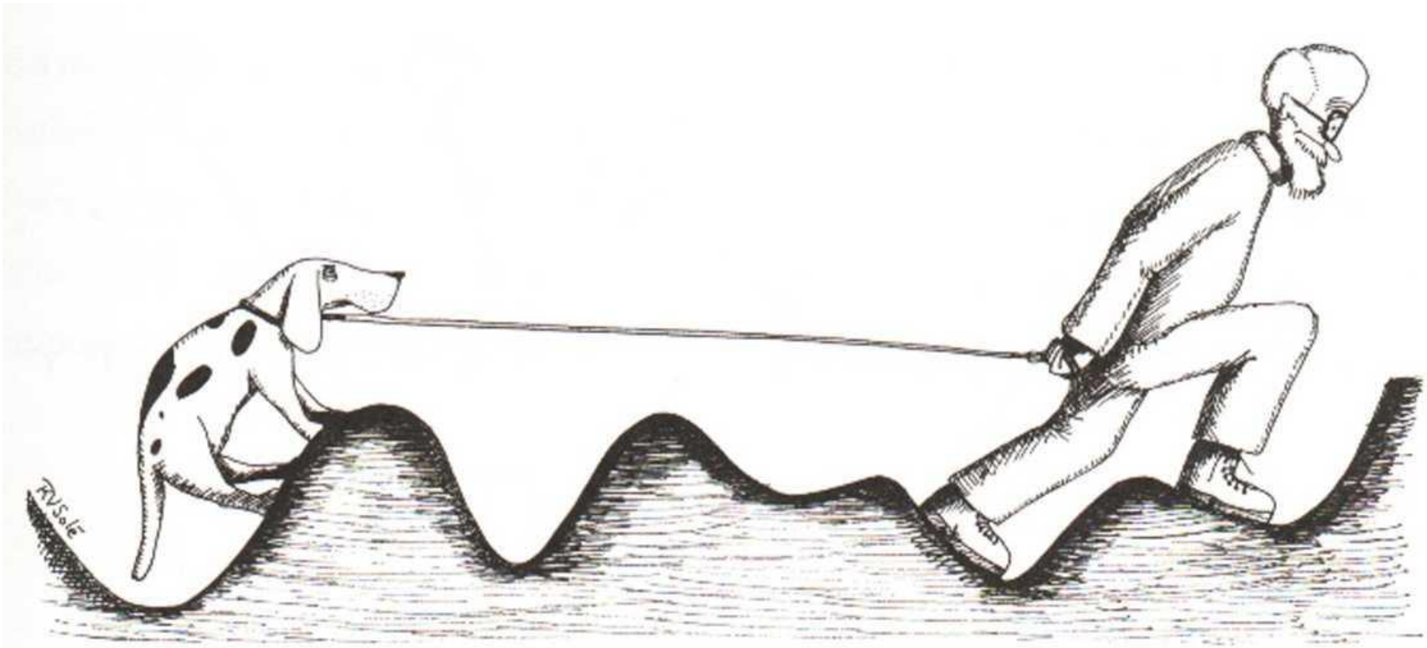






Kanuti River, Alaska

Why the general idea is so common



Richard Solé

Many equilibrium states plus the application of a force

Further reading

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